The Concept of Invariant Geometry of Second Order

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Abstract

The study of higher order Lagrange spaces founded on the notion of bundle of velocities of order k has been given by Radu Miron and Gheorghe Atanasiu in [2]. The bundle of accelerations corresponds in this study to k=2. In this paper we shall introduce the notion of general invariant frames, the reprezentation of geometrical objects, we shall construct an N-linear invariant connection, the movement equations of the invariant frame and the transformations laws in changing invariant frames. Further we study the invariant covariant derivatives, the torsions and curvatures tensors of a N-linear connection and the structure equations.

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1 General Invariant Frames

Let us consider the bundle $E = Osc^2 M$, a nonlinear connection N with the coefficients $(\underset{(1)}{N^i{}_j}, \underset{(2)}{N^i{}_j})$ and the duals $(\underset{(1)}{M^i{}_j}, \underset{(2)}{M^i{}_j})$ and the direct decomposition

$$T_u(\,Osc^2 M\,) = N_0(u) \oplus N_1(u) \oplus V_2(u) \quad \forall u \in E \tag{1.1}$$

Let be

$$\begin{aligned} e_\alpha^{(0)} &: u \in E \to e_\alpha^{(0)}(u) \subset N_0(u) \\ e_\alpha^{(1)} &: u \in E \to e_\alpha^{(1)}(u) \subset N_1(u) \\ e_\alpha^{(2)} &: u \in E \to e_\alpha^{(2)}(u) \subset V_2(u) \end{aligned} \tag{1.2}$$

h, v_1, v_2 frames over E. We have $e_\alpha^{(A)}(u) = e_\alpha^{(A)i} \left.\dfrac{\delta}{\delta y_i^{(A)}}\right|_u$ where $y_i^{(0)} = x_i$. Denote by $f^{(A)\alpha}$ the duals of $e_\alpha^{(A)}$, $A = 0, 1, 2$

The duality conditions are

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$$<e^{(A)i}_{\alpha}, f^{(B)\alpha}_{j}> = \delta^i_j \delta^A_B \quad (A, B = 0, 1, 2) \tag{1.3}$$

We call $\mathbf{R} = (e^{(0)i}_{\alpha}, e^{(1)i}_{\alpha}, e^{(2)i}_{\alpha})$ general invariant frames over E and $\mathbf{R}^* = (f^{(0)\alpha}_{i}, f^{(1)\alpha}_{i}, f^{(2)\alpha}_{i})$ his dual. The representation of the adapted basis of (1.1) in $\mathbf{R}$ is given by

$$\frac{\delta}{\delta x^i} = f^{(0)\alpha}_{i} \frac{\delta}{\delta s^{(0)\alpha}} \quad \frac{\delta}{\delta y^{(A)i}} = f^{(A)\alpha}_{i} \frac{\delta}{\delta s^{(A)\alpha}} \quad A = 1, 2 \tag{1.4}$$

and for the cobasis

$$\delta x^i = e^{(0)i}_{\alpha} \delta s^{(0)\alpha} \quad ; \delta y^{(A)i} = e^{(A)i}_{\alpha} \delta s^{(A)\alpha} \quad A = 1, 2; \tag{1.5}$$

having the relations

$$\left\langle \frac{\delta}{\delta s^{(A)\alpha}}, \delta s^{(B)\beta} \right\rangle = \delta^\beta_\alpha \delta^B_A \qquad (A, B = 0, 1, 2) \tag{1.6}$$

This representation lead us to an invariant frames transformation group $\mathbf{R} \to \bar{\mathbf{R}}$ with the analitycal expressions

$$\overline{e}^{(A)i}_{\alpha} = \overset{A}{C}{}^{\beta}_{\alpha} \, (x, y^{(1)}, y^{(2)}), e^{(A)i}_{\beta} \quad ; \quad f^{(B)\alpha}_{j} = \overline{\overset{B}{C}}{}^{\alpha}_{\beta} \, \overline{f}^{(B)\beta}_{j} \tag{1.7}$$

isomorphic with the multiplicative nonsingular matrix group

$$\begin{pmatrix} \overset{0}{C}{}^{\alpha}_{\beta} & 0 & 0 \\ 0 & \overset{1}{C}{}^{\alpha}_{\beta} & 0 \\ 0 & 0 & \overset{2}{C}{}^{\alpha}_{\beta} \end{pmatrix}$$

Observation 1.1 The frames $\mathbf{R}$ and $\mathbf{R}^*$ are non-holonomic and thru them we can introduce the non-holonomic sistem of coordinates $(s^{(0)\alpha}, s^{(1)\alpha}, s^{(2)\alpha})$ in Vranceanu sense. The definition of the Lie bracket leads us to the introduction of the non-holonomy coefficients of Vranceanu

$$\left[\frac{\delta}{\delta s^{(A)\alpha}}, \frac{\delta}{\delta s^{((B)\beta}} \right] = \underset{(AB)}{\overset{C}{W}}{}^{\gamma}_{\alpha\beta} \frac{\delta}{\delta s^{(C)\gamma}} \tag{1.8}$$

$(A, B, C = 0, 1, 2; \quad A \leq B; \quad$ summation on $C)$.

The calculus of the coefficients $\underset{(AB)}{\overset{C}{W}}{}^{\gamma}_{\alpha\beta}$ shows us that $\underset{(AA)}{\overset{A}{W}}{}^{\gamma}_{\alpha\beta}$ act like a nonlinear connection having the expressions

$$\underset{(AA)}{\overset{A}{W}}{}^{\gamma}_{\alpha\beta} = f^{(A)\alpha}_{k} \left(\frac{\delta e^{(A)k}_{\alpha}}{\delta s^{(A)\beta}} - \frac{\delta e^{(A)k}_{\beta}}{\delta s^{(A)\alpha}} \right),$$

the coefficients $\underset{(BB)}{\overset{A}{W}}{}^{\gamma}_{\alpha\beta}$ act like tensors (torsion tensors). The others are non-holonomic objects like

$$\underset{(AB)}{\overset{A}{W}}{}^{\gamma}_{\alpha\beta} = -f^{(A)\alpha}_{\ \ k}\frac{\delta e^{(A)k}_{\alpha}}{\delta s^{(B)\beta}} \text{ and } \underset{(BA)}{\overset{A}{W}}{}^{\gamma}_{\alpha\beta} = -f^{(B)\alpha}_{\ \ k}\frac{\delta e^{(B)k}_{\alpha}}{\delta s^{(A)\beta}} - \underset{(AA)}{B}{}^{p}_{kl}\ e^{(B)k}_{\ \alpha} e^{(A)l}_{\ \beta} f^{(A)\gamma}_{\ \ p}$$

where $\underset{(AA)}{B}{}^{p}_{kl}$ are torsion tensors corresponding to Berwald connection.

2 The reprezentation of geometric objects in non-holonomic frames

Let $X \in \chi(E)$ be a vector field, consider the adapted basis to the decomposition (1.1) and the frames $\mathbf{R}$ and $\mathbf{R}^*$. Then for X we have the following reprezentations:

$$X = X^{(0)i}\frac{\delta}{\delta x^i} + X^{(1)i}\frac{\delta}{\delta y^{(1)i}} + X^{(2)i}\frac{\delta}{\delta y^{(2)i}} \tag{2.1}$$

and

$$X = X^{(0)\alpha}\frac{\delta}{\delta s^{\alpha}} + X^{(1)\alpha}\frac{\delta}{\delta s^{(1)\alpha}} + X^{(2)\alpha}\frac{\delta}{\delta s^{(2)\alpha}}.$$

So, the following relations hold $X^{(A)i} = e^{(A)i}_{\alpha}X^{(A)\alpha}$ and $X^{(A)\alpha} = f^{(A)\alpha}_{i}X^{(A)i}$ $A = 0, 1, 2$. For any local coordinates transformations we have

$$\bar{X}^{(A)\alpha} = \bar{f}^{(A)\alpha}_{\ \ i}\bar{X}^{(A)i} = f^{(A)\alpha}_{\ \ k}\delta^{k}_{i}X^{(A)i} = X^{(A)\alpha}. \tag{2.2}$$

Proposition 2.1 *The non-holonomic components of the vector field X are the invariant components of the h, v_1 and v_2 projections of the vector field X on the three distributions of (1.1).*

For a 1-form field $\omega \in \chi^*$ denoting $\omega^{(0)} = \omega \cdot h$, $\omega^{(1)} = \omega \cdot v_1$ and $\omega^{(2)} = \omega \cdot v_2$ we have in the same fashion

$$\begin{aligned} \omega &= \omega^{(0)}_{i}\delta x^i + \omega^{(1)}_{i}\delta y^{(1)i} + \omega^{(2)}_{i} y^{(2)i} \\ \omega &= \omega^{(0)}_{\alpha}\delta s^{(0)\alpha} + \omega^{(1)}_{\alpha}\delta s^{(1)\alpha} + \omega^{(2)}_{\alpha} s^{(2)\alpha} \end{aligned} \tag{2.3}$$

So $\omega^{(A)}_{\alpha} = e^{(A)i}_{\alpha}\omega^{(A)}_{i}$ and conversely $\omega^{(A)}_{i} = f^{(A)\alpha}_{i}\omega^{(A)}_{\alpha}$ and we obtain that $\omega^{(A)}_{\alpha}$ are invariant at any local change of coordinates. $A = 0, 1, 2$ In this manner we prove that the non-holonomic components of any tensor field are invariant at local change of coordinates.

Definition 2.1 A liniar connection D is called N-linear connection if
i) D preserves by parallelism the horizontal distribution N_0
ii) the 2-tangent structure J is absolut parallel with respect to D, that is
$D_X J = 0 \quad \forall X \in \chi(E)$

If we consider a N-linear connection D with the coefficients

$$D\Gamma(N) = \left(L^{i}{}_{jk}, C^{i}_{jk}, C'^{i}_{j}k\right)$$

then

Proposition 2.2 *In the considered non-holonomic frames the components of the N-linear connection D are given by*

$$\text{(2.4)} \qquad \overset{0A}{L}{}^{\gamma}_{\beta\alpha} = f^{(A)\gamma}_{m}\left(\frac{\delta e^{(A)m}_{\ \ \beta}}{\delta s^{(0)\alpha}} + e^{(0)i}_{\ \ \alpha} e^{(A)j}_{\ \ \beta} L^{m}_{ij}\right), \quad A = \{0,1,2\}$$

$$\overset{BA}{C}{}^{\gamma}_{\beta\alpha} = f^{(A)\gamma}_{m}\left(\frac{\delta e^{(A)m}_{\ \ \beta}}{\delta s^{(B)\alpha}} + e^{(B)i}_{\ \ \alpha} e^{(A)j}_{\ \ \beta} \underset{(B)}{C^{m}_{ij}}\right), \quad A = \{0,1,2\};\ B = \{1,2\}.$$

As we can see in the non-holonomic frame $\mathbf{R}$ we have nine coefficients instead of three essentials.

Proposition 2.3 *Let us consider a N-linear connection D having in the frames* $\mathbf{R}$ *and* $\mathbf{R}^*$ *the coefficient given by proposition (2.2). The movement equations of the frames* $\mathbf{R}$ *and* $\mathbf{R}^*$ *are*

$$\text{(2.5)} \qquad e^{(A)i}_{\ \ \alpha|m} = \overset{0A}{L}{}^{\gamma}_{\beta\alpha}\, e^{(A)i}_{\ \ \gamma} f^{(0)\beta}_{\ \ m}$$

$$e^{(A)i}_{\ \ \alpha} \overset{(B)}{|} = \overset{AB}{C}{}^{\gamma}_{\beta\alpha}\, e^{(A)i}_{\ \ \gamma} f^{(B)\beta}_{\ \ m} \qquad A = \{0,1,2\}, \{B = 1,2\}$$

$$\text{(2.6)} \qquad f^{(A)\gamma}_{\ \ i|m} = -\,\overset{0A}{L}{}^{\gamma}_{\beta\alpha}\, f^{(A)\alpha}_{\ \ i} f^{(0)\beta}_{\ \ m}$$

$$f^{(A)\alpha}_{\ \ i} \overset{(B)}{|} = -\,\overset{AB}{C}{}^{\gamma}_{\beta\alpha}\, f^{(A)\alpha}_{\ \ i} f^{(B)\beta}_{\ \ m} \qquad A = \{0,1,2\}, B = \{1,2\}$$

Now we can introduce the notion of $h,\ v_1,\ v_2$ invariant covariant derivation operators denoted by $_{)}$ and $\overset{(B)}{)}$, $B = 1, 2$. If we consider $X \in \chi(E)$ a vector field and $\omega \in \chi^*(E)$ a 1-form field then

Definition 2.2 *The* $h,\ v_1,\ v_2$ *invariant covariant derivatives of the invariant projections of the vector field* X *and the 1-form* ω *are given by:*

$$\text{(2.7)} \qquad X^{(A)\alpha}_{\ \ \ )\beta} = \frac{\delta X^{(A)\alpha}}{\delta s^{(0)\beta}} + \overset{0A}{L}{}^{\alpha}_{\varphi\beta}\, X^{(A)\varphi}$$

$$X^{(A)\alpha} \overset{(B)}{)}_{\beta} = \frac{\delta X^{(A)\alpha}}{\delta s^{(B)\beta}} + \overset{BA}{C}{}^{\alpha}_{\varphi\beta}\, X^{(A)\varphi}$$

(2.8)
$$\omega^{(A)}_{\alpha}{}_{)\beta} = \frac{\delta \omega^{(A)}_{\alpha}}{\delta s^{(0)\beta}} - \overset{0A}{L}{}^{\varphi}_{\alpha\beta}\,\omega^{(A)}_{\varphi}$$
$$\omega^{(A)}_{\alpha}\overset{(B)}{)}_{\beta} = \frac{\delta \omega^{(A)}_{\alpha}}{\delta s^{(B)\beta}} - \overset{BA}{C}{}^{\varphi}_{\alpha\beta}\,\omega^{(A)}_{\varphi}$$

By direct calculation we prove

Theorem 2.1 *The h, v_1 and v_2 invariant covariant derivatives of the components $X^{(A)\alpha}$ and $\omega^{(A)}_{\alpha}$ are the invariant components of the covariant derivatives*

What we have introduced for vector and 1-forms fields remains true also for tensor fields for example the h invariant covariant derivative of the invariant components of the tensor field T^{α}_{β} is

$$T^{\alpha}_{\beta}{}_{)\gamma} = \frac{\delta T^{\alpha}_{\beta}}{\delta^{(0)\gamma}} + \overset{(00)}{L}{}^{\alpha}_{\varphi\gamma}\,T^{\varphi}_{\beta} - \overset{(00)}{L}{}^{\varphi}_{\beta\gamma}\,T^{\alpha}_{\varphi}.$$

3. Torsion and Curvature d-tensor Fields

The torsion tensor of the N-linear connection D on E

(3.1)
$$\mathcal{T}(\mathcal{X},\mathcal{Y}) = \mathcal{D}_{\mathcal{X}}\mathcal{Y} - \mathcal{D}_{\mathcal{Y}}\mathcal{X} - [\mathcal{X},\mathcal{Y}], \qquad \forall\, \mathcal{X},\mathcal{Y} \in \chi(\mathcal{E})$$

in the invariant frame **R** , has a number of horizontal and vertical components corresponding to D^h , D^{v_1} , D^{v_2}

Theorem 3.1. *The torsion tensor of a N-linear connection D in the invariant frame* **R** *is characterized by the d-tensor fields with local components*

(3.2)
$$\begin{cases} \underset{(0)}{T}{}^{\gamma}{}_{\beta\alpha} = \overset{(00)}{L}{}^{\gamma}{}_{\beta\alpha} - \overset{(00)}{L}{}^{\gamma}{}_{\alpha\beta} - \underset{(00)}{\overset{(0)}{W}}{}^{\gamma}{}_{\beta\alpha} \\ \underset{(0A)}{R}{}^{\gamma}{}_{\beta\alpha} = \underset{(00)}{\overset{(A)}{W}}{}^{\gamma}{}_{\beta\alpha} \end{cases}$$

(3.3)
$$\begin{cases} \underset{(1)}{K}{}^{\gamma}{}_{\beta\alpha} = -\underset{(1)}{\overset{(10)}{C}}{}^{\gamma}{}_{\beta\alpha} - \underset{(01)}{\overset{(0)}{W}}{}^{\gamma}{}_{\beta\alpha} \\ \underset{(11)}{P}{}^{\gamma}{}_{\beta\alpha} = \overset{(01)}{L}{}^{\gamma}{}_{\beta\alpha} + \underset{(01)}{\overset{(1)}{W}}{}^{\gamma}{}_{\beta\alpha} \\ \underset{(12)}{P}{}^{\gamma}{}_{\beta\alpha} = \underset{(01)}{\overset{(2)}{W}}{}^{\gamma}{}_{\beta\alpha} \end{cases}$$

$$
(3.4)\quad \begin{cases}
\underset{(2)}{K}{}^{\gamma}{}_{\beta\alpha} = -\overset{(20)}{\underset{(2)}{C}}{}^{\gamma}{}_{\beta\alpha} - \overset{(0)}{\underset{(02)}{W}}{}^{\gamma}{}_{\beta\alpha} \\
\underset{(21)}{P}{}^{\gamma}{}_{\beta\alpha} = \overset{(1)}{\underset{(02)}{W}}{}^{\gamma}{}_{\alpha\beta} - \overset{(1)}{\underset{(01)}{W}}{}^{\gamma}{}_{\beta\alpha} \\
\underset{(22)}{P}{}^{\gamma}{}_{\beta\alpha} = \overset{(02)}{L}{}^{\gamma}{}_{\beta\alpha} - \overset{(2)}{\underset{(02)}{W}}{}^{\gamma}{}_{\beta\alpha}
\end{cases}
$$

$$
(3.5)\quad \begin{cases}
\underset{(11)}{Q}{}^{\gamma}{}_{\beta\alpha} = \overset{(11)}{\underset{(1)}{C}}{}^{\gamma}{}_{\beta\alpha} - \overset{(11)}{\underset{(1)}{C}}{}^{\gamma}{}_{\alpha\beta} - \overset{(1)}{\underset{(11)}{W}}{}^{\gamma}{}_{\beta\alpha} \\
\overset{(2)}{\underset{(21)}{Q}}{}^{\gamma}{}_{\beta\alpha} = \overset{(2)}{\underset{(11)}{W}}{}^{\gamma}{}_{\beta\alpha}
\end{cases}
$$

$$
(3.6)\quad \begin{cases}
\underset{(12)}{Q}{}^{\gamma}{}_{\beta\alpha} = \overset{(21)}{\underset{(2)}{C}}{}^{\gamma}{}_{\beta\alpha} - \overset{(1)}{\underset{(12)}{W}}{}^{\gamma}{}_{\beta\alpha} \\
\underset{(22)}{Q}{}^{\gamma}{}_{\beta\alpha} = -\overset{(1)}{\underset{(12)}{C}}{}^{\gamma}{}_{\alpha\beta} - \overset{(2)}{\underset{(12)}{W}}{}^{\gamma}{}_{\beta\alpha}
\end{cases}
$$

$$
(3.7)\quad \begin{cases}
\underset{(2)}{S}{}^{\gamma}{}_{\beta\alpha} = \overset{(22)}{\underset{(2)}{C}}{}^{\gamma}{}_{\beta\alpha} - \overset{(22)}{\underset{(2)}{C}}{}^{\gamma}{}_{\alpha\beta} - \overset{(2)}{\underset{(22)}{W}}{}^{\gamma}{}_{\beta\alpha}.
\end{cases}
$$

Theorem 3.2. *The components given by Theorem 2.1 are the invariant components of the d-tensor fields of torsion of the N-linear connection D*

The curvature tensor field $\mathcal{R}$ of the N-linear connection D on $Osc^2(M)$ has the expression

$$
(3.8)\quad \mathcal{R}(\mathcal{X},\mathcal{Y}) = [\,\mathcal{D}_{\mathcal{X}}\,,\,\mathcal{D}_{\mathcal{Y}}\,]\,\mathcal{Z} - \mathcal{D}_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z}.
$$

Theorem 3.3. *The curvature tensor field $\mathcal{R}$ of a N-linear connection D in the invariant frame* **R** *is characterized by the following d-tensor fields on $Osc^2(M)$:*

$$
(3.9)\quad \begin{aligned}
R_{\gamma}{}^{\varphi}{}_{\beta\alpha} = {}& \frac{\delta \overset{(00)}{L}{}^{\varphi}_{\gamma\beta}}{\delta s^{(0)\alpha}} - \frac{\delta \overset{(00)}{L}{}^{\varphi}_{\gamma\alpha}}{\delta s^{(0)\beta}} + \overset{(00)}{L}{}^{\eta}_{\gamma\beta}\overset{(00)}{L}{}^{\varphi}_{\eta\alpha} - \overset{(00)}{L}{}^{\eta}_{\gamma\alpha}\overset{(00)}{L}{}^{\varphi}_{\eta\beta} - \\
& -\overset{(0)}{\underset{(00)}{W}}{}^{\psi}_{\beta\alpha}\ \overset{(00)}{L}{}^{\varphi}_{\gamma\psi} + \overset{(1)}{\underset{(00)}{W}}{}^{\psi}_{\beta\alpha}\ \overset{(10)}{\underset{(1)}{C}}{}^{\varphi}_{\gamma\psi} + \overset{(2)}{\underset{(00)}{W}}{}^{\psi}_{\beta\alpha}\ \overset{(20)}{\underset{(2)}{C}}{}^{\varphi}_{\gamma\psi}
\end{aligned}
$$

$$
\begin{aligned}
\underset{(1)}{P}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = {} & \frac{\delta \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\gamma\beta}}{\delta s^{(0)\alpha}} - \frac{\delta \overset{(00)}{L}{}^{\varphi}_{\gamma\alpha}}{\delta s^{(1)\beta}} + \underset{(1)}{\overset{(10)}{C}}{}^{\eta}_{\gamma\beta}\, \overset{(00)}{L}{}^{\varphi}_{\eta\alpha} - \overset{(00)}{L}{}^{\eta}_{\gamma\alpha}\, \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\eta\beta} - \\
& - \underset{(01)}{\overset{(0)}{W}}{}^{\psi}_{\beta\alpha}\, \overset{(00)}{L}{}^{\varphi}_{\gamma\psi} + \underset{(01)}{\overset{(1)}{W}}{}^{\psi}_{\beta\alpha}\, \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\gamma\psi} + \underset{(01)}{\overset{(2)}{W}}{}^{\psi}_{\beta\alpha}\, \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\gamma\psi}
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
\underset{(2)}{P}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = {} & \frac{\delta \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\gamma\beta}}{\delta s^{(0)\alpha}} - \frac{\delta \overset{(00)}{L}{}^{\varphi}_{\gamma\alpha}}{\delta s^{(2)\beta}} + \underset{(2)}{\overset{(20)}{C}}{}^{\eta}_{\gamma\beta}\, \overset{(00)}{L}{}^{\varphi}_{\eta\alpha} - \overset{(00)}{L}{}^{\eta}_{\gamma\alpha}\, \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\eta\beta} - \\
& - \underset{(02)}{\overset{(0)}{W}}{}^{\psi}_{\beta\alpha}\, \overset{(00)}{L}{}^{\varphi}_{\gamma\psi} + \underset{(02)}{\overset{(1)}{W}}{}^{\psi}_{\beta\alpha}\, \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\gamma\psi} + \underset{(02)}{\overset{(2)}{W}}{}^{\psi}_{\beta\alpha}\, \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\gamma\psi}
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
\underset{(11)}{S}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = {} & \frac{\delta \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\gamma\beta}}{\delta s^{(1)\alpha}} - \frac{\delta \underset{(1)}{\overset{(01)}{C}}{}^{\varphi}_{\gamma\alpha}}{\delta s^{(1)\beta}} + \underset{(1)}{\overset{(10)}{C}}{}^{\eta}_{\gamma\beta}\, \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\eta\alpha} - \underset{(1)}{\overset{(10)}{C}}{}^{\eta}_{\gamma\alpha}\, \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\eta\beta} - \\
& - \underset{(11)}{\overset{(1)}{W}}{}^{\psi}_{\beta\alpha}\, \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\gamma\psi} - \underset{(11)}{\overset{(2)}{W}}{}^{\psi}_{\beta\alpha}\, \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\gamma\psi}
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
\underset{(21)}{S}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = {} & \frac{\delta \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\gamma\beta}}{\delta s^{(1)\alpha}} - \frac{\delta \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\gamma\alpha}}{\delta s^{(2)\beta}} + \underset{(2)}{\overset{(20)}{C}}{}^{\eta}_{\gamma\beta}\, \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\eta\alpha} - \underset{(1)}{\overset{(10)}{C}}{}^{\eta}_{\gamma\alpha}\, \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\eta\beta} - \\
& - \underset{(12)}{\overset{(1)}{W}}{}^{\psi}_{\beta\alpha}\, \underset{(1)}{\overset{(10)}{C}}{}^{\varphi}_{\gamma\psi} - \underset{(12)}{\overset{(2)}{W}}{}^{\psi}_{\beta\alpha}\, \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\gamma\psi}
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
\underset{(22)}{S}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = {} & \frac{\delta \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\gamma\beta}}{\delta s^{(2)\alpha}} - \frac{\delta \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\gamma\alpha}}{\delta s^{(2)\beta}} + \underset{(2)}{\overset{(20)}{C}}{}^{\eta}_{\gamma\beta}\, \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\eta\alpha} - \underset{(2)}{\overset{(20)}{C}}{}^{\eta}_{\gamma\alpha}\, \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\eta\beta} - \\
& - \underset{(22)}{\overset{(2)}{W}}{}^{\psi}_{\beta\alpha}\, \underset{(2)}{\overset{(20)}{C}}{}^{\varphi}_{\gamma\psi}.
\end{aligned}
\tag{3.14}
$$

Theorem 3.4. *The components given by Theorem 2.3 are the invariant components of the d-tensor fields of curvature of the N-linear connection D*

Theorem 3.5. *In the frame* **R** *the essential components of the curvature tensor field* $\mathcal{R}$ *are those given by Theorem 2.3.*

4. Structure Equations

Let (C,c) $c : I \longrightarrow Osc^2 M$, $C = Im\ c$ be a smooth curve parametrically given on $Osc^2 M$ and let $\dot{c}$ be the tangent field.

Proposition 4.1. *The covariant differential of the vector field X in the frame* **R** *is:*

$$DX = \left\{ \left(\frac{\delta X^{(A)\alpha}}{\delta s^{(0)\beta}} \delta s^{(0)\beta} + \frac{\delta X^{(A)\alpha}}{\delta s^{(1)\beta}} \delta s^{(1)\beta} + \frac{\delta X^{(A)\alpha}}{\delta s^{(2)\beta}} \delta s^{(2)\beta} \right) + X^{(A)\gamma} \overset{(A)}{\omega}{}^{\alpha}_{\ \gamma} \right\} \frac{\delta}{\delta s^{(A)\alpha}}$$

$A \in \{0, 1, 2\}$ *summation on A)* *where*

$$(4.15) \qquad \overset{(A)}{\omega}{}^{\alpha}_{\ \gamma} = \overset{(0A)}{L}{}^{\alpha}_{\gamma\beta}\, \delta s^{(0)\beta} + \underset{(1)}{\overset{(1A)}{C}}{}^{\alpha}_{\gamma\beta}\, \delta s^{(1)\beta} + \underset{(2)}{\overset{(2A)}{C}}{}^{\alpha}_{\gamma\beta}\, \delta s^{(2)\beta}$$

1-forms $\overset{(A)}{\omega}{}^{(\alpha}_{\ \gamma}$ will be called invariant 1-forms of connection for the N-linear connection D, depending only on D.

Theorem 4.1. *The exterior differentials of the 1-forms* $\delta s^{(A)\alpha}$ *are given by*

$$d(\delta s^{(0)\gamma}) = \frac{1}{2} \underset{(00)}{\overset{(0)}{W}}{}^{\gamma}_{\beta\alpha}\, \delta s^{(0)\alpha} \wedge \delta s^{(0)\beta} + \underset{(01)}{\overset{(0)}{W}}{}^{\gamma}_{\beta\alpha}\, \delta s^{(0)\alpha} \wedge \delta s^{(1)\beta} + \underset{(02)}{\overset{(0)}{W}}{}^{\gamma}_{\beta\alpha}\, \delta s^{(0)\alpha} \wedge \delta s^{(2)\beta}$$

$$d(\delta s^{(1)\gamma}) = \frac{1}{2} \underset{(00)}{\overset{(1)}{W}}{}^{\gamma}_{\beta\alpha}\, \delta s^{(0)\alpha} \wedge \delta s^{(0)\beta} + \underset{(01)}{\overset{(1)}{W}}{}^{\gamma}_{\beta\alpha}\, \delta s^{(0)\alpha} \wedge \delta s^{(1)\beta} + \underset{(02)}{\overset{(1)}{W}}{}^{\gamma}_{\beta\alpha}\, \delta s^{(0)\alpha} \wedge \delta s^{(2)\beta} + \frac{1}{2} \underset{(11)}{\overset{(1)}{W}}{}^{\gamma}_{\beta\alpha}\, \delta s^{(1)\alpha} \wedge \delta s^{(1)\beta} + \underset{(12)}{\overset{(1)}{W}}{}^{\gamma}_{\beta\alpha}\, \delta s^{(1)\alpha} \wedge \delta s^{(2)\beta}$$

$$d(\delta s^{(2)\gamma}) = \frac{1}{2} \overset{(2)}{\underset{(00)}{W^{\gamma}_{\beta\alpha}}} \delta s^{(0)\alpha} \wedge \delta s^{(0)\beta} + \overset{(2)}{\underset{(01)}{W^{\gamma}_{\beta\alpha}}} \delta s^{(0)\alpha} \wedge \delta s^{(1)\beta} +$$

$$+ \overset{(2)}{\underset{(02)}{W^{\gamma}_{\beta\alpha}}} \delta s^{(0)\alpha} \wedge \delta s^{(2)\beta} + \frac{1}{2} \overset{(2)}{\underset{(11)}{W^{\gamma}_{\beta\alpha}}} \delta s^{(1)\alpha} \wedge \delta s^{(1)\beta} + \overset{(2)}{\underset{(12)}{W^{\gamma}_{\beta\alpha}}} \delta s^{(1)\alpha} \wedge \delta s^{(2)\beta} +$$

$$+\frac{1}{2} \overset{(2)}{\underset{(22)}{W^{\gamma}_{\beta\alpha}}} \delta s^{(2)\alpha} \wedge \delta s^{(2)\beta}. \tag{4.16}$$

Using the invariant 1-forms of connection for the N-linear connection D we prove the following fundamental theorem:

Theorem 4.2. *The structure equations of the N-linear connection D on the total space E in invariant frames are*

$$d(\delta s^{(A)\alpha}) - \delta s^{(A)\beta} \wedge \overset{(A)}{\omega^{\alpha}_{\beta}} = - \overset{(A)}{\Omega^{\alpha}} \tag{4.17}$$

$$d(\overset{(A)}{\omega^{\alpha}_{\beta}}) - \overset{(A)}{\omega^{\gamma}_{\beta}} \wedge \overset{(A)}{\omega^{\alpha}_{\gamma}} = - \overset{(A)}{\Omega^{\alpha}_{\beta}} \tag{4.18}$$

(A=0,1,2), where the 2-forms of torsion $\overset{(A)}{\Omega^{\alpha}}$ *are given by*

$$\overset{(0)}{\Omega^{\alpha}} = \frac{1}{2} \underset{(0)}{T^{\alpha}_{\beta\gamma}} \delta s^{(0)\beta} \wedge \delta s^{(0)\gamma} + \underset{(1)}{K^{\alpha}_{\beta\gamma}} \delta s^{(0)\beta} \wedge \delta s^{(1)\gamma} + \underset{(2)}{K^{\alpha}_{\beta\gamma}} \delta s^{(0)\beta} \wedge \delta s^{(2)\gamma} +$$

$$+ \overset{(1)}{\Omega^{\alpha}} = \frac{1}{2} \underset{(01)}{R^{\alpha}_{\beta\gamma}} \delta s^{(0)\beta} \wedge \delta s^{(0)\gamma} + \underset{(11)}{P^{\alpha}_{\beta\gamma}} \delta s^{(0)\beta} \wedge \delta s^{(1)\gamma} + \underset{(21)}{P^{\alpha}_{\beta\gamma}} \delta s^{(0)\beta} \wedge \delta s^{(2)\gamma} +$$

$$+\frac{1}{2} \underset{(1)}{S^{\alpha}_{\beta\gamma}} \delta s^{(1)\beta} \wedge \delta s^{(1)\gamma} + \underset{(2)}{K^{\alpha}_{\beta\gamma}} \delta s^{(1)\beta} \wedge \delta s^{(2)\gamma} +$$

$$+ \overset{(2)}{\Omega^{\alpha}} = \frac{1}{2} \underset{(02)}{R^{\alpha}_{\beta\gamma}} \delta s^{(0)\beta} \wedge \delta s^{(0)\gamma} + \underset{(12)}{P^{\alpha}_{\beta\gamma}} \delta s^{(0)\beta} \wedge \delta s^{(1)\gamma} + \underset{(22)}{P^{\alpha}_{\beta\gamma}} \delta s^{(0)\beta} \wedge \delta s^{(2)\gamma}$$

$$+\frac{1}{2} \underset{(21)}{Q^{\alpha}_{\beta\gamma}} \delta s^{(1)\beta} \wedge \delta s^{(1)\gamma} + \underset{(22)}{Q^{\alpha}_{\beta\gamma}} \delta s^{(1)\beta} \wedge \delta s^{(2)\gamma} +$$

$$+\frac{1}{2} \underset{(2)}{S^{\alpha}_{\beta\gamma}} \delta s^{(2)\beta} \wedge \delta s^{(2)\gamma} \tag{4.19}$$

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