QR-Hypersurfaces of Quaternionic Kähler Manifolds

Vittorio Mangione

Dedicated to the Memory of Grigorios TSAGAS (1935-2003), President of Balkan Society of Geometers (1997-2003)

Abstract

We prove that the basic manifold of a submersion from a QR-hypersurface of a quaternionic Kähler manifold to an almost quaternionic Hermitian manifold is quaternionic Kähler. Then we prove some results involving the sectional curvatures.

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Introduction

Real hypersurfaces of quaternionic space forms have been studied by many authors ([1], [2], [3], [4], [5], [11], [12]) under conditions concerning their shape operator. It is known that real hypersurface of quaternionic Kähler manifolds are not CR-hypersurface in general ([2]).
The study of CR-submanifolds of a quaternionic Kähler manifolds has been carried out in the paper [1]. S. Kobayashi considered the similarity between the total space of a Riemannian submersion and a CR-submanifold of a Kähler manifold in terms of distributions ([9]). In this paper we study Riemannian submersions from QR-hypersurface of a quaternionic Kähler manifold over an almost quaternionic Hermitian manifold (second section). In the last section we study some curvature properties induced on the basic manifold by the submersion.

1 Hypersurfaces of quaternionic Kähler manifolds

We say that a $4(m+1)-dimensional$ manifold $\tilde{M}$ with a metric $\tilde{g}$ is a *quaternionic Kähler manifold* $(m \geq 1)$ if there exists a $3-dimensional$ vector bundle V of tensors of type $(1,1)$ on $\tilde{M}$ satisfying the following conditions:

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(a) In any coordinate neighborhood $\tilde{U}$ on $\tilde{M}$ there is a local basis of almost Hermitian structures $\{\mathcal{J}_a, \tilde{g}\}$, such that $\mathcal{J}_a^2 = -Id$, $a \in \{1,2,3\}$ and $\mathcal{J}_a \circ \mathcal{J}_b = -\mathcal{J}_b \circ \mathcal{J}_a = \mathcal{J}_c$ for any cyclic permutation (a,b,c) of $(1,2,3)$.

(b) For any local section φ of V and any tangent vector X to $\tilde{M}$, $\tilde{\nabla}_X\varphi$ is also a local section in V, where $\tilde{\nabla}$ denotes the Levi-Civita connection of $\tilde{g}$.

Condition (b) is equivalent to the following:

(b′) There exist local $1-forms$ ω_{ab}, $a,b \in \{1,2,3\}$ on $\tilde{\mathcal{U}}$ such that $\omega_{ab} + \omega_{ba} = 0$, and

$$\tilde{\nabla}_x \mathcal{J}_a = \omega_{ab}(x)\mathcal{J}_b + \omega_{ac}(x)\mathcal{J}_c \tag{1}$$

for any cyclic permutation (a,b,c) of $(1,2,3)$.

Given two local bases $\{\mathcal{J}_a\}$ and $\{\mathcal{J}'_a\}$ of V defined on coordinate neighborhoods $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{U}}'$ such that $\tilde{\mathcal{U}} \cap \tilde{\mathcal{U}}' \neq \emptyset$, we have on $\tilde{\mathcal{U}} \cap \tilde{\mathcal{U}}'$:

$$\mathcal{J}'_a = \sum_{b=1}^{3} C_{ab}\mathcal{J}_b \tag{2}$$

where $[C_{ab}]$ is an element of the special orthogonal group $SO(3)$ (see [8]).
Let M be an orientable hypersurface of $\tilde{M}$ and ξ a unit normal field defined on M. On $\tilde{\mathcal{U}}$ $\xi_a = -\mathcal{J}_a(\xi)$, $a \in \{1,2,3\}$ defines a tangent vector field to M. Similarly, we define ξ'_a on $\tilde{\mathcal{U}}'$ and on $\tilde{U} \cap \tilde{\mathcal{U}}' \neq \emptyset$ we have:

$$\xi'_a = \sum_{b=1}^{3} C_{ab}\xi_b,\ b \in \{1,2,3\} \tag{3}$$

so that one obtains a distribution $\mathcal{V}$ on M which is locally represented by $\{\xi_a\}$, $1 \leq a \leq 3$, on $\tilde{\mathcal{U}}$. Let $\mathcal{H}$ be the orthogonal complementary distribution to $\mathcal{V}$ with respect to the Riemannian metric g induced by $\tilde{g}$ on M.
We see that for each $x \in M$, $\mathcal{H}_x$ is $\mathcal{J}_a - invariant$, but $\mathcal{V}_x$ is not an anti-invariant subspace of $T_x\tilde{M}$ with respect $\mathcal{J}_a$, $a = \{1,2,3\}$. It is easy see that $\mathcal{J}_a(\mathcal{V}_x) = T_xM^{\perp}$, $x \in M$, where $T_xM^{\perp}$ is the normal space at x to the hypersurface M in $\tilde{M}$. In general, when the previous conditions are satisfied, we say that M is a $QR-hypersurface$ of $\tilde{M}$(see [3]). Now, let B be the second fundamental form of M in $\tilde{M}$. Then, for any E, $F \in \Gamma(TM)$ we have the Gauss formula

$$\tilde{\nabla}_E F = \nabla_E F + B(E,F), \tag{4}$$

where $\tilde{\nabla}$ and ∇ are the Levi-Civita connections on $\tilde{M}$ and M, respectively.
If L denotes the fundamental tensor of Weingarten with respect to ξ, we have the Weingarten formula

$$\tilde{\nabla}_E \xi = -L(E), \tag{5}$$

and for any E, $F \in \Gamma(TM)$ the following formula

$$g(L(E),F) = g(B(E,F),\xi) \tag{6}$$

holds.

The integrability of the distributions $\mathcal{V}$ and $\mathcal{H}$ on M has been studied by A. Bejancu ([2]).
We recall that the vertical distribution $\mathcal{V}$ is integrable if and only if

$$B(U, X) = 0 \tag{7}$$

for any $U \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$.
If (7) is satisfied, we say that M is a *mixed geodesic* $QR-hypersurface$ of $\tilde{M}$.

2 Riemannian submersions of QR-hypersurfaces

Let M be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold $\tilde{M}$. We denote by $(M', g', \mathcal{J}'_a)$, $a \in \{1, 2, 3\}$, an almost quaternionic Hermitian manifold (i.e. satisfying the condition (a)). We say that a Riemannian submersion π: $M \to M'$ is a *QR-submersion* if the following conditions are satisfied:

i) $\mathcal{V}$ is the kernel of π_*;

ii) for each $x \in M$, $\pi_* : \mathcal{H}_x \to T_{\pi(x)}M'$ is an isometry with respect to each complex structure of H_x and $T_{\pi(x)}M'$, where $T_{\pi(x)}M'$ denotes the tangent space to M' at $\pi(x)$.

As in the paper [10], the letters U, V, W, W' will always denote vertical vector fields and X, Y, Z, Z' horizontal vector fields. A horizontal vector field X on M is said to be *basic* if it is $\pi-related$ to a vector field X' on M'.
We denote by T and A O'Neill's fundamental tensors (see [13], [11]).

Lemma 2.1 *Let X and Y be basic vector fields on M. Then the following conditions hold:*

a) The horizontal component $h[X, Y]$ of $[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$;

b) $h(\nabla_X Y\ )$ is basic vector field corresponding to $\nabla'_{X'}Y'$ where ∇ and ∇' are the the Levi-Civita connections on M and M', respectively;

c) $[X, U] \in \Gamma(\mathcal{V})$, for any vertical field $U \in \Gamma(\mathcal{V})$;

where h denotes the horizontal component of a vector E on M.

We define a skew-symmetric tensor field C by

$$\tilde{\nabla}_x Y = h\tilde{\nabla}_x Y + C(X, Y) \tag{1}$$

for all $X, Y \in \Gamma(\mathcal{H})$.
The second fundamental form B of M in $\tilde{M}$ is:

$$B(E,F) = \tilde{\nabla}_E F - \nabla_E F \tag{2}$$

for all $E, F \in \Gamma(TM)$.

Theorem 2.2 *Let M be a mixed geodesic QR-hypersuface of a quaternionic Kähler manifold $\tilde{M}$. If $\pi : M \to M'$ is a QR-submersion of M on an almost quaternionic Hermitian manifold, then M' is a quaternionic Kähler manifold.*

Proof. By using Gauss formula and (1) we obtain

$$h\nabla_X \mathcal{J}_a Y - \mathcal{J}_a(h\nabla_X Y) = \omega_{ab}(X)\mathcal{J}_b Y + \omega_{ac}(X)\mathcal{J}_c Y, \tag{3}$$

for any local basic vector fields X, Y on M and for any cyclic permutation (a,b,c) of $(1,2,3)$. Then we can define $1 - forms$ ω'_{ab} on M' by

$$\omega'_{ab}(X') \circ \pi = \omega_{ab}(X),\ a,b,c \in \{1,2,3\}, \tag{4}$$

for any local vector field X' on M' and X a real basic vector field on M such that $\pi_* X = X'$.
On the other hand, by the definition of a QR-submersion we have

$$\pi_* \circ \mathcal{J}_a = \mathcal{J}'_a \circ \pi_*. \tag{5}$$

Using Lemma 2.1, from (3)-(5) we obtain

$$h(\nabla'_X \mathcal{J}'_a)Y' = \omega'_{ab}(X')\mathcal{J}'_b Y' + \omega'_{ac}(X')\mathcal{J}'_c Y',$$

where ∇' is the Levi-Civita connection on M' and X', Y' any local vector fields on M'. We conclude that $(M', \mathcal{J}'_a, g')$ is a quaternionic Kähler manifold. □

3 Totally umbilical QR-hypersurfaces

In the sequel we shall denote by $\langle \cdot , \cdot \rangle$ the scalar product induced on the tangent spaces of M and $\tilde{M}$ by the Riemannian metric g. We recall that a hypersurface M of $\tilde{M}$ is totally umbilical if the first and the second fundamental forms are proportional, that is

$$B(E,F) \ = \ \langle E,F \rangle H \tag{1}$$

for any E, $F \in \Gamma(TM)$, where H is the mean curvature vector of M, defined by the formula,

$$H = \frac{1}{4m+3} TraceB. \tag{2}$$

We have the Gauss equation:

$$\tilde{R}(E,E',F,F') = R(E,E'F,F') - \langle B(E,F), B(E',F')\rangle + \\ + \langle B(E,F'), B(F,E')\rangle. \tag{3}$$

Taking account of the formula (1), the Gauss equation for a totally umbilical hypersurface M in $\tilde{M}$ becomes:

$$\tilde{R}(E,E',F,F') = R(E,E',F,F') - (\langle E,F\rangle\langle E',F'\rangle + \\ -\langle E,F'\rangle\langle F,F'\rangle)\|H\|^2, \tag{4}$$

where $\|H\|^2 = \langle H,H\rangle$.
We see that, if M is a totally umbilical QR-hypersurface of $\tilde{M}$, then it is a mixed geodesic QR-hypersurface, i.e. $B(V,X) = 0$ for any $V \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$. Consequently, the vertical distribution $\mathcal{V}$ is integrable.

Moreover, it is easy to check that each leaf of $\mathcal{V}$ is totally geodesic in M (see, for example [3], p. 121). Then we conclude that the first fundamental tensor T of the Riemannian submersion $\pi : M \to M'$ vanishes, because $T_U V$ is the second fundamental form of each fibre for any $U, V \in \Gamma(\mathcal{V})$ (see [7], [13]).
Let us now recall the following two Gray-O'Neill curvature equations for a Riemannian submersion:

$$R(U,V,U',V') = \hat{R}(U,V,U',V') + \langle T_U V', T_V U'\rangle + \\ -\langle T_V V', T_U U'\rangle, \tag{5}$$

$$R(X,Y,X',Y') = R^*(X,Y,X',Y') + 2\langle C(X,Y), C(X',Y')\rangle + \\ +\langle C(Y,X'), C(X,Y')\rangle - \langle C(X,X'), C(Y,Y')\rangle, \tag{6}$$

for all $U,V,U',V' \in \Gamma(\mathcal{V})$ and $X,Y,X',Y' \in \Gamma(\mathcal{H})$, where, for any quadruplet of horizontal vector fields (X,Y,X',Y'), $R^*(X,Y,X',Y') = R'(\pi_* X, \pi_* Y, \pi_* X', \pi_* Y') \circ \pi$, with R^* Riemannian curvature on the fibres of $\mathcal{H}$. Here R' is the Riemannian curvature of the metric g' on M'.

Lemma 3.1 *Let M be a totally umbilical, not totally geodesic, QR-hypersurface of a quaternionic Kähler manifold. Then the tensor field C wich measures the integrability of the horizontal distribution $\mathcal{H}$, is given by the formula*

$$C(X,Y) = \|H\| \sum_{a=1}^{3} \langle X, \mathcal{J}_a Y\rangle \xi_a. \tag{7}$$

Proof. Using (1), (4), (5) and (7), we obtain

$$\mathcal{J}_a(LX) - \nabla_X \xi_a \;=\; \omega_{ac}(X)\xi_b - \omega_{ab}(X)\xi_c, \tag{8}$$

for any $X \in \Gamma(\mathcal{H})$. Now, by (6) in (8), we have

$$\langle \nabla_X Y, \xi_a\rangle \;=\; \langle B(X, \mathcal{J}_a Y), \xi\rangle, \tag{9}$$

for any $X, Y \in \Gamma(\mathcal{H})$, and $a \in \{1,2,3\}$. Taking into account that the mean curvature vector H of M is a global vector field and it is non vanishing on M (see [3]), we take $\xi = \dfrac{H}{\|H\|}$. Then we have

$$C(X,Y) = V\nabla_X Y = \|H\| \sum_{a=1}^{3} \langle X, \mathcal{J}_a Y\rangle \xi_a, \tag{10}$$

from which formula (7) follows. □

Theorem 3.1 *Let M be a totally umbilical, not totally geodesic, QR-hypersurface of a quaternionic Kähler manifold. Then,*

a) $\tilde{K}(U,V) = K(U,V) - \|H\|^2$, *where* $\{U,V\}$ *is an orthonormal basis of the vertical* $2-$*plane* α, $\alpha \subset \mathcal{V}_x, x \in M$, *and* $\tilde{K}, K$ *denote the sectional curvatures of* α *on* $\tilde{M}, M$, *respectively.*

b) $K(X,Y) = K'(X',Y') - 3\|H\|^2 \sum_{a=1}^{3} \langle X, \mathcal{J}_a Y\rangle^2$, *where* X, Y *is an orthonormal basis of a horizontal* $2-$*plane* $\alpha \subset \mathcal{H}_x$, $K(X,Y)$ *denoting the sectional curvature of* α, *and* $K'(X',Y')$ *denotes the sectional curvature in* M' *of the* $2-$*plane spanned by* $X' = \pi_* X$ *and* $Y' = \pi_* Y$.

Proof. Property a) is easily obtained from (4) and (5). From (6), as an immediate consequence of the skew-symmetry of C, we have

$$R(X,Y,X,Y) = R'(X',Y'.X',Y') - 3\|C(X,Y)\|^2. \tag{11}$$

Lemma 3.1 and (11) directly give b). □

We recall that a totally umbilical, not totally geodesic, hypersurface M of a Riemannian manifold $\tilde{M}$ is an extrinsic hypersphere if the mean curvature vector field H is parallel with respect to the linear normal connection $\nabla^{\perp}$ or, equivalently, $\|H\| = c$ is a constant $c \neq 0$ on M.
Then we have the following

Theorem 3.2 *Let M be an extrinsic hypersurface of a flat quaternionic Kähler manifold* $\tilde{M}$ *and* $\pi : M \to M'$ *a QR-submersion of M on a quaternionic Kähler manifold M'. Then M' is a quaternionic Kähler manifold with constant quaternionic sectional curvature* $c > 0$.

Proof. By (4), (6) and Lemma 3.1 we have

$$\begin{aligned} R'(X',Y')Z' = \|H\|^2\{&g'(Y',Z')X' - g'(X',Z')Y' + \\ &+ \sum_{a=1}^{3}(g'(\mathcal{J}'_a Y',Z')\mathcal{J}'_a X' - g(\mathcal{J}'_a X',Z')\mathcal{J}'_a Y') + \\ &+ 2g'(X',\mathcal{J}'_a Y')\mathcal{J}'_a Z'). \end{aligned}$$

where $\|H\|$ is a constant on M' and $X', Y', Z' \in \Gamma(TM')$. □

Remark There exist no proper totally umbilical QR-submanifolds in positively or negatively curved quaternionic Kähler manifolds (see [3]).

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Dipartimento di Matematica, Università
degli Studi di Parma
Via M. D'Azeglio, 85A, 43100 Parma, Italia.
email: vittorio.mangione@unipr.it