The Spectrum of the Symmetric Space $SP(l)/SU(l)$

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Abstract

The aim of the present paper is to determine the spectrum of the Laplace operator on the functions on $SP(l)/SU(l)$.

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1 Intoduction

Let (M, g) be a compact Riemannian manifold of dimension n. From this Riemannian manifold we obtain the vector spaces $\Lambda^q(M, R), q = 0, 1, ..., n$. If we apply the Laplace operator $\Delta = d\delta + \delta d$ on $\Lambda^q(M, R)$ we obtain the spectrum $Sp^q(M, g)$.

One problem in the spectral theory is to determine $Sp^q(M, g)$, where (M, g) is a known compact Riemannian manifold.

The aim of the present paper is to determine $Sp^0(M, g)$, that means the spectrum of Δ on the functions on M, where $M = SP(l)/SU(l)$ and g is the Riemannian metric on M, coming from the Killing-Cartan form of the Lie algebra $sp(l)$.

2 Basic elements for symmetric spaces

Let $M = G/H$ be a compact homogenous space. One of the problems concerning of the spectrum is to calculate

(2.1) $$Sp^q(M = G/H, g), q = 0, 1, ..., dim\, M$$

Let t and v be the Lie algebras of G and H respectively. If m is the tangent space of M at its origin, we have the relation

(2.2) $$t = v \oplus m$$

Let $\langle , \rangle$ be a positively definite inner product on the vector space m which comes from the Riemannian metric g, which is an $ad(H)$-invariant. Conversely, an inner

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product on m which is an $ad(H)$-invariant gives a G-invariant metric on the homogeneous space M, which is also called homogeneous manifold.

We assume that the Lie group G is compact and semisimple and the Lie subgroup H of G is its closed subgroup.

Let b be the Cartan subalgebra of t. The set of the simple roots which are linear forms on the b_{C}, that is elements of b^*_{C}, is given as follows:

$$\Psi = \{\psi_1, ..., \psi_l\}\,, \tag{2.3}$$

where $l = \dim b$.

We denote by Λ the set of all roots of b_{C},

$$\Lambda = \left\{ \lambda = \sum_{i=1}^{l} r_i \psi_i / r_i \in \mathbf{Z} \right\} \tag{2.4}$$

and with Λ^+ the set of all positive roots, so we have

$$\Lambda^+ = \left\{ \lambda = \sum_{i=1}^{l} r_i \psi_i / r_i \geq 0 \right\}.$$

For each element $p \in b^*$, we take $p^* = \dfrac{2p}{< p, p >}$ so we can construct the following set:

$$\Psi^* = \{\psi^*_1, ..., \psi^*_l\} \tag{2.5}$$

where:

$$\psi^*_1 = \frac{2\psi_1}{\langle \psi_1, \psi_1 \rangle}, ..., \psi^*_l = \frac{2\psi_l}{\langle \psi_l, \psi_l \rangle}$$

and consequently for each λ, we take λ^* defined as follows:

$$\lambda^* = \frac{2\lambda}{< \lambda, \lambda >}. \tag{2.6}$$

Now, we can define the fundamental weights $\mu_i \in b, i = 1, ..., l$, via

$$< \mu_i, \psi_j >= \delta_{ij}, i = 1, ...l, j = 1, ..., l \tag{2.7}$$

We construct the set of the weights

$$B = \left\{ \mu = \sum_{i=1}^{l} m_i \mu_i / m_i \in \mathbf{Z}, m_i \geq 0 \right\}. \tag{2.8}$$

If we consider the set

$$\{I = \mu \in b / < \mu, \psi^*_i > \in \mathbf{Z}, i = 1, ..., l\}\,, \tag{2.9}$$

then B is equivalent to the set

(2.10) $$N_1 = \{\mu \in / < \mu, \lambda > \geq 0, \forall \lambda \in \Lambda\},$$

or

(2.11) $$N_2 = \{\mu \in I / < \mu, \psi_i > \geq 0, i = 1, ..., l\}.$$

The above relations are used for the calculations of the spectrum of a symmetric space $M = G/H$, where G is a simply connected compact semisimple Lie group and H is a closed subgroup of G, where the Riemannian metric on M comes from the Killing-Cartan form $\langle , \rangle$ on the Lie algebra t of G. If v is the Lie algebra of H, then we have

$$t = v \oplus s,$$

where v is the eigenspace with the eigenvalue $+1$ and s is the eigenspace with the eigenvalue -1.

Let m be the maximal abelian subspace of s and Λ_m^+ the system of positive roots on $m_{\mathbb{C}}$.

We define the subset k of v as follows

(2.12) $$k = \{x \in v/[x, m] = 0\},$$

and we consider the Cartan subalgebra b_1 of k.

The following relaton is valid

$$b = b_1 + m,$$

and therefore b is a Cartan subalgebra of t. We define the Λ_m^+ as follows:

(2.13) $$\Lambda_m^+ = \left\{\lambda/m : \lambda \in \Lambda^+ \text{ and } \lambda/m \neq 0\right\},$$

and we construct the set

(2.14) $$B_m = \left\{\mu \in^* | \frac{< \mu, \lambda >}{< \lambda, \lambda >} \geq 0, \quad \forall \lambda \in \Lambda_m^+\right\}.$$

The following theorem is valid

Theorem. *Let $M = G/H$ the compact symmetric space of Riemann, where G is a simply connected, compact, semisimple Lie group and H is a closed subgroup of G, where the metric g on M comes from the Killing-Cartan form of the Lie algebra t of G. The spectrum $Sp(M, g)$ is given by*

(2.15) $$Sp(M, g) = \{< \mu, \mu > +2 < \mu, \delta_m >, \mu \in B_m\},$$

where δ is the half of the sum of the positive roots and δ_m is the restriction of δ on m, which is the maximal abelian subspace of s (the eigenspace with eigenvalue -1). The multiplicity of the eigenvalue is given by

(2.16) $$P(\mu) = \prod_{\lambda \in \Lambda^+} \frac{< \mu + \delta, \lambda >}{< \delta, \lambda >}.$$

3 The spectrum $Sp^0(M,g)$, where $M = SP(l)/SU(l)$

Let $sp(l)$ and $su(l)$ be the Lie algebras of $SP(l)$ and $SU(l)$ respectively. Then we have

$$sp(l) = su(l) \oplus m = h \oplus m,$$

where the direct sum is valid if the Lie algebras are considered as vector spaces and m is the tangent space of the $SP(l)/SU(l)$ at its origin.

The Lie algebra $sp(l)$ has the form

$$sp(l) = \left\{ X \in M_{2l} | XK + KX^T = O, \quad \text{where} \quad K = \begin{bmatrix} O & I_l \\ I_l & O \end{bmatrix} \right\}.$$

An element of the Lie algebra $sp(l)$ is split into an element of $su(l)$ and an element of the tangent space m in the following way

$$X \in sp(l) \Rightarrow X = \begin{bmatrix} \underset{l \times l}{\mathrm{A}} & \underset{l \times l}{\mathrm{B}} \\ \underset{l \times l}{\mathrm{C}} & \underset{l \times l}{\mathrm{A}^T} \end{bmatrix},$$

where B and C are symmetric martices of order l.

The matrix A can be written $A = \begin{bmatrix} a_{11} & \underset{1 \times (l-1)}{\mathrm{A}_{12}} \\ \underset{(l-1) \times 1}{\mathrm{A}_{21}} & \underset{(l-1) \times (l-1)}{\mathrm{A}_{22}} \end{bmatrix}$.

Therefore the vector X takes the form

$$X = \begin{bmatrix} a_{11} & \underset{1 \times (l-1)}{\mathrm{A}_{12}} & \multicolumn{2}{c}{\underset{l \times l}{\mathrm{B}}} \\ \underset{(l-1) \times 1}{\mathrm{A}_{21}} & \underset{(l-1) \times (l-1)}{\mathrm{A}_{22}} & & \\ \multicolumn{2}{c}{\underset{l \times l}{\mathrm{C}}} & -a_{11} & -A_{21}^T \\ & & -A_{12}^T & -A_{22}^T \end{bmatrix} =$$

$$= \begin{bmatrix} -tr(A_{22}) & A_{12} & \multicolumn{2}{c}{O} \\ A_{21} & A_{22} & & \\ \multicolumn{2}{c}{O} & tr(A_{22}) & -A_{21}^T \\ & & -A_{12}^T & -A_{22}^T \end{bmatrix} \oplus$$

$$\oplus \begin{bmatrix} tr(A_{22}) + a_{11} & O & \multicolumn{2}{c}{B} \\ O & O & & \\ \multicolumn{2}{c}{C} & -tr(A_{22}) - a_{11} & O \\ & & O & O \end{bmatrix}$$

Hence the tangent space m takes the following form

$$m = \left\{ \begin{bmatrix} k & O & \multicolumn{2}{c}{B} \\ O & O & & \\ \multicolumn{2}{c}{C} & -k & O \\ & & O & O \end{bmatrix}, k \in \mathbf{R} \right\},$$

and the Lie algebra h becomes

$$\left\{\left[\begin{array}{cccc} -tr(A_{22}) & A_{12} & & \\ A_{21} & A_{22} & \multicolumn{2}{c}{O} \\ & & tr(A_{22}) & -A_{21}^T \\ \multicolumn{2}{c}{O} & -A_{12}^T & -A_{22}^T \end{array}\right]\right\}.$$

Therefore the Lie algebra $sp(l)$ can be written $sp(l) = h \oplus m$. The roots of $sp(l)$ are given by $\{\pm e_i \pm e_k, \pm 2e_i | i < k, i, k = 1, ..., l\}$, where $e_i = \frac{1}{4l+4} \cdot (E_i - E_{i+l})\,, i = 1, ..., l$ and E_i is a matrix of order $2l$ which has at the position (i, i) the number one and everywhere else the number zero.

Taking into consideration the form of the tangent space m, the quantities e_i take the following form:

- for $i = 1, e_i = \frac{1}{4l+4} \cdot (E_1 - E_{1+l}) = O \oplus \frac{1}{4l+4} \cdot (E_1 - E_{l+1})$
- for $2 \le i \le l, e_i = \frac{1}{4l+4} \cdot (E_i - E_{i+l}) = \frac{1}{4l+4} \cdot (E_i - E_{i+l}) \oplus O.$

At first we construct the weights of $sp(l)$,

$$w_i = e_1 + e_2 + ... + e_i \Rightarrow w_{i_m} = e_1 = \frac{1}{4l+4} \cdot (E_1 - E_{1+l})\,, \quad i = 1, ..., l\,.$$

Then we construct the quantity

$$\mu = m_1 w_{1_m} + ... + m_l w_{l_m} \quad \text{where} \quad m_1, ..., m_l \in Z^+\,.$$

Taking into consideration the form of the weights restricted to the tangent space, the quantity μ takes the form $\mu = (m_1 + m_2 + ... + m_l) \cdot e_1 = m \cdot e_1$.

Now, we construct the half of the sum of all positive roots δ

$$2\delta = 2le_1 + (2l-2)\, e_2 + ... + 4e_{l-1} + 2e_l.$$

Subsequently the restriction of the half of the sum of all positive roots to the tangent space takes the following form $\delta_m = l \cdot e_1$. At first we calculate the quantities which appeare in formula (17)

$$\langle \mu, \mu \rangle = \langle m \cdot e_1, m \cdot e_1 \rangle = m^2 \langle e_1, e_1 \rangle = \frac{m^2}{4l+4};$$

$$\langle \mu, \delta_m \rangle = \langle m \cdot e_1, l \cdot e_1 \rangle = m \cdot l \cdot \langle e_1, e_1 \rangle = \frac{ml}{4l+4}.$$

So, the spectrum takes the form

$$Sp(SP(l)/SU(l), g) = \left\{ \frac{m^2 + 2ml}{4l+4}, m \in \mathbf{Z} \right\}. \tag{3.1}$$

The positive roots are

i) $e_i - e_j, 1 \le i < j \le l$

ii) $e_i + e_j, 1 \le i < j \le l$

iii) $2e_i, 1 \le i \le l$

and for

$$\mu + \delta = me_1 + le_1 + (l-1)e_2 + ... + e_l = (m+l)e_1 + (l-1)e_2 + ... + e_l,$$

we construct

$$P_1 = \prod_{1\le i<j\le l} \frac{<\mu+\delta, e_i - e_j>}{<\delta, e_i - e_j>} = \prod_{2\le j\le l} \frac{<\mu+\delta, e_1 - e_j>}{<\delta, e_1 - e_j>} = \prod_{2\le j\le l} \frac{m+j-1}{j-1} = \begin{pmatrix} m+l-1 \\ l-1 \end{pmatrix};$$

$$P_2 = \prod_{1\le i<j\le l} \frac{<\mu+\delta, e_i + e_j>}{<\delta, e_i + e_j>} = \prod_{2\le j\le l} \frac{<\mu+\delta, e_1 + e_j>}{<\delta, e_1 + e_j>} = \prod_{2\le j\le l} \prod \frac{m+2l-j+1}{2l-j+1};$$

$$P_3 = \prod_{1\le i\le l} \frac{<\mu+\delta, 2e_i>}{<\delta, 2e_i>} = \frac{<\mu+\delta, 2e_1>}{<\delta, 2e_1>} = \frac{m+l}{l}.$$

Having taken into consideration the relation (18) the multiplicity of the eigenvalue is taken then

$$d\mu^2 = P_1P_2P_3 = \prod_{k=1}^{2l-1} \frac{m+k}{k} = \begin{pmatrix} m+2l-1 \\ 2l-1 \end{pmatrix}. \tag{3.2}$$

We conclude with the following

Theorem. *The spectrum $Sp(M,g)$, where $M = SP(L)/SU(l)$, is given by the relation (19). The multiplicity of the eigenvalue is given by the relation (20).*

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