Weakly ϕ-Symmetric Contact Metric Spaces

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Abstract

The examples, that we denote by G_w, given in [2] of contact metric spaces which are weakly locally ϕ -symmetric, but not strongly, satisfy the stronger condition that their contact metric structure is homogeneous. In this paper we give the first example of weakly locally ϕ-symmetric space which is not homogeneous, consequently these spaces form a larger class. Moreover, we show that the examples G_w are the only 3-dimensional weakly, but not strongly, locally ϕ-symmetric spaces which have constant scalar curvature and vertical Ricci curvature.

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1 Introduction

A locally symmetric Sasakian manifold (or K-contact) manifold is of constant sectional curvature 1 (see [7],[10]). For this reason, Takahashi [9] introduced the notion of a locally ϕ-symmetric space. It is a Sasakian manifold satisfying the condition

$$(\nabla_X R)(Y, Z, V, W) = 0 \tag{1.1}$$

for all vector fields X, Y, Z, V, W orthogonal to the characteristic vector field ξ. This notion has been for the most part explored only in the Sasakian context and it is not clear what the corresponding notion should be for a general contact metric manifold.

D.E.Blair, T. Koufogiorgos and R.Sharma [1] extended the Takahashi's notion to a general contact metric manifold (M, η, g, ξ, ϕ) by using the same curvature condition (1.1); moreover they proved that a 3-dimensional contact metric manifold , with $Q\phi = \phi Q$, is weakly locally ϕ-symmetric iff it is of constant scalar curvature. In the Sasakian case, the condition (1.1) is satisfied if and only if all characteristic reflections are (local) isometries. Then E. Boeckx and L. Vanhecke [3] gave the following new generalization : a contact metric manifold is called locally ϕ-simmetric if and only if all characteristic reflections are (local) isometries. To distinguish between the two definitions, since the first is weaker than the second, following [2] we speak of *weakly locally ϕ-symmetric spaces* (for the first one) and *strongly locally ϕ-symmetric* (for the second one).

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In [4] was determined all 3-dimensional strongly locally ϕ-symmetric spaces and proved that a contact metric three-manifold is strongly locally ϕ-symmetric if and only if it is a locally homogeneous contact metric manifold satisfying the condition $\sigma(X) = 0\ \forall X \in Ker\eta$, where $\sigma(X)$ denotes the *vertical Ricci curvature* $\rho(X, \xi) = g(Q\xi, X)$. Recently E. Boeckx, P. Bunken and L. Vanhecke [2] give the first examples of contact metric spaces which are weakly locally ϕ-symmetric, but not strongly. These examples are non-unimodular Lie group of dimension three, that we denote by G_w, equipped with a left invariant contact metric structure which depends by a parameter $w \in R$, $w < 0$. We note that the parameter w is completely determined by the Webster scalar curvature (see Remark 4.1).

In this paper we show that the spaces G_w are the only weakly locally ϕ-symmetric, but not strongly, with constant scalar curvature and vertical Ricci curvature $\sigma(X)$ (see Theorem 4.1). The examples G_w satisfy the stronger condition that their contact metric structure is homogeneous. So, one natural question is to see if there exist weakly locally ϕ-symmetric three-spaces which are not homogeneous. We give a positive answer to this question (see Theorem 4.3), consequently the weakly locally ϕ-symmetric spaces form a larger class. In the last section, we show that the unit tangent sphere bundle of a Riemannian two-manifold (M, G) is weakly locally ϕ-symmetric if and only if (M, G) has constant sectional curvature.

2 Preliminaries on contact metric manifolds

In this section we collect some basic facts about contact metric manifolds. All manifolds are assumed to be connected and smooth. A $(2n+1)$-dimensional manifold M has an *almost contact structure* if it admits a vector field ξ (the *characteristic field*), a one-form η and a $(1,1)$-tensor field ϕ satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi.$$

Then one can always find a Riemannian metric g which is compatible with the structure, that is, such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X and Y. (ξ, η, ϕ, g) is called an *almost contact metric structure* and (M, ξ, η, ϕ, g) an *almost contact metric manifold*. If additionally it holds $d\eta(X, Y) = g(X, \phi Y)$, then (M, ξ, η, ϕ, g) is called a *contact metric manifold*. In what follows we denote by ∇ the Levi Civita connection and by R the corresponding Riemann curvature tensor given by $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ for all smooth vector fields X, Y. Moreover, we denote by ρ the Ricci tensor of type $(0,2)$, by Q the corresponding endomorphism field and by r the scalar curvature. We note that $\sigma(X) := g(Q\xi, X) = 0\ \forall X \in Ker\eta$ iff $Q\xi$ is parallel to ξ, moreover $Q\phi = \phi Q$ implies $Q\xi$ is parallel to ξ. The tensor $h = \frac{1}{2} L_\xi \phi$, where L denotes the Lie derivative, is symmetric and satisfies $-\phi h = \nabla\xi + \phi = h\phi$. A contact metric space is said to be a *K-contact manifold* if ξ is a Killing vector field, or equivalently, $h = 0$. For a three-dimensional contact metric manifold, the *Webster scalar curvature* W (see [5]) and the $\phi-$*sectional curvature*H are given by

$$W = \frac{1}{8}(r - \rho(\xi, \xi) + 4) \qquad 2H = r - 4(1 - \lambda^2) = r - 2\rho(\xi, \xi), \tag{2.1}$$

moreover, the contact metric structure is K-contact iff it is Sasakian.

Next, let (M, ξ, η, ϕ, g) be a three-dimensional contact metric manifold and m a point of M. Then there exists a local orthonormal basis $\{\xi, e_1, e_2 = \phi e_1\}$ of smooth eigenvectors of h in a neigborhood of m. Now, let U_1 be the open subset of M where $h \neq 0$ and let U_2 be the open subset of points $m \in M$ such that $h = 0$ in a neighborhood of m. $U_1 \cup U_2$ is an open dense subset of M. On U_1 we put $he_1 = \lambda e_1$ and hence, from (2.4) we have $he_2 = -\lambda e_2$ where λ is a non-vanishing smooth function. Then we have

Lemma 2.1 [4] *On U_1 we have*

$$
\begin{aligned}
&\nabla_\xi e_1 = -ae_2\,, && \nabla_\xi e_2 = ae_1\,,\\
&\nabla_{e_1}\xi = -(\lambda+1)e_2\,, && \nabla_{e_2}\xi = -(\lambda-1)e_1\,,\\
&\nabla_{e_1}e_1 = \frac{1}{2\lambda}\{(e_2)(\lambda)+A\}e_2\,, && \nabla_{e_2}e_2 = \frac{1}{2\lambda}\{e_1(\lambda)+B\}e_1\,,\\
&\nabla_{e_1}e_2 = -\frac{1}{2\lambda}\{(e_2)(\lambda)+A\}e_1 + (\lambda+1)\xi\,, &&\\
&\nabla_{e_2}e_1 = -\frac{1}{2\lambda}\{e_1(\lambda)+B\}e_2 + (\lambda-1)\xi\,, &&
\end{aligned}
\tag{2.2}
$$

$$
[e_1, e_2] = -\frac{1}{2\lambda}\{(e_2)(\lambda)+A)\}e_1 + \frac{1}{2\lambda}\{(e_1)(\lambda)+B\}e_2 + 2\xi\,, \tag{2.3}
$$

where $A = \rho(\xi, e_1)$, $B = \rho(\xi, e_2)$ and a is a smooth function.

Finally, we recall that the components of the Ricci operator Q, with respect to $\{\xi, e_1, e_2 = \phi e\}$, are given by (see [8])

$$
\begin{cases}
Q\xi = 2(1-\lambda^2)\xi + Ae_1 + Be_2,\\
Qe_1 = A\xi + (\frac{r}{2} - 1 + \lambda^2 + 2a\lambda)e_1 + \xi(\lambda)e_2,\\
Qe_2 = B\xi + \xi(\lambda)e_1 + (\frac{r}{2} - 1 + \lambda^2 - 2a\lambda)e_2,
\end{cases}
$$

from which it follows easily

$$
(\nabla_\xi Q)\xi = -4\lambda\xi(\lambda)\xi + \{\xi(A) + aB\}e_1 + \{\xi(B) - aA\}e_2, \tag{2.4}
$$

$$
\begin{aligned}
(\nabla_{e_1}Q)e_1 = {} & \{e_1(A) + (\lambda+1)\xi(\lambda) - \frac{B}{2\lambda}[e_2(\lambda) + A]\}\xi\\
& + \{e_1(\frac{r}{2} + \lambda^2 + 2a\lambda) - \frac{\xi(\lambda)}{\lambda}[e_2(\lambda) + A]\}e_1\\
& + \{e_1\xi(\lambda) + 2a(e_2)(\lambda) + (2a - \lambda - 1)A\}e_2,
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
(\nabla_{e_2}Q)e_2 &= \{e_2(B) + (\lambda-1)\xi(\lambda) - \frac{A}{2\lambda}[e_1(\lambda)+B]\}\xi \\
&+\{e_2(\xi)\lambda - 2ae_1(\lambda) + (1-\lambda-2a)B\}e_1 \\
&+\{e_2(\frac{r}{2}+\lambda^2-2a\lambda) - \frac{\xi(\lambda)}{\lambda}[e_1(\lambda)+B]\}e_2,
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
&(\nabla_{e_1}Q)e_2 = \{e_1(B) + (\lambda+1)(\frac{r}{2}+3\lambda^2-3-2a\lambda) + \\
&\frac{A}{2\lambda}[e_2(\lambda)+A]\}\xi + \{e_1\xi(\lambda) + 2ae_2(\lambda) + A(2a-\lambda-1)\}e_1 \\
&+\{e_1(\frac{r}{2}+\lambda^2-2a\lambda) - 2B(\lambda+1) + \frac{\xi(\lambda)}{\lambda}[e_2(\lambda)+A]\}e_2,
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
&(\nabla_{e_2}Q)e_1 = \{e_2(A) + (\lambda-1)(\frac{r}{2}+3\lambda^2-3+2a\lambda) + \\
&+\frac{B}{2\lambda}[e_1(\lambda)+B]\}\xi + \{e_2(\frac{r}{2}+\lambda^2+2a\lambda) - 2A(\lambda-1) \\
&+\frac{\xi(\lambda)}{\lambda}[(e_1)(\lambda)+B]\}e_1 + \{e_2\xi(\lambda) - 2ae_1(\lambda) + B(1-2a-\lambda)\}e_2.
\end{aligned} \tag{2.8}
$$

3 A characterization of weakly locally ϕ-symmetric contact metric three-manifolds

In the sequel we denote by M a contact metric three-manifold and by (η, g, ϕ, ξ) its contact metric structure.

Lemma 3.1 *A contact metric three-manifold M is weakly locally ϕ-symmetric if and only if*

$$
\begin{cases} e_1(H) = 2B(\lambda+1) \\ e_2(H) = 2A(\lambda-1). \end{cases} \tag{3.1}
$$

where H is the ϕ-sectional curvature and λ is the eigenvalue corresponding to the eigenvector e_1.

Proof. From (1.1) follows that M is weakly locally ϕ-symmetric if and ony if

$$(\nabla_V R)(X,Y,Z) = g((\nabla_V R)(X,Y,Z),\xi)\xi$$

for any $X,Y,Z,V \in Ker\,\eta$. Since $\dim M = 3$, we have the well-known formula

$$
\begin{aligned}
&R(X,Y)Z = g(X,Z)QY - g(Y,Z)QX + \rho(X,Z)Y - \rho(Y,Z)X + \\
&-\frac{r}{2}\{g(X,Z)Y - g(Y,Z)X\},
\end{aligned}
$$

for all X,Y,Z vector fields on M. Therefore, we have

$$
\begin{aligned}
(\nabla_{e_1}R)(e_1,e_2,e_1) \;=\; & (\nabla_{e_1}Q)e_2 + g((\nabla_{e_1}Q)e_1,e_1)e_2 \\
& -g((\nabla_{e_1}Q)e_2,e_1)e_1 - e_1(\frac{r}{2})e_2,
\end{aligned}
$$

$$\begin{aligned}(\nabla_{e_1}R)(e_1,e_2,e_2) &= -(\nabla_{e_1}Q)e_1 + g((\nabla_{e_1}Q)e_1,e_2)e_2 \\ &\quad -g((\nabla_{e_1}Q)e_2,e_2)e_2 + e_1(\frac{r}{2})e_1, \\ (\nabla_{e_2}R)(e_1,e_2,e_1) &= (\nabla_{e_2}Q)e_2 + g((\nabla_{e_2}Q)e_1,e_1)e_2 \\ &\quad -g((\nabla_{e_2}Q)e_2,e_1)e_1 - e_2(\frac{r}{2})e_2, \\ (\nabla_{e_2}R)(e_1,e_2,e_2) &= -(\nabla_{e_2}Q)e_1 + g((\nabla_{e_2}Q)e_1,e_2)e_2 \\ &\quad -g((\nabla_{e_2}Q)e_2,e_2)e_1 + e_2(\frac{r}{2})e_1.\end{aligned}$$

Consequently $(\nabla_{e_1}R)(e_1,e_2,e_1)$ and $(\nabla_{e_2}R)(e_1,e_2,e_1)$ are parallel to ξ if and only if holds the following

$$\left\{\begin{array}{l}\frac{1}{2}e_1(r) = g(\nabla_{e_1}Q)e_1,e_1) + g(\nabla_{e_1}Q)e_2,e_2) \\ \frac{1}{2}e_2(r) = g(\nabla_{e_2}Q)e_1,e_1) + g(\nabla_{e_2}Q)e_2,e_2).\end{array}\right. \tag{3.2}$$

Imposing that the other components $(\nabla_{e_1}R)(e_1,e_2,e_2)$ and $(\nabla_{e_2}R)(e_1,e_2,e_2)$ are parallel to ξ we get the same condition (3.2). If M is Sasakian, then $Q\xi = 2\xi$, $Qe_1 = (\frac{r}{2}-1)e_1$, $Qe_2 = \frac{r}{2}-1)e_2$, from which it follows

$$\xi(r) = g((\nabla_\xi Q)\xi,\xi) + g((\nabla_{e_1}Q)e_1,\xi) + g((\nabla_{e_2}Q)e_2,\xi) = 0,$$

and hence

$$r = const. \Longleftrightarrow e_1(r) = e_2(r) = 0.$$

But $r = 4+2H$, so $\xi(H) = \xi(r) = 0$ and $r = const. \Leftrightarrow H = const. \Leftrightarrow e_1(H) = e_2(H) = 0$. Moreover (see [11]): M is locally ϕ-symmetric $\Leftrightarrow r = const$. Therefore, we get the statement of Lemma 3.1, since for M Sasakian $A = B = 0$.

Now assume that M is not Sasakian. From (2.5)-(2.8) we have

$$\begin{aligned}g((\nabla_{e_1}Q)e_1,e_1) &= \frac{e_1(r)}{2} + e_1(\lambda^2) + e_1(2a\lambda) - \frac{\xi(\lambda)}{\lambda}\{e_2(\lambda)+A\}, \\ g((\nabla_{e_1}Q)e_2,e_2) &= \frac{e_1(r)}{2} + e_1(\lambda^2) - e_1(2a\lambda) + \frac{\xi(\lambda)}{\lambda}\{e_2(\lambda)+A\} - 2B(\lambda+1), \\ g((\nabla_{e_2}Q)e_1,e_1) &= \frac{e_2(r)}{2} + e_2(\lambda^2) + e_2(2a\lambda) + \frac{\xi(\lambda)}{\lambda}\{e_2(\lambda)+B\} - 2A(\lambda-1), \\ g((\nabla_{e_2}Q)e_2,e_2) &= \frac{e_2(r)}{2} + e_2(\lambda^2) - e_2(2a\lambda) - \frac{\xi(\lambda)}{\lambda}\{e_1(\lambda)+B\}.\end{aligned}$$

Then, using 3.2, M is weakly locally ϕ-symmetric if and ony if

$$\left\{\begin{array}{l}\frac{1}{2}e_1(r) + 2e_1(\lambda^2) = 2B(\lambda+1) \\ \frac{1}{2}e_2(r) + 2e_2(\lambda^2) = 2A(\lambda-1).\end{array}\right. \tag{3.3}$$

Then, by (2.1), (3.3) is equivalent to (3.1).

Corollary 3.2 *If $Q\xi$ is parallel to ξ, then M is weakly locally ϕ-symmetric if and only if it has constant ϕ-sectional curvature.*

4 Main results

Theorem 4.1 *Let M be a 3-dimensional contact metric manifold. Then M is weakly locally ϕ-symmetric with constant scalar curvature and vertical Ricci curvature if, and only if, either M is strongly locally ϕ-symmetric or it is locally isometric to a Lie group G_w.*

Proof. The necessary condition is trivial. We show the sufficient condition. In the Sasakian case, the two definition are equivalent, so we have to consider only the non Sasakian case. Then the set $U_1 \neq \emptyset$ where we suppose $\lambda < 0$. Since $r = const.$, from Lemma 3.1 we have

$$\begin{cases} 2\lambda e_1(\lambda) = B(\lambda+1) \\ 2\lambda e_2(\lambda) = A(\lambda-1), \end{cases} \tag{4.1}$$

and hence

$$2\lambda[e_1, e_2](\lambda^2) = 2\lambda(\lambda-1)e_1(A) - 2\lambda(\lambda+1)e_2(B) + 2AB. \tag{4.2}$$

Moreover, by Lemma 2.1 and (4.1), we have

$$2\lambda[e_1, e_2](\lambda^2) = -2AB + 8\lambda^2\xi(\lambda), \tag{4.3}$$

and, by (4.1) and $0 = \xi(r) = g((\nabla_\xi Q)\xi, \xi) + g((\nabla_{e_1} Q)e_1, \xi) + g((\nabla_{e_2} Q)e_2, \xi)$, we get

$$\begin{aligned} 8\lambda^2\xi(\lambda) &= 4\lambda e_1(A) + 4\lambda e_2(B) - 2(Be_2(\lambda) + Ae_1(\lambda)) - 4AB \\ &= 4\lambda e_1(A) + 4\lambda e_2(B) - 6AB. \end{aligned} \tag{4.4}$$

From (4.2) and (4.3) we have

$$4\lambda^2\xi(\lambda) = \lambda(\lambda-1)e_1(A) - \lambda(\lambda+1)e_2(B) + 2AB \tag{4.5}$$

which, using (4.4), gives

$$\lambda(\lambda-3)e_1(A) - \lambda(\lambda+3)e_2(B) + 5AB = 0. \tag{4.6}$$

Since $\rho(\xi, e_i) = const.$, from (4.6) and (4.5) we get $AB = 0$ *and* $\xi(\lambda) = 0$. Now, we consider separately the cases $A = B = 0$; $A \neq 0$, $B = 0$; $A = 0$, $B \neq 0$.

Case A=B=0. In this case, (4.1) gives $e_1(\lambda) = e_2(\lambda) = 0$ and hence, since $\xi(\lambda) = 0$, we have λ constant. Now, using the formula

$$e_i(r) = g((\nabla_\xi Q)\xi, e_i) + g((\nabla_{e_1} Q)e_1, e_i) + g((\nabla_{e_2} Q)e_2, e_i), \tag{4.7}$$

for i=1,2, and (2.4)-(2.6), since r and λ are constant, we get $e_1(a) = e_2(a) = 0$. Moreover, $\xi(a) = [e_1, e_2](a) = 0$. So, also a is constant. Then, applying Theorem 3.1 of [8] and theorem 5.1 of [4], we get that M is strongly locally locally ϕ-symmetric.

Case A=0, B≠ 0. From (4.1) we have $e_2(\lambda) = 0$ *and* $e_1(\lambda) = \dfrac{B}{2\lambda}(\lambda+1)$. But, see lamma 2.1, $(a+\lambda-1)e_1(\lambda) = [\xi, e_2](\lambda) = 0$. Therefore either $a = 1-\lambda$ or $e_1(\lambda) = 0$. Assume $a = 1-\lambda$. Then lemma 2.1 gives

$$
(4.8)\qquad
\begin{aligned}
&\nabla_\xi e_1 = (\lambda-1)e_2\,, && \nabla_\xi e_2 = (1-\lambda)e_1\,,\\
&\nabla_{e_1}\xi = -(\lambda+1)e_2\,, && \nabla_{e_2}\xi = -(\lambda-1)e_1\,,\\
&\nabla_{e_1}e_1 = 0, && \nabla_{e_2}e_2 = \frac{B(1+3\lambda)}{4\lambda^2}e_1\,,\\
&\nabla_{e_1}e_2 = (\lambda+1)\xi, && \nabla_{e_2}e_1 = -\frac{B(1+3\lambda)}{4\lambda^2}e_2 + (\lambda-1)\xi,
\end{aligned}
$$

and

$$
(4.9)\qquad [e_1,e_2] = \frac{B(1+3\lambda)}{4\lambda^2}e_2 + 2\xi.
$$

Consequently using (4.8),

$$
\begin{aligned}
R(e_1,e_2)e_1 &= -\nabla_{e_1}\nabla_{e_2}e_1 + \nabla_{e_2}\nabla_{e_1}e_1 + \nabla_{[e_1e_2]}e_1 =\\
&= \left\{-\frac{B^2}{16\lambda^4}(15\lambda^2+16\lambda+5) + (\lambda-1)(\lambda+3)\right\}e_2 + B\xi.
\end{aligned}
$$

On the other hand $2g(R(e_1,e_2)e_1,e_2) = 2H = r - 4(1-\lambda^2)$, therefore we obtain

$$
8\lambda^4\{r + 2(\lambda-1)^2\} + B^2(15\lambda^2+16\lambda+5) = 0.
$$

This equation, since B, r are constant, implies $\lambda = \mathit{const}.(\neq 0)$ and hence, by (4.1), $\lambda = -1$ and $a = 2$. Assuming $e_1(\lambda) = 0$, we have $\lambda = \mathit{const}. = -1$. In this case

$$
Q\xi = Be_2, \qquad Qe_1 = \left(\frac{r}{2} - 2a\right)e_1, \qquad Qe_2 = B\xi + \left(\frac{r}{2} + 2a\right)e_2,
$$

and hence applying formula (4.7), for i=1,2, we get

$$
(4.10)\qquad e_2(a) = 0, \qquad 2e_1(a) = (2-a)B.
$$

Moreover $[e_1,e_2] = -\frac{B}{2}e_2 + 2\xi$, so (4.10) gives

$$
-\frac{B}{2}e_2(a) + 2\xi(a) = [e_1,e_2](a) = e_1e_2(a) - e_2e_1(a) = -e_2\left\{\frac{2-a}{2}B\right\} = 0,
$$

from which we have $\xi(a) = 0$. Then by (2.2)

$$
(a-2)e_1(a) = [\xi,e_2](a) = \xi e_2(a) - e_1\xi(a) = 0.
$$

gives $a = \mathit{const}.$, and by (4.10), $a = 2$. Thus, we have

$$
[e_1,e_2] = -\frac{B}{2}e_2 + 2\xi,\ \ [\xi,e_2] = 0,\ \ [e_1,\xi] = 2e_2.
$$

So, M is locally isometric to a Lie group G_w (see [2],[8]).

Case A≠0, B=0. We show that this case can not occur. $A \neq 0$, $B = 0$ and $\lambda < 0$, by (4.1), imply

$$
e_1(\lambda) = 0,\ \ e_2(\lambda) = \frac{A(\lambda-1)}{2\lambda} \neq 0.
$$

Then computing $R(e_1,e_2)e_1$ as in the before case, we get $\lambda = \mathit{const}.$ which contradicts $e_2(\lambda) \neq 0$.

Corollary 4.2 *A 3-dimensional homogeneous contact metric manifold is weakly, but not strongly, locally ϕ-symmetric iff it is locally isometric to a Lie group G_w.*

Remark 4.1 (i) If in the proof of theorem 4.1 we assume $\lambda > 0$, then can not occur the case $A = 0, B \neq 0$. (ii) The non unimodular Lie group G_w is associated to the Lie algebra

$$[e_1, e_2] = \alpha e_2 + 2\xi, \ \ [e_1, \xi] = 2e_2, \ \ [\xi, e_2] = 0,$$

hence it is determined by the Milnor's isomorphism invariant D [6] given by: $D = -\frac{8\gamma}{\alpha^2} = -\frac{16}{\alpha^2} < 0$. In our case $\alpha = -\frac{B}{2}$. On the other hand, computing the Webster scalar curvature of G_w, using (2.1), we find $W = -\frac{\alpha^2}{4} - \frac{1}{2} < 0$. So D, and hence G_w, is determined by the Webster scalar curvature W.

Theorem 4.3 *There exists a weakly locally ϕ-symmetric space with constant scalar curvature and non constant vertical Ricci curvature. In particular such space is neither locally homogeneous nor strongly locally ϕ-symmetric.*

Proof. Consider the 3-dimensional manifold $M_1 = \{x \in R^3 : x_1 \neq 0\}$. In the sequel we denote by $\partial_i, i = 1, 2, 3$, the partial derivative $\frac{\partial}{\partial x_i}$. Let η the 1-form defined by

$$\eta = x_1 x_2 dx_1 + dx_3.$$

η is a contact form because

$$\eta \wedge d\eta = -x_1 dx_1 \wedge dx_2 \wedge dx_3.$$

The characteristic vector field of (M_1, η) is $\xi = \partial_3$. In fact

$$\eta(\partial_3) = 1, \ \ (d\eta)(\partial_3, \cdot) = x_1 dx_2 \wedge dx_1)(\partial_3, \cdot) = 0.$$

It is not difficult to see that the contact distribution is generated by the global vector fields

$$e_1 = -\frac{2}{x_1}\partial_2, \ \ e_2 = \partial_1 - \frac{4x_3}{x_1}\partial_2 - x_1 x_2 \partial_3.$$

The vector fields e_1, e_2, ξ satisfy

$$[\xi, e_1] = 0, \ \ [\xi, e_2] = 2e_1, \ \ [e_1, e_2] = 2\xi + \frac{1}{x_1}e_1. \tag{4.11}$$

Now, consider the Riemannian metric g defined by

$$g(\xi, e_1) = g(\xi, e_2) = g(e_1, e_2) = 0, \ \ g(\xi, \xi) = g(e_1, e_1) = g(e_2, e_2) = 1,$$

and the tensor ϕ defined by

$$\phi(\xi) = 0, \ \ \phi(e_1) = e_2, \ \ \phi(e_2) = -e_1.$$

The tensors η, g and ϕ satisfy

$$(d\eta)(\xi, e_i) = 0 = g(\xi, \phi e_i), \;\; (d\eta)(e_i, e_i) = 0 = g(e_i, \phi e_i),$$

$$(d\eta)(e_1, e_2) = \frac{1}{2}\{e_1\eta(e_2) - e_2\eta(e_1) - \eta([e_1, e_2])\} = -1 = g(e_1, \phi e_2).$$

Then (η, g, ϕ) is a contact metric structure on M_1. Moreover the tensor h satisfies

$$h(e_1) = \frac{1}{2}\{[\xi, e_2] - \phi[\xi, e_1]\} = e_1, \;\; h(e_2) = h\phi e_1 = -\phi h(e_1) = -e_2.$$

Thus $\lambda = +1$ and $(e_1, e_2, e_3 = \xi)$ is an orthonormal ϕ-basis of eigenvector for h. Since $(e_1, e_2, e_3 = \xi)$ is an orthonormal basis, the Levi-Civita connection is defined by the formula

$$\nabla_{e_i} e_j = \frac{1}{2}\sum_k -\{g(e_i, [e_j, e_k]) + g(e_j, [e_k, e_i]) + g(e_k, [e_i, e_j])\}e_k.$$

Then, by (4.11), we get

$$\begin{array}{lll} \nabla_\xi \xi = 0, & \nabla_{e_1}\xi = -2e_2, & \nabla_{e_2}\xi = 0, \\ \nabla_\xi e_1 = -2e_2, & \nabla_{e_1} e_1 = -\frac{1}{x_1}e_2, & \nabla_{e_2} e_1 = 0, \\ \nabla_\xi e_2 = 2e_1, & \nabla_{e_1} e_2 = \frac{1}{x_1}e_1 + 2\xi, & \nabla_{e_2} e_2 = 0. \end{array} \tag{4.12}$$

Using (4.12) we obtain

$$R(e_1, e_2)e_1 = -4e_2, \qquad R(\xi, e_1)e_2 = 0, \qquad R(\xi, e_2)e_1 = -\frac{2}{x_1}e_2,$$

from which $H = -4$, $B = 0$, and $A = -\frac{2}{x_1}$. Moreover $\lambda = 1$, then

$$\begin{cases} e_1(H) = 0 = 2B(\lambda + 1) \\ e_2(H) = 0 = 2A(\lambda - 1), \end{cases}$$

and hence, using Lemma 3.1, (M_1, η, g) is a weakly locally ϕ-symmetric. Of course such space is neither homogeneous nor strongly locally ϕ-symmetric because A is not a constant function. This conclude the proof. **Remark 4.2** The main result of [5] says that every compact and orientable three-manifold has a contact metric structure whose Webster scalar curvature W is either a constant ≤ 0 or it is strictly positive everywhere. Theorem 4.2 gives an example of non-compact contact metric three-manifold with $W = const. = -\frac{1}{2} < 0$ with the geometric property that the basis $\{e_1, e_2, \xi\}$ is parallel along the integral curves of the vector field e_2.

5 The unit tangent sphere bundle of a surface

Let (M, G) be a 2-dimensional Riemannian manifold. Consider on M isothermal local coordinate (x_1, x_2) on M. Then the Riemannian metric G is given by

$$G = e^{2f}((dx_1)^2 + (dx_2)^2)$$

where f is a C^∞ function on M. Let TM be the tangent sphere bundle. The immersion of the unit tangent sphere bundle $T^1M = \{z = (p, v) \in TM : e^{2f}((v_1)^2 + (v_2)^2) = 1\}$ into TM is defined by

$$(y_1, y_2, \theta) \longrightarrow (x_1, x_2, v_1, v_2) = (y_1, y_2, e^{-f} cos\theta, e^{-f} sin\theta).$$

Let(η, g, ξ, ϕ) the standard contact metric structure on T^1M. Then $\xi = 2\xi'$ where ξ' is geodesic flow given by

$$\xi' = v_1 \frac{\partial}{\partial y_1} + v_2 \frac{\partial}{\partial y_2} + (v_1 f_2 - v_2 f_1) \frac{\partial}{\partial \theta}.$$

where $f_1 = \frac{\partial f}{\partial x_1}$ and $f_2 = \frac{\partial f}{\partial x_2}$. Moreover setting

$$e_2 = 2\frac{\partial}{\partial \theta} = 2\left\{-v_2 \frac{\partial}{\partial v_1} + v_1 \frac{\partial}{\partial v_2}\right\},$$

and

$$e_1 = 2U = 2\left\{-v_2 \frac{\partial}{\partial y_1} + v_1 \frac{\partial}{\partial y_2} - (v_2 f_2 + v_1 f_1) \frac{\partial}{\partial \theta}\right\}$$

then $(\xi, e_1, e_2 = \phi e_1)$ is a local orthonormal ϕ-basis of T^1M. Denote by ∇ the Levi-Civita connection of (T^1M, g). The Gaussian curvature k of (M, G) considered as a function on T^1M is defined by $k(p, v) = k(p)$. Using the Christoffel symbols of (M, G), we find

$$\begin{array}{ll} \nabla_\xi \xi = \nabla_{e_1} e_1 = \nabla_{e_2} e_2 = 0, & \nabla_{e_1} \xi = -\nabla_\xi e_1 = -k e_2, \\ \nabla_{e_2} e_1 = (k-2)\xi,\ \nabla_{e_2} \xi = (2-k) e_1, & \nabla_\xi e_2 = -k e_1,\ \nabla_{e_1} e_2 = k\xi. \end{array}$$

Cosequently, we get

$$\begin{aligned} &R(e_1, e_2)e_1 = -e_1(k)\xi + k^2 e_2, \\ &R(\xi, e_1)e_2 = -\xi(k)\xi - e_1(k)e_1, \\ &R(\xi, e_2)e_1 = -\xi(k)\xi, \\ &he_1 = \frac{1}{2}\{[\xi, e_2] - \phi[\xi, e_1]\} = (k-1)e_1, \quad he_2 = -\phi h e_1 = (1-k)e_2, \end{aligned}$$

from which

$$\left\{ \begin{array}{l} H = g(R(e_1, e_2)e_1, e_2) = k^2 \\ B = \rho(\xi, e_2) = g(R(\xi, e_1)e_2, e_1) = -e_1(k) \\ A = \rho(\xi, e_1) = g(R(\xi, e_2)e_1, e_2) = 0 \\ \lambda = k - 1. \end{array} \right.$$

Then $e_2(H) = e_2(k^2) = 0 = 2A(\lambda - 1)$ and

$$e_1(H) = 2B(\lambda + 1) \Leftrightarrow e_1(k^2) = 0.$$

Moreover $2\xi(k^2) = [e_1, e_2](k^2)$. So, by lemma 3.1, T^1M is weakly locally ϕ-symmetric if and only if (M, G) has constant curvature. Hence we get the following theorem.

Theorem 5.1 *The unit tangent sphere bundle T^1M equipped with the standard contact metric structure is weakly locally ϕ-symmetric if and only if the base manifold has constant Gaussian curvature.*

Remark 5.1. Let $M(c)$ be a 2-dimensional Riemannian manifold of constant Gaussian curvature c. Then the universal covering of $T^1(M)$ is a simply connected Lie group equipped with a left invariant contact metric structure, more precisely we get : $SU(2)\ if\ c > 0$, $\tilde{SL}(2,R)\ if\ c < 0$, $\tilde{E}(2)\ if\ c = 0$, the universal covering of the isometry groups of S^2, H^2 and E^2, respectively.

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