Variational Calculus on Sub-Riemannian Manifolds

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Dedicated to the Memory of Grigorios TSAGAS (1935-2003),
President of Balkan Society of Geometers (1997-2003)

Abstract

We provide invariant formulas for the Euler-Lagrange equation associated to sub-Riemannian geodesics. They use the concept of curvature and horizontal connection introduced and studied in the paper.

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1 Introduction

The geodesic is a concept which comes from Riemannian geometry. It is the curve with the minimum energy $E = \int_0^1 \frac{1}{2}|\dot{c}(s)|^2\, ds$ between two given points. At least two kind of constraints can be considered to act on the curve: holonomic and non-holonomic. A holonomic constraint is when the energy is perturbed by a potential $U(c)$ and the energy becomes $E = \int_0^1 (\frac{1}{2}|\dot{c}|^2 + U(c))\, ds$. The equation geodesic in this case is $\nabla_{\dot{c}}\dot{c} = -U'(c)$.

The other kind of constraints are the nonholonomic ones (see [1], [8], [9]). These are constraints on the velocity of the curve. The energy to be minimized is $E = \int_0^1 \left(\frac{1}{2}|\dot{c}|^2 + !\,(\dot{c})\right) ds$. The paper deals with a presentation of the variational calculus for the case when ! is a 1-form of type (1:1) such that (1:3) does not vanish. It is said that these kind of sub-Riemannian manifolds are of step 2. They are also called Heisenberg manifolds (see [2]). In general a sub-Riemannian manifold is said to be of step k if $k-1$ iterations need for the brackets of X_j in order to span the whole tangent space.

In section 5 we shall deal with examples of sub-Riemannian manifolds of superior type.

The idea of the paper is to consider the solutions of the Euler-Lagrange system as geodesics in a certain connection with certain perturbation given by the curvature tensor defined in section 2. Section 3 shows that the classical Hamilton-Jacobi equation still holds if the gradient is modified into a horizontal gradient. The relationship

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between the symplectic and sub-Riemannian structures is pointed out in section 4. Section 5 provides a few examples of sub-Riemannian manifolds and their geodesic equations. Some of these equations were solved in [3,4,6].

Consider a nonintegrable 2-dimensional distribution $x \to \mathcal{H}_x$ in $\mathbb{R}^3 = \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}_t$ defined as $\mathcal{H} = \ker \omega$, where ω is a 1-form on $\mathbb{R}^3$. The distribution $\mathcal{H}$ is called the *horizontal distribution*. We shall assume the 1-form $\omega = \omega^1 dx_1 + \omega^2 dx_3 + \omega^3 dt$ has the coefficient $\omega^3 \neq 0$ so that dividing by it we may assume

$$\omega = -A_1(x)dx_1 + A_2(x)dx_2 + dt \tag{1.1}$$

with $A_1 = -\omega^1$, and $A_2 = \omega^2$. One may verify that

$$\omega(X_1) = \omega(X_1) = 0$$

where

$$X_1 = \partial_{x_1} + A_1(x)\partial_t\,; \qquad X_2 = \partial_{x_2} - A_2(x)\partial_t \tag{1.2}$$

The vector fields X_1, X_2 span the horizontal distribution $\mathcal{H}$ and they are called *horizontal vector fields*.

Suppose the 2-form

$$\Omega := d\omega = \Big(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_1}\Big) dx_1 \wedge dx_2 \tag{1.3}$$

does not vanish. Then

$$[X_1; X_2] = -\Big(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_1}\Big)\partial_t \in \mathcal{H} \tag{1.4}$$

and then $\mathcal{H}$ is not integrable, by Frobenius theorem.

Consider the positive definite metric $g : \mathcal{H} \times \mathcal{H} \to \mathcal{F}$ in which the vector fields $\{X_1; X_2\}$ are orthonormal. The metric g is called the *sub-Riemannian metric* defined by the vector fields X_1 and X_2.

A curve $s \to c(s) = (x_1(s); x_2(s); t(s))$ is called *horizontal curve* if $\dot c(s) \in \mathcal{H}_{c(s)}$, for every s. As

$$\begin{aligned} \dot c(s) &= \dot x_1(s)\partial_{x_1} + \dot x_2(s)\partial_{x_2} + \dot t(s)\partial_t \\ &= \dot x_1(s)X_1 + \dot x_2(s)X_2 + \omega(\dot c(s))\partial_t \end{aligned}$$

then $c(s)$ is a horizontal curve iff

$$\omega(\dot c) = \dot t - A_1(c)\dot x_1 + A_2(c)\dot x_2 = 0 \tag{1.5}$$

The length of c with respect to the metric g is

$$l(c) = \int_0^1 \sqrt{g(\dot c(s); \dot c(s))}\,ds = \int_0^1 \sqrt{\dot x_1(s)^2 + \dot x_2(s)^2}\,ds \tag{1.6}$$

Given two points O and P there is at lest a horizontal curve connecting them (see [5]). The Carnot-Carathéodory distance is defined as

$$d_C(O; P) = \inf\{l(c); c(0) = O; c(1) = P; c \text{ horizontal}\} \tag{1.7}$$

The horizontal curve with minimum length are called *sub-Riemannian geodesics* and can be described using the Hamiltonian formalism as in the following (see [7]).
Consider the sub-elliptic operator

$$\Delta_X = \frac{1}{2}\Big(X_1^2 + X_1^2\Big) \tag{1.8}$$

and define the Hamiltonian as the principal symbol of Δ_X

$$H(x;t;»;µ) = \frac{1}{2}\Big(»_1 + A_1(x)µ\Big)^2 + \frac{1}{2}\Big(»_2 - A_2(x)µ\Big)^2 \tag{1.9}$$

The projections on the $(x;t)$-space of the solution of the Hamilton's system

$$\dot{x} = H_\xi\ ; \quad \dot{t} = H_\theta \tag{1.10}$$

$$\dot{»} = -H_x\ ; \quad \dot{µ} = -H_t \tag{1.11}$$

with the boundary conditions

$$x(0) = t(0) = 0\ ; \quad x(1) = x; t(1) = t \tag{1.12}$$

are called *sub-Riemannian geodesics* between the origin and $(x;t)$.
From $\dot{t} = H_\theta$ we get

$$\dot{t} = A_1\dot{x}_1 - A_2\dot{x}_2 \tag{1.13}$$

i.e. the sub-Riemannian geodesics are horizontal curves.

2 Connection and curvature

The horizontal connection

The *horizontal connection* is defined as

$$D : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \tag{2.14}$$

$$D(V;W) = D_V W = \sum_{k=1,2} V g(W;X_k)X_k \tag{2.15}$$

Proposition 2.1 D *is a linear metric connection.*

Proof. One needs to verify the Leibnitz rule

$$D_V(fW) = V(f)W + f\,D_V W \tag{2.16}$$

and the condition

$$Ug(V;W) = g(D_U V;W) + g(V;D_U W) \tag{2.17}$$

For the first part,

$$D_V(fW) = \sum V\,g(fW;X_k)\,X_k =$$

$$= \sum V(f)\,g(W;X_k)X_k + f\sum V g(W;X_k)X_k = V(f)W + f\,D_V W$$

To show the second part,

$$
\begin{aligned}
&g(D_U V; W) + g(V; D_U W) = \\
&= g\Big(\sum U\, g(V; X_i) X_i; W\Big) + g\Big(V; \sum U\, g(W; X_i) X_i\Big) = \\
&= g\Big(\sum U(V^i) X_i; W\Big) + g\Big(V; \sum U(W^i) X_i\Big) = \\
&= \sum U(V^i) W^i + \sum U(W^i) V^i = U\Big(\sum V^i W^i\Big) = U\, g(V; W)
\end{aligned}
$$

where $V = \sum V^i X_i$ and $W = \sum W^i X_i$.

Let $Z = Z^1 X_1 + Z^2 X_2$ be a horizontal vector field. The *horizontal divergence* is defined as

$$
\operatorname{div}_{\mathcal{H}} Z = \operatorname{trace}_g(V \to D_V Z) = \tag{2.18}
$$

$$
\sum_k g(X_k; D_{X_k} Z) = \sum_k \Big(X_k(Z^j) X_j\Big)^k = \sum_k X_k(Z^k) = \sum_k X_k\, g(Z; X_k):
$$

Define also the *X-gradient* of a function f as

$$
\nabla_X f = \sum_k X_k(f)\, X_k: \tag{2.19}
$$

Then

$$
\frac{1}{2} \operatorname{div}_{\mathcal{H}} \nabla_X = \Delta_X f \tag{2.20}
$$

The curvature tensor. Let $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be given by

$$
\mathcal{K}(U) = \sum_k \Omega(U; X_k) X_k: \tag{2.21}
$$

$\mathcal{K}$ is $\mathcal{F}(R^3)$-linear and can be considered as a (1,1)-*tensor of curvature*.

The following result shows that $\mathcal{K}$ is skew-selfadjoint.

Proposition 2.2 *For every* $U; W \in \mathcal{H}$

$$
g\Big(\mathcal{K}(U); W\Big) + g\Big(U; \mathcal{K}(W)\Big) = 0: \tag{2.22}
$$

Proof. We show first that

$$
g\Big(\mathcal{K}(U); W\Big) = \Omega(U; W); \tag{2.23}
$$

and using the skew-symmetry of Ω we get (2.22).

Indeed,

$$
\begin{aligned}
g\Big(\mathcal{K}(U); W\Big) &= g\Big(\sum_k \Omega(U; X_k) X_k; W\Big) = \\
&= \sum_k g(X_k; W) \Omega(U; X_k) = \Omega(U; W):
\end{aligned}
$$

Corollary 2.3 *For any* $U \in \mathcal{H}$,

$$g\Big(\mathcal{K}(U), U\Big) = 0. \tag{2.24}$$

The last result suggests that in the case of a 2-dimensional distribution, the curvature $\mathcal{K}$ is proportional with a rotation of angle $\pi/2$.
Define the rotation $\mathcal{J} : \mathcal{H} \to \mathcal{H}$ as

$$\mathcal{J}(X_1) = X_2\,; \qquad \mathcal{J}(X_2) = -X_1 \tag{2.25}$$

Then

$$\mathcal{K}(X_1) = \Omega(X_1, X_2)X_2 = \Omega(X_1, X_2)\mathcal{J}(X_1)$$

$$\mathcal{K}(X_2) = \Omega(X_2, X_1)X_1 = \Omega(X_1, X_2)\mathcal{J}(X_2)$$

We arrived at the following formula for the curvature

$$\mathcal{K}(U) = \Omega(X_1, X_2)\mathcal{J}(U), \qquad \forall U \in \mathcal{H} \tag{2.26}$$

If the matrix Ω_{ij} is non-degenerate i.e. $\Big(\dfrac{\partial A_1}{\partial x_1} + \dfrac{\partial A_2}{\partial x_1}\Big) \neq 0$, then $\mathcal{K}(U) \neq 0$ for $U \neq 0$.

If V is not a horizontal vector field then the curvature can still be defined using

$$\mathcal{K}(V) = \sum_k \Omega(V, X_k)X_k \tag{2.27}$$

This is because the right hand side depends only on the horizontal part of V. Indeed, consider the vector field

$$V = V^1\partial_{x_1} + V^2\partial_{x_2} + V^3\partial_t$$

A computation shows

$$V = \underbrace{V^1X_1 + V^2X_2}_{=V_H} + \omega(V)\partial_t$$

Then

$$\Omega(V, X_k) = \Omega(V_H, X_k) + \omega(V)\underbrace{\Omega(\partial_t, X_k)}_{=0}$$

Hence $\mathcal{K}(V) = \mathcal{K}(V_H)$.

3 The Euler-Lagrange equation

The Legendre transform of the Hamiltonian (1.9) leads to the following Lagrangian

$$L(x, t, \dot{x}, \dot{t}) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \mu(\dot{t} - A_1(x)\dot{x}_1 + A_2(x)\dot{x}_2), \tag{3.28}$$

where μ is constant because

$$\dot{\mu} = -\frac{\partial H}{\partial t} = -\frac{dH}{dt} = 0. \tag{3.29}$$

We deal now with a minimization problem with constraints given by

$$L(c, \dot{c}) = \frac{1}{2} g(\dot{c}, \dot{c}) + \mu\, \omega(\dot{c}) \tag{3.30}$$

A computation shows the Euler-Lagrange system of equations

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{c}} = \frac{\partial L}{\partial c}, \qquad c = (x_1, x_2, t) \tag{3.31}$$

becomes

$$\ddot{x}_1 = \mu\Big(\frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1}\Big)\dot{x}_2 \tag{3.32}$$

$$\ddot{x}_2 = -\mu\Big(\frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1}\Big)\dot{x}_1 \tag{3.33}$$

If the velocity of the geodesic is given by $\dot{c}(s) = \dot{x}_1(s)X_1 + \dot{x}_2(s)X_2$, the system $(3.32) - (3.33)$ can be written as

$$\ddot{x}_1 X_1 + \ddot{x}_2 X_2 = \mu\Big(\frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1}\Big)(\dot{x}_2 X_1 - \dot{x}_1 X_2) \tag{3.34}$$

The right hand side has the meaning of curvature. Indeed, using (2.25) and (2.26) the right hand side of (3.34) yields

$$-\mu\, \Omega(X_1, X_2)\, \mathcal{J}(\dot{c}) = -\mu \mathcal{K}(\dot{c}). \tag{3.35}$$

For the left hand side of (3.34) consider the acceleration defined by the horizontal connection along $\dot{c}(s)$

$$D_{\dot{c}}\dot{c} = \sum_k \dot{c}\, g(\dot{c}, X_k) X_k = \dot{c}(\dot{x}_1)X_1 + \dot{c}(\dot{x}_2)X_2 = \ddot{x}_1 X_1 + \ddot{x}_2 X_2.$$

Hence the Euler-Lagrange system of equations can be written globally as

$$D_{\dot{c}}\dot{c} = -\mu \mathcal{K}(\dot{c}) \tag{3.36}$$

In sub-Riemannian geometry the acceleration of the geodesics is equal to the curvature. This keeps the geodesics into the horizontal distribution. Like in Riemannian geometry, we have

Corollary 3.1 *The length of velocity* $\dot{c}$ *in the sub-Riemannian metric* g *is constant.*

Proof. Since D is a metric connection,

$$\dot{c}\, g(\dot{c}, \dot{c}) = 2g(D_{\dot{c}}\dot{c}, \dot{c}) = -2\mu g(\mathcal{K}(\dot{c}), \dot{c}) = 0,$$

by Corollary 2.3.

The Hamilton-Jacobi equation.

Lemma 3.2 *Let* $c(s) = (x_1(s), x_2(s), t(s))$ *be a horizontal curve and a smooth function* $f \in \mathcal{F}(\mathbb{R}^3)$. *Then*

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + g(\dot c, \nabla_X f). \tag{3.37}$$

Proof.

$$\begin{aligned}
\frac{df}{ds} &= \frac{\partial f}{\partial s} + \frac{\partial f}{\partial x_1}\dot x_1 + \frac{\partial f}{\partial x_2}\dot x_2 + \frac{\partial f}{\partial t}\dot t = \\
&= \frac{\partial f}{\partial s} + \Big(X_1 f - A_1(x)\frac{\partial f}{\partial t}\Big)\dot x_1 + \Big(X_2 f + A_2(x)\frac{\partial f}{\partial t}\Big)\dot x_2 + \frac{\partial f}{\partial t}\dot t = \\
&= \frac{\partial f}{\partial s} + (X_1 f)\,\dot x_1 + (X_2 f)\,\dot x_2 + \frac{\partial f}{\partial t}\,\omega(\dot c) = \frac{\partial f}{\partial s} + g(\dot c, \nabla_X f).
\end{aligned}$$

In the following we need to find the minimum of

$$I = \int_0^\tau \frac{1}{2}(\dot x_1(s)^2 + \dot x_2(s)^2)\, ds = \int_0^\tau \frac{1}{2}|\dot c(s)|_g^2\, ds$$

over the horizontal curves $c(s)$ with fixed ends.

Let $S(x, t) \in \mathcal{F}$ be the solution for the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \tau} + \frac{1}{2}|\nabla_X S|^2 = 0, \quad S(O) = 0. \tag{3.38}$$

Consider the integral

$$J = \int_0^1 \frac{1}{2}|\dot c(s)|_g^2\, ds - dS \tag{3.39}$$

Using Lemma 3.2

$$\begin{aligned}
J &= \int_0^\tau \Big(\frac{1}{2}|\dot c(s)|_g^2 - \frac{\partial S}{\partial s} - g(\nabla_X S, \dot c)\Big)\, ds = \\
&= \int_0^\tau \left(\frac{1}{2}|\dot c - \nabla_X S|_g^2 - \Big(\frac{\partial S}{\partial s} + \frac{1}{2}|\nabla_X S|^2\Big)\right) ds = \int_0^\tau \frac{1}{2}|\dot c - \nabla_X S|_g^2\, ds
\end{aligned} \tag{3.40}$$

The integrals I and J reach the minimum for the same horizontal curve and this occurs for a curve with the velocity

$$\dot c = \nabla_X S \tag{3.41}$$

Theorem 3.3 *A horizontal curve* $c(s)$ *is energy-minimizing iff* (3.41) *holds.*

Using (2.20) we get

Corollary 3.4 *The horizontal divergence of the geodesic flow is*

$$\mathrm{div}_{\mathcal{H}}\, \dot c = 2\Delta_X S \tag{3.42}$$

The Hamiltonian. The Hamiltonian $H : T^*M \to R$ is defined as

$$H(x;p) = \frac{1}{2}\sum_k p(X_k)^2$$

If $p = df$,

$$H(x;df) = \frac{1}{2}\sum df(X_k)^2 = \frac{1}{2}\sum X_k(f)^2 = \frac{1}{2}|\nabla_X f|^2:$$

For $f = S$,

$$H(x;dS) = \frac{1}{2}|\nabla_X S|^2 = \frac{1}{2}|\dot{c}|^2 = \frac{1}{2}:$$

We also have

$$H(x;!) = \frac{1}{2}\sum !(X_i)^2 = 0:$$

The eiconal equation. Consider the energy associated to a function $f \in \mathcal{F}(R^3)$ defined as

$$H(\nabla f) = H(x;df) = \frac{1}{2}|\nabla_X f|^2 = \frac{1}{2}\Big((X_1 f)^2 + (X_2 f)^2\Big) \tag{3.43}$$

The front wave is given by the level curves of the energy and it is described by the eiconal equation

$$H(\nabla f) = k; \quad \text{positive constant} \tag{3.44}$$

with the initial condition

$$f(O) = 0 \tag{3.45}$$

If $k = 0$, then f is the constant function equal to zero. Indeed, suppose that f is not constant. There is a point p such that $(\mathrm{grad} f)_p \neq 0$. Then $\Sigma_c = f^{-1}(c)$ will be a surface through p, where $c = f(p)$. As $X_i(f) = 0$, then X_i are tangent to Σ_c on a neighborhood of p and hence Σ_c becomes integral surface for the horizontal distribution $\mathcal{H}$ around p, which is nonintegrable, contradiction.

If $k \neq 0$, consider the geodesics starting at origin $c(0) = O$, parametrized such that $|\dot{c}(s)|_g^2 = 2k$. If S is the action along $c(s)$, by (3.41) we have

$$H(\nabla S) = \frac{1}{2}|\nabla_X S|_g^2 = \frac{1}{2}|\dot{c}|_g^2 = k:$$

Jacobi vector fields and curvature. Let $c(s)$ be a subRiemannian geodesic which starts at origin and let P be the first conjugate point with 0 along $c(s)$. Denote by $V(s)$ a Jacobi vector field along $c(s)$ and by $S(s)$ the action between 0 and $c(s)$.

Proposition 3.5

$$\int_0^1 \mathcal{K}(V(s))(S(s))\,ds = 0; \tag{3.46}$$

where $P = c(1)$ *and* $\mathcal{K}$ *is the curvature.*

Proof. Let $c_\epsilon = F_\epsilon(c)$ be a smooth variation of c, such that for every ϵ, c_ϵ is a sub-Riemannian geodesic. As c_ϵ is a horizontal curve, then

$$0 = \int_0^1 \omega(\dot{c}_\epsilon(s))\, ds = \int_{c_\epsilon} \omega = \int_{F_\epsilon(c)} \omega = \int_c F_\epsilon^* \omega$$

Then

$$\frac{d}{d\epsilon} \int_c F_\epsilon^* \omega = 0$$

or,

$$\int_c L_V \omega = 0 ;$$

where V is the Jacobi vector field associat to the variation $(c_\epsilon)_\epsilon$. As V is zero at the end points of c,

$$\int_c d(i_V \omega) = \int_{\partial c} i_V \omega = \omega(V)(0) - \omega(V)(1) = 0 :$$

Cartan decomposition yields

$$L_V \omega = d(i_V \omega) + i_V(d\omega);$$

and then

$$\int_c i_V \Omega = 0 \;;$$

which can also be written as

$$\int_0^1 \Omega(V(s); \dot{c}(s))\, ds = 0 :$$

Using $\dot{c} = \sum \dot{c}^j X_j$ and $\dot{c}^j(s) = X_j\,(S)V$ -6.58 TD[4((