On the Covering Space and the Automorphism Group of the Covering Space

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

In this paper some properties of covering space and the automorphism group of the covering space are obtained.

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1 Introduction

Let $\widetilde{X}$ be a connected space, X be a space and let $p : \widetilde{X} \longrightarrow X$ be a continious map. If for every $x \in X$ has an path connected open neighbourhood U such that $p^{-1}(U)$ is open in $\widetilde{X}$ and each component of $p^{-1}(U)$ is mapped topologically onto U by p then p is called a *covering map*. In this case the pair $(\widetilde{X}, p)$ is called a *covering space* of X.

Let $(\widetilde{X}, p)$ be a covering space of X, $\widetilde{x}_0 \in \widetilde{X}$,and $p(\widetilde{x}_0) = x_0$. Then, for any path α in X with initial point x_0, there exists a unique path β in $\widetilde{X}$ with initial point $\widetilde{x}_0$ such that $p\beta = \alpha$.

Let $(\widetilde{X}, p)$ be a covering space of X and $x \in X$. For any point $\widetilde{x} \in p^{-1}(x)$ and any $\alpha \in \pi(X, x)$ we define $\widetilde{x}\alpha \in p^{-1}(x)$ as follows: From above there exists a unique path class $\widetilde{\alpha}$ in $\widetilde{X}$ such that $p_*(\widetilde{\alpha}) = \alpha$ and the initial point of $\widetilde{\alpha}$ is the point $\widetilde{x}$. Define $\widetilde{x}\alpha$ to be the terminal point of the path class $\widetilde{\alpha}$. Then it is easily verify that

$$\begin{aligned} (\widetilde{x}\alpha)\beta &= \widetilde{x}(\alpha\beta), \\ \widetilde{x}e &= \widetilde{x}. \end{aligned}$$

Thus $\pi(X, x)$ be a group of right operators on the set $p^{-1}(x)$. Moreover the group $\pi(X, x)$ acts transitively on the set $p^{-1}(x)$. To show this let $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x)$. Since $\widetilde{X}$ is arcwise connected, there exists a path class $\widetilde{\alpha}$ in $\widetilde{X}$ with initial point $\widetilde{x}_0$ and terminal point $\widetilde{x}_1$. Let $p_*(\widetilde{\alpha}) = \alpha$. Then, α is an equivalence class of closed paths, and obviously $\widetilde{x}_0\alpha = \widetilde{x}_1$. Thus, the set $p^{-1}(x)$ is a homogeneous right $\pi(X, x)-$ space.

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From the definition, we see that for any point $\widetilde{x} \in p^{-1}(x)$, the isotropy subgroup corresponding to this point is precisely the subgroup $p_*\pi(\widetilde{X}, \widetilde{x})$ of $\pi(X, x)$. Hence $p^{-1}(x)$ is isomorphic to the space of cosets, $\pi(X, x)/\ p_*\pi(\widetilde{X}, \widetilde{x})$, and the number of sheets of the covering is equal to the index of the subgroup $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$

If X is simply connected then the fundamental group $\pi(X, x)$ is trival and the index of $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$ is 1. So $(\widetilde{X}, p)$ is an one-sheeted covering of X and therefore p is a homeomorphism. Similarly if $\widetilde{X}$ is simply connected then $\pi(\widetilde{X}, \widetilde{x})$ is trival and the index of $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$ is equal to the order of $\pi(X, x)$.

A *covering transformation* of a covering space $(\widetilde{X}, p)$ of X is a homeomorphism $h : \widetilde{X} \longrightarrow \widetilde{X}$ such that $ph = p$. The set of all covering transformations of $(\widetilde{X}, p)$ form a group denoted by $A(\widetilde{X}, p)$.

Let $(\widetilde{X}, p)$ be a covering space of X and $p(\widetilde{x}_1) = p(\widetilde{x}_2) = x$ for $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$ and $x \in X$. Let consider the homomorphisms

$$\begin{aligned} p_* &: \pi(\widetilde{X}, \widetilde{x}_1) \to \pi(X, x), \\ p_* &: \pi(\widetilde{X}, \widetilde{x}_2) \to \pi(X, x) \end{aligned}$$

Let $\{\gamma_i : i \in I\}$ be a path class in $\widetilde{X}$ with inital point $\widetilde{x}_1$ and terminal point $\widetilde{x}_2$. Define

$$u_* : \pi(\widetilde{X}, \widetilde{x}_1) \to \pi(\widetilde{X}, \widetilde{x}_2)$$

to be $u_*(\alpha) = \gamma^{-1}\alpha\gamma$ for $\gamma \in \{\gamma_i : i \in I\}$.Then we have the commutative diagram in Figure 1.

$$\begin{tikzcd} \pi(\widetilde{X}, \widetilde{x}_1) \arrow[r, "p_*"] \arrow[d, "u_*"'] & \pi(X, x) \arrow[d, "v_*"] \\ \pi(\widetilde{X}, \widetilde{x}_2) \arrow[r, "p_*"'] & \pi(X, x) \end{tikzcd}$$

Figura 1.

Here $v_*(\beta) = (p_*\gamma)^{-1}\beta(p_*\gamma)$. Since $p_*\gamma$ is a closed it is a path in $\pi(X, x)$. So the images of the fundamental groups $\pi(\widetilde{X}, \widetilde{x}_1)$ and $\pi(\widetilde{X}, \widetilde{x}_2)$ under p_* are conjugate subgroups of $\pi(X, x)$.

Lemma 1.1 *Let $(\widetilde{X}, p)$ be a path connected covering space of a locally pathwise connected space X and $p(\widetilde{x}_1) = p(\widetilde{x}_2) = x$ for $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$ and $x \in X$. Then $p_*\pi(\widetilde{X}, \widetilde{x}_1)$ and $p_*\pi(\widetilde{X}, \widetilde{x}_2)$ are conjugate subgroups of $\pi(X, x)$ iff there exists a $\varphi \in A(\widetilde{X}, p)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$.*[4]

Lemma 1.2 *Let* $(\widetilde{X}_1, p_1)$ *and* $(\widetilde{X}_2, p_2)$ *be two covering space of a locally pathwise connected space* X. *Then these two covering are isomorphic iff there exists a homeomorphism* $h : \widetilde{X}_1 \to \widetilde{X}_2$ *such that* $p_2 h = p_1$.[4]

Lemma 1.3 *If two space are homeomorphic, then their fundamental groups are isomorphic, i.e. if* $h : (X, x) \to (Y, y)$ *is a homeomorphism, then* $h_* : \pi(X, x) \to \pi(Y, y)$ *is an isomorphism.*[3]

Lemma 1.4 *If* $(\widetilde{X}_1, p_1)$ *and* $(\widetilde{X}_2, p_2)$ *are two simply connected covering space of a locally pathwise connected space* X, *then there exists a homeomorphism* $h : (\widetilde{X}_1, p_1) \longrightarrow (\widetilde{X}_2, p_2)$ *such that* $p_2 h = p_1$.[2]

Lemma 1.5 *Let* $(\widetilde{X}_1, p_1)$ *and* $(\widetilde{X}_2, p_2)$ *be two covering space of a locally pathwise connected space* X. *Then there exists a morphism* φ *from* $(\widetilde{X}_1, p_1)$ *into* $(\widetilde{X}_2, p_2)$ *such that* $\varphi(\widetilde{x}_1) = \widetilde{x}_2$ *iff*

$$p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1) = p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2). \text{ [4]}$$

Lemma 1.6 *Let* $(\widetilde{X}, p)$ *be path connected covering space of a locally pathwise connected space* X. *Then* p *is a homeomorphism iff* $p_*\pi(\widetilde{X}, \widetilde{x}) = \pi(X, x)$. [6]

From Lemma 1.1 we have if $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ are two covering spaces of a locally pathwise connected space X ,then these two coverings are isomorphic iff $p_{1*}\pi(\widetilde{X}_1, \widetilde{x}_1)$ and $p_{2*}\pi(\widetilde{X}_2, \widetilde{x}_2)$ are conjugate subgroups of $\pi(X, x)$.

Let X be a connected space. The category of connected spaces of X has objects which are covering projections $p : \widetilde{X} \to X$, where $\widetilde{X}$ is connected, and morphisms

$$p_1 : \widetilde{X}_1 \to X , \quad p_2 : \widetilde{X}_2 \to X \text{ and } f : \widetilde{X}_1 \to \widetilde{X}_2$$

such that $p_2 f = p_1$.

Definition. Let X be a connected space and $\widetilde{X}$ be a locally path connected space. A universal covering space of X is an object $p : \widetilde{X} \to X$ of the category of connected covering spaces of X such that for any object $p_1 : \widetilde{X}_1 \to X$ there is a morphism $f : \widetilde{X} \to \widetilde{X}_1$ such that $p_1 f = p$. [6]

Lemma 1.7 *If* $(\widetilde{X}, p)$ *is an universal covering space of a locally pathwise connected space* X,*then the automorphism group* $A(\widetilde{X}, p)$ *is isomorphic to the fundamental group* $\pi(X, x)$, *and the number of the sheets of the covering is equal to the order of the fundamental group* $\pi(X, x)$. [4]

Let $\widetilde{X}$ be be a pathwise connected space and let $(\widetilde{X}, p)$ be a covering space of a locally pathwise connected space X . Then $(\widetilde{X}, p)$ is a *regular covering space* of X iff $p_*\pi(\widetilde{X}, \widetilde{x})$ is a normal subgroup of $\pi(X, x)$.

Lemma 1.8 *Let* $p : \widetilde{X} \to X$ *be a covering map such that* $p(\widetilde{x}_1) = p(\widetilde{x}_2)$ *for* $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$. *Then* p *is regular iff* $p_*\pi(\widetilde{X}, \widetilde{x}_1) = p_*\pi(\widetilde{X}, \widetilde{x}_2)$.[6]

Lemma 1.9 *Let $\widetilde{X}$ be be a pathwise connected space and let $(\widetilde{X},p)$ is a regular covering space of a locally pathwise connected space X . Then X homeomorphic to the quatient space $\widetilde{X}\ /\ A(\widetilde{X},p)$.* [6]

Let G be a group of homeomorphisms of X. If for every $x \in X$, there exists a neighbourhood V of x such that $gV \cap V = \theta$, for all $g \in G$ different from the unity of G, then we say G *acts discontiniously* on X.

Lemma 1.10 *Let G be a discontinious proper group of homeomorphisms of a locally pathwise connected space X and let $q : X \to X/G$ be a naturel projection defined by $q(x) = [x]$. Then $(\widetilde{X},q)$ is a regular covering space of X/G and $A(\widetilde{X},q)$ is isomorphic to G.* [6]

2 On the covering space and its automorphism group

In this paper we obtain some properties of covering spaces and their automorphism and fundamental groups.

Theorem 2.1 *Let $(\widetilde{X}_1,p_1)$ and $(\widetilde{X}_2,p_2)$ be two universal covering space of a locally pathwise connected space X . Then*

1. *these two covering spaces are homeomorphic, and therefore the fundamental groups $\pi(\widetilde{X}_1,\widetilde{x}_1)$ and $\pi(\widetilde{X}_2,\widetilde{x}_2)$ are isomorphic.*
2. *$A(\widetilde{X}_1,p_1)$ and $A(\widetilde{X}_2,p_2)$ are isomorphic.*

Proof. 1. Let $(\widetilde{X}_1,p_1)$ and $(\widetilde{X}_2,p_2)$ be two universal covering space of X . Then from Lemma 1.4. there exists a homeomorphism $h : (\widetilde{X}_1,p_1) \longrightarrow (\widetilde{X}_2,p_2)$ such that $p_2h = p_1$. Therefore these two covering are isomorphic. Therefore form Lemma 1.3. the fundamental groups $\pi(\widetilde{X}_1,\widetilde{x}_1)$ and $\pi(\widetilde{X}_2,\widetilde{x}_2)$ are isomorphic.

2. Let $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2) = x$ for $\widetilde{x}_1 \in \widetilde{X}_1$ and $\widetilde{x}_2 \in \widetilde{X}_2$. Since $(\widetilde{X}_1,p_1)$ and $(\widetilde{X}_2,p_2)$ are universal covering space of X, from Lemma 1.8. the automorphism group $A(\widetilde{X}_1,p_1)$ is isomorphic to the fundamental group $\pi(X,x)$, and the automorphism group $A(\widetilde{X}_2,p_2)$ is isomorphic to the fundamental group $\pi(X,x)$. Therefore $A(\widetilde{X}_1,p_1)$ and $A(\widetilde{X}_2,p_2)$ are isomorphic.

Theorem 2.2 *Let $(\widetilde{X},p)$ be an universal covering space of a locally pathwise connected space X and $p(\widetilde{x}_1) = p(\widetilde{x}_2) = x$ for $\widetilde{x}_1,\widetilde{x}_2 \in \widetilde{X}$. Then there exists an automorphism $\varphi \in A(\widetilde{X},p)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$, i.e. $A(\widetilde{X},p)$ acts transitively on the set $p^{-1}(x)$.*

Proof. Since $(\widetilde{X},p)$ is an universal covering space of X , there is a path γ in $\widetilde{X}$ with initial point $\widetilde{x}_1$ and terminal point $\widetilde{x}_2$. Using γ define

$$u_* : \pi(\widetilde{X},\widetilde{x}_1) \to \pi(\widetilde{X},\widetilde{x}_2)$$

to be $u_*(\alpha) = \gamma^{-1}\alpha\gamma$. Then u_* is an isomorphism and thus $p_*\pi(\widetilde{X},\widetilde{x}_1)$ and $p_*\pi(\widetilde{X},\widetilde{x}_2)$ are conjugate subgroups of $\pi(X,x)$. Therefore from Lemma 1.1. $\varphi(\widetilde{x}_1) = \widetilde{x}_2$.

Theorem 2.3 *Let $(\widetilde{X}, p)$ be a covering space of a locally pathwise connected space X. Then there exists a $\varphi \in A(\widetilde{X}, p)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$ iff $(\widetilde{X}, p)$ is a regular covering space of X.*

Proof. Let assume that there exists a $\varphi \in A(\widetilde{X}, p)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$. Then from Lemma 1.5 $p_*\pi(\widetilde{X}, \widetilde{x}_1) = p_*\pi(\widetilde{X}, \widetilde{x}_2)$. Thus from Lemma 1.9 $(\widetilde{X}, p)$ is a regular covering space of X.

Conversely let $(\widetilde{X}, p)$ is a regular covering space of X. Then from Lemma 1.9 $p_*\pi(\widetilde{X}, \widetilde{x}_1) = p_*\pi(\widetilde{X}, \widetilde{x}_2)$, and from Lemma 1.5 there exists a $\varphi \in A(\widetilde{X}, p)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$.

Theorem 2.4 *Universal covering is regular.*

Proof. Let $(\widetilde{X}, p)$ be an universal covering space of a locally pathwise connected space X, and let $\widetilde{x}_1$ and $\widetilde{x}_2$ be two points of $\widetilde{X}$ such that $p(\widetilde{x}_1) = p(\widetilde{x}_2) = x$. Since $(\widetilde{X}, p)$ is an universal covering space of X, there exists a path γ in $\widetilde{X}$ with initial point $\widetilde{x}_1$ and terminal point $\widetilde{x}_2$. Define $u_* : \pi(\widetilde{X}, \widetilde{x}_1) \to \pi(\widetilde{X}, \widetilde{x}_2)$ to be $u_*(\alpha) = \gamma^{-1}\alpha\gamma$. Then u_* is an isomorphism and thus $p_*\pi(\widetilde{X}, \widetilde{x}_1)$ and $p_*\pi(\widetilde{X}, \widetilde{x}_2)$ are conjugate subgroups of $\pi(X, x)$. Thus from Lemma 1.1. there exists a $\varphi \in A(\widetilde{X}, p)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$. Therefore from above Theorem $(\widetilde{X}, p)$ is regular covering space of X.

Let $(\widetilde{X}, p)$ is a regular covering space of a locally pathwise connected space X. Then we know from Lemma 1.10 that X is homeomorphic to the quatient space $\widetilde{X}/A(\widetilde{X}, p)$, i.e. there exists a homeomorphism $r : \widetilde{X}/A(\widetilde{X}, p) \to X$.

Theorem 2.5 *If $(\widetilde{X}, p)$ is an universal covering space of a locally pathwise connected space X, then the order of the automorphism group $A\left(\widetilde{X}/A(\widetilde{X}, p), r\right)$ of the quatient space $\widetilde{X}/A(\widetilde{X}, p)$ is equal to the number of the sheets of the covering $(\widetilde{X}, p)$ of X.*

Proof. Since universal covering space is regular $(\widetilde{X}, p)$ is a regular covering space of X. So there exists a homeomorphism $r : \widetilde{X}/A(\widetilde{X}, p) \to X$. On the other hand since $(\widetilde{X}, p)$ is an universal covering space of X, $(\widetilde{X}, q)$ is an universal covering space of $\widetilde{X}/A(\widetilde{X}, p)$, and the number of the sheets of the covering is equal to the order of the group $\pi(X, x)$. r is an universal covering map since p and q are universal covering. So from Lemma 1.8. $A\left(\widetilde{X}/A(\widetilde{X}, p), r\right)$ is isomorphic to the fundamental group $\pi(X, x)$. Thus the number of the sheets of the covering $(\widetilde{X}, p)$ of X is equal the the order of the automorphism group $A\left(\widetilde{X}/A(\widetilde{X}, p), r\right)$.

Theorem 2.6 *If $(\widetilde{X}, p)$ is an universal covering space of a locally pathwise connected space X and $A(\widetilde{X}, p)$ is the discontinious proper group of homeomorphisms of $\widetilde{X}$, then the fundamental groups $\pi(X, x)$ and $\pi\left(\widetilde{X}/A(\widetilde{X}, p), [\widetilde{x}]\right)$ are isomorphic under the naturel projection*

$$q : \widetilde{X} \to \widetilde{X}/A(\widetilde{X}, p)$$

defined by $q(\widetilde{x}) = [\widetilde{x}]$.

Proof. Since $A(\widetilde{X},p)$ be the discontinious proper group of homeomorphisms of $\widetilde{X}$, q is a regular map and the automorphism group $A(\widetilde{X},q)$ is isomorphic to $A(\widetilde{X},p)$. Since $(\widetilde{X},p)$ is an universal covering space, $(\widetilde{X},q)$ is an universal covering space of $\widetilde{X}/A(\widetilde{X},p)$. Therefore the automorphism group $A(\widetilde{X},p)$ is isomorphic to the fundamental group $\pi(X,x)$ and $A(\widetilde{X},q)$ is isomorphic to $\pi\left(\widetilde{X}/A(\widetilde{X},p),[\widetilde{x}]\right)$. Hence the fundamental groups $\pi(X,x)$ and $\pi(\tilde{X}/A(\tilde{X},p),[\tilde{x}])$ are isomorphic since $A(\tilde{X},q)$ is isomorphic to $A(\tilde{X},p)$.

Let $(\widetilde{X},p)$ be a universal covering space of a locally pathwise connected space X . Then from above Theorem we have following corollaries.

Corollary 2.7 *The number of the sheets of the covering is equal to the order of the fundamental group* $\pi\left(\widetilde{X}/A(\widetilde{X},p),[\widetilde{x}]\right)$.

Corollary 2.8 $A(\widetilde{X},p)$ *is isomorphic to* $\pi\left(\widetilde{X}/A(\widetilde{X},p),[\widetilde{x}]\right)$.

Theorem 2.9 *Let* $(\widetilde{X}_1,p_1)$ *and* $(\widetilde{X}_2,p_2)$ *be two universal covering space of a simply connected space* X *and let* G_1 *and* G_2 *be discontinious proper group of homeomorphisms of* $\widetilde{X}_1$ *and* $\widetilde{X}_2$, *respectively. Then the diagram in Figure 2 is commutative.*

Proof. Since X is simply connected, p_1 and p_2 are homeomorphisms and therefore p_{1*} and p_{2*} are isomorphisms. On the other hand, since $(\widetilde{X}_1,p_1)$ and $(\widetilde{X}_2,p_2)$ are universal covering space of X, there exists a homeomorphism $h:\widetilde{X}_1\to\widetilde{X}_2$ such that $p_2h=p_1$. So $h_*:\pi(\widetilde{X}_1,\widetilde{x}_1)\to\pi(\widetilde{X}_2,\widetilde{x}_2)$ is an isomorphism for $\widetilde{x}_1\in\widetilde{X}_1$ and $\widetilde{x}_2\in\widetilde{X}_2$. Since these covering space are universal $A(\widetilde{X}_1,p_1)$ and $A(\widetilde{X}_2,p_2)$ are isomorphic to the fundamental group $\pi(X,x)$, i.e. there exist the isomorphisms φ_{1*} and φ_{2*}. From Theorem 2.6. there exist the isomorphisms r_{1*} and r_{1*}. Hence the diagram is commutative.

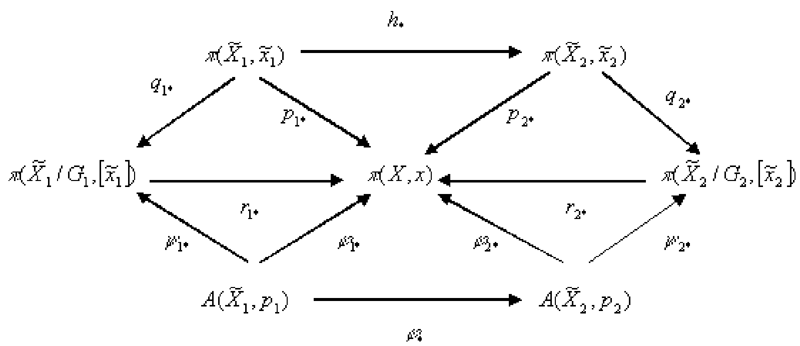

Figure 2.

Theorem 2.10 *If* $(\widetilde{X},p)$ *is an universal covering space of a locally pathwise connected space* X, *then the automorphism group* $A\left(\widetilde{X}/A(\widetilde{X},p),r\right)$ *of the quatient space* $\widetilde{X}/A(\widetilde{X},p)$ *is isomorphic to the automorphism group* $A(\widetilde{X},p)$.

Proof. Since the universal covering is regular this covering is regular. Therefore there exists a homeomorphism $r : \widetilde{X}/A(\widetilde{X}, p) \to X$. On the other hand since this covering is universal from Lemma 1.8. the automorphism group $A(\widetilde{X}, p)$ is isomorphic to the fundamental group $\pi(X, x)$. Moreover from Theorem 2.5. the automorphism group $A\left(\widetilde{X}/A(\widetilde{X}, p), r\right)$ is isomorphic to the $\pi(X, x)$. Therefore $A\left(\widetilde{X}/A(\widetilde{X}, p), r\right)$ is isomorphic to the automorphism group $A(\widetilde{X}, p)$.

From above Theorem

Corollary 2.11 *If* $(\widetilde{X}_1, p_1)$ *and* $(\widetilde{X}_2, p_2)$ *are two universal covering spaces of a locally pathwise connected space* X *then* $\widetilde{X}_1/A(\widetilde{X}_1, p_1)$ *and* $\widetilde{X}_2/A(\widetilde{X}_2, p_2)$ *are isomorphic.*

Proof. Let $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ be two universal covering spaces of X. Then there exists a homeomorphism $h : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 h = p_1$. On the other hand since these coverings are regular there exist homeomorphisms $r_1 : \widetilde{X}_1/A(\widetilde{X}_1, p_1) \to X$ and $r_2 : \widetilde{X}_2/A(\widetilde{X}_2, p_2) \to X$. Hence $\widetilde{X}_1/A(\widetilde{X}_1, p_1)$ and $\widetilde{X}_2/A(\widetilde{X}_2, p_2)$ are isomorphic.

Theorem 2.12 *Let* $(\widetilde{X}, p)$ *be a covering space of a locally pathwise connected space* X. *Then the number of the elements in the orbit* $[\widetilde{x}]$ *of the point* $\widetilde{x} \in p^{-1}(x)$ *is equal to the number of the sheets of the covering* $(\widetilde{X}, p)$ *of* X.

Proof. We know that the number of the elements in the orbit $[\widetilde{x}]$ of the point $\widetilde{x} \in p^{-1}(x)$ is equal to the index of the isotropy subgroup which corresponding to $\widetilde{x}$ in $\pi(X, x)$. Moreover isotropy subgroup which corresponding $\widetilde{x}$ is the subgroup $p_*\pi(\widetilde{X}, \widetilde{x})$ of $\pi(X, x)$. Since the number of the sheets of the covering is equal to the index of the subgroup $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$, the number of the sheets of the covering is equal to the elements in the orbit $[\widetilde{x}]$ of the point $\widetilde{x} \in p^{-1}(x)$.

Theorem 2.13 *If* $(\widetilde{X}, p)$ *is an universal covering space of a locally pathwise connected space* X, *then the order of the fundamental group* $\pi\left(\widetilde{X}/A(\widetilde{X}, p), [\widetilde{x}]\right)$ *of the quatient space* $\widetilde{X}/A(\widetilde{X}, p)$ *is equal to the number of sheets of the covering* $(\widetilde{X}, p)$ *of* X.

Proof. Since universal covering is regular this covering is regular, and therefore there exists a homeomorphism $r : \widetilde{X}/A(\widetilde{X}, p) \to X$. Thus r_* is an isomorphism, i.e. the fundamental groups $\pi\left(\widetilde{X}/A(\widetilde{X}, p), [\widetilde{x}]\right)$ and $\pi(X, x)$ are isomorphic. Since this covering is universal $A(\widetilde{X}, p)$ is isomorphic to the fundamental group $\pi(X, x)$, and the number of the sheets of the covering is equal to the order of the fundamental group $\pi(X, x)$. Therefore the number of the sheets of the covering is equal to the order of the fundamental group $\left(\widetilde{X}/A(\widetilde{X}, p), [\widetilde{x}]\right)$ since $\pi\left(\widetilde{X}/A(\widetilde{X}, p), [\widetilde{x}]\right)$ and $\pi(X, x)$ are isomorphic.

Definition. Let $(\widetilde{X}, p)$ be a covering space of a locally pathwise connected space X. If the automorphism group $A(\widetilde{X}, p)$ acts transitively on the set $p^{-1}(x)$, for every $x \in X$, then $(\widetilde{X}, p)$ is called *Galois*.

Theorem 2.14 *Regular covering is Galois.*

Proof. Let $(\widetilde{X}, p)$ be a regular covering space of X. Then from Theorem 2.3. there exist a $\varphi \in A(\widetilde{X}, p)$ such that $\varphi(\widetilde{x}_1) = \widetilde{x}_2$ for $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$, i.e. $A(\widetilde{X}, p)$ acts transitively on the set $p^{-1}(x)$ for $x \in X$. Thus from definition $(\widetilde{X}, p)$ is Galois covering of X.

We know from Theorem 2.4. that universal covering is regular. Thus from above Theorem

Corollary 2.15 *Universal covering is Galois.*

Theorem 2.16 *If $(\widetilde{X}, p)$ is a $2-$ sheeted covering space of a locally pathwise connected space X, then this covering is Galois.*

Proof. Let $(\widetilde{X}, p)$ be $2-$ sheeted covering space of X. Since the number of the sheets of the covering is equal to the index of the subgroup $p_*\pi(\widetilde{X}, \widetilde{x})$ in $\pi(X, x)$, $p_*\pi(\widetilde{X}, \widetilde{x})$ is a normal subgroup of $\pi(X, x)$. Therefore $(\widetilde{X}, p)$ is a regular covering space of X and thus $(\widetilde{X}, p)$ is a Galois covering of X.

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