On Finslerian Hypersurfaces Given by β-Changes

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Abstract

In 1984 C.Shibata has dealt with a change of Finsler metric which is called a β-change of metric [12]. For a β-change of Finsler metric, the differential one-form β play very important roles. In 1985 M.Matsumoto studied the theory of Finslerian hypersurfaces [6]. In there various types of Finslerian hypersurfaces are treated and they are called a hyperplane of the 1st kind, a hyperplane of the 2nd kind and a hyperplane of the 3rd kind.

The purpose of the present paper is to give some relations between the original Finslerian hypersurface and another Finslerian hypersurface given by the β-change of Finsler metrics under certain conditions.

The terminology and notations are referred to the Matsumoto's monograph [8].

Mathematics Subject Classification: 53C60, 53B40
Key words: Finslerian hypersurface, hyperplane, curvature, Finsler metric

1 Preliminaries

Let M^n be an n-dimensional smooth manifold and $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with a fundamental function $L(x,y)$ on M^n. Then the metric tensor $g_{ij}(x,y)$ and Cartan's C-tenson $C_{ijk}(x,y)$ are given by

$$g_{ij} = (\partial^2 L^2/\partial y^i \partial y^j)/2, \quad C_{ijk} = (\partial g_{ij}/\partial y^k)/2, \tag{1.1}$$

and we can introduce in F^n the Cartan connection $C\Gamma = (F_j{}^i{}_k\ , N^i{}_j, C_j{}^i{}_k\ )$.

A hypersurface M^{n-1} of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where u^α are Gaussian coordinates on M^{n-1} and Greek indices run from 1 to n-1. Here, we shall assume that the matrix consisting of the pojection factors $B_\alpha{}^i = \partial x^i/\partial u^\alpha$ is of rank n-1. The following notations are also employed : $B^i_{\alpha\beta} := \partial^2 x^i/\partial u^\alpha \partial u^\beta$, $B^i_{0\beta} := v^\alpha B^i_{\alpha\beta}$, $B^{ij\ldots}_{\alpha\beta\ldots} := B^i_\alpha B_\beta{}^j \ldots$. If the supporting element y^i at a point (u^α) of M^{n-1} is assumed to be tangential to M^{n-1}, we may then write $y^i = B^i_\alpha(u)v^\alpha$, so that v^α is thought of as the supporting element of M^{n-1} at the point (u^α). Since the function $\underline{L}(u,v) := L(x(u), y(u,v))$ gives rise to a Finsler metric of M^{n-1}, we get an $(n$-1)-dimensional Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$.

Balkan Journal of Geometry and Its Applications, Vol.7, No.2, 2002, pp. 49-55.

At each point (u^α) of F^{n-1}, the unit normal vector $N^i(u,v)$ is defined by

$$g_{ij}B^i_\alpha N^j = 0, \quad g_{ij}N^iN^j = 1. \tag{1.2}$$

If (B^α_i, N_i) is the inverse matrix of (B^i_α, N^i), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0, \quad N^iB^\alpha_i = 0, \quad N^iN_i = 1, \tag{1.3}$$

and further

$$B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j. \tag{1.4}$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get $B^\alpha_i = g^{\alpha\beta}g_{ij}B_\beta{}^j$, $N_i = g_{ij}N^j$.

For the induced Cartan connection $IC\Gamma = (N^\alpha_\beta, F_\beta{}^\alpha{}_\gamma, C_\beta{}^\alpha{}_\gamma)$ on F^{n-1}, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature vector H_α are given by

$$\begin{aligned} H_{\alpha\beta} &:= N_i(B^i_{\alpha\beta} + F_j{}^i{}_k\, B^{jk}_{\alpha\beta}) + M_\alpha H_\beta, \\ H_\alpha &:= N_i(B^i_{0\alpha} + N^i{}_jB^j_\alpha), \end{aligned} \tag{1.5}$$

where $M_\alpha := C_{ijk}B^i_\alpha N^jN^k$ and $B^i_{0\alpha} = B^i_{\beta\alpha}\, v^\beta$.

Contracting $H_{\beta\alpha}$ by v^β, we immediately get

$$H_{0\alpha} := H_{\beta\alpha}v^\beta = H_\alpha. \tag{1.6}$$

Further we have put

$$M_{\alpha\beta} := C_{ijk}B^{ij}_{\alpha\beta}N^k, \quad Q_{\alpha\beta} := C_{ijk\,|0}B^{ij}_{\alpha\beta}N^k, \quad Q_{\alpha\beta\gamma} := C_{ijk|0}B^{ijk}_{\alpha\beta\gamma}. \tag{1.7}$$

The Gauss equation with respect to $IC\Gamma$ is written as

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= R_{ijkh}B^{ijkh}_{\alpha\beta\gamma\delta} + P_{ijkh}(B^k_\gamma H_\delta - B^k_\delta H_\gamma)B^{ij}_{\alpha\beta}N^h + \\ &+ (H_{\alpha\gamma}H_{\beta\delta} - H_{\alpha\delta}H_{\beta\gamma}). \end{aligned} \tag{1.8}$$

2 Hypersurfaces given by the β-change of a Finsler metric

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space with a fundamental function $L(x,y)$. For a differential one-form $\beta(x,dx) = b_i(x)dx^i$ on M^n, we shall consider a change of Finsler metric which is defined by $L(x,y) \longrightarrow \bar{L}(x,y) = f(L(x,y), \beta(x,y))$, where $f(L,\beta)$ is a positively homogeneous function of L and β of degree one. This is called a β-change of the metric. Then we can introduce in $\bar{F}^n = (M^n, \bar{L})$ the Cartan connection $C\bar{\Gamma} = (\bar{F}_j{}^i{}_k, \bar{N}^i{}_j, \bar{C}_j{}^i{}_k)$ from a β-change of the metric.

For the later use, we prepare here the following two lemmas.

Lemma 1 (Shibata[12]). *If the covariant vector $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$ on F^n, the difference tensor $D_j{}^i{}_k$ $(:= \bar{F}_j{}^i{}_k - F_j{}^i{}_k)$ vanishes.*

This lemma leads us to $\bar{N}^i_j = N^i{}_j$ from $D^i{}_j \equiv \bar{N}^i_j - N^i{}_j = D_j{}^i{}_k\, y^k = D_j{}^i 0$.

Lemma 2 (Shibata[12]). *Assume that the covariant vector $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$ on F^n. Then the h-curvature tensor $\bar{R}_h{}^i{}_{jk}(x,y)$*

of $\bar{F}^n$, obtained from F^n by the β-change, vanishes if and only if the h-curvature tensor $R_h{}^i{}_{jk}(x,y)$ of F^n vanishes.

We now consider a Finslerian hypersurface $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ of F^n and another Finslerian hypersurface $\bar{F}^{n-1} = (M^n, \underline{\bar{L}}(u,v))$ of the $\bar{F}^n$ given by the β-change. Let N^i be a unit normal vector at each point of F^{n-1}, and (B_i^α, N_i) be the inverse matrix of (B_α^i, N^i). The functions $B_\alpha^i(u)$ may be considered as components of n-1 linearly independent vectors tangent to F^{n-1} and they are invariant under the β-change. And so a unit normal vector $\bar{N}^i(u,v)$ of $\bar{F}^{n-1}$ is uniquely determined by

$$\bar{g}_{ij} B_\alpha^i \bar{N}^j = 0, \qquad \bar{g}_{ij} \bar{N}^i \bar{N}^j = 1. \tag{2.1}$$

The fundamental tensor $\bar{g}_{ij} = (\partial^2 \bar{L}^2 / \partial y^i \partial y^j)/2$ of the Finsler space $\bar{F}^n$ given by a β-change is as follows [12]:

$$\bar{g}_{ij}(x,y) = p g_{ij}(x,y) + p_0 b_i b_j + p_{-1}(b_i y_j + b_j y_i) + p_{-2} y_i y_j, \tag{2.2}$$

where we put $p = f f_L / L$

$$\begin{aligned} &p_0 = f f_{\beta\beta} + f_\beta{}^2, \qquad p_{-1} = (f f_{L\beta} + f_L f_\beta)/L, \\ &p_{-2} = (f f_{LL} + f_L{}^2 - f f_L / L) L^2, \end{aligned} \tag{2.3}$$

and subscripts L, β denote partial differetiations by L, β respectively. Now contracting (1.2) by v^α, we immediately get

$$y_i N^i = 0. \tag{2.4}$$

Further contracting (2.2) by $N^i N^j$ and paying attention to (1.2) and (2.4), we have

$$\bar{g}_{ij} N^i N^j = p + p_0 (b_i N^i)^2. \tag{2.5}$$

Then we obtain

$$\bar{g}_{ij}(\pm N^i / \sqrt{p + p_0 (b_i N^i)^2})(\pm N^j / \sqrt{p + p_0 (b_i N^i)^2}) = 1, \tag{2.6}$$

provided $p + p_0 (b_i N^i)^2 > 0$. Therefore we can put

$$\bar{N}^i = N^i / \sqrt{p + p_0 (b_i N^i)^2}, \tag{2.7}$$

where we have chosen the sign "+" in order to fix an orientation.

Using (1.2) and (2.4), the first condition of (2.1) gives us

$$(b_i N^i)(p_0 b_j B_\alpha^j + p_{-1} y_j B_\alpha^j) = 0. \tag{2.8}$$

Now, assuming that $p_0 b_j B_\alpha^j + p_{-1} y_j B_\alpha^j = 0$ and contracting this by v^α, we find $p_0 \beta + p_{-1} L^2 = 0$. By (2.3) this equation lead us to $f f_\beta = 0$, where we have used $L f_{L\beta} + \beta f_{\beta\beta} = 0$ and $L f_L + \beta f_\beta = f$ owing to the homogeneity of f. Thus we have $f_\beta = 0$ because of $f \neq 0$. This fact means $\bar{L} = f(L)$ and contradicts the definition of a β-change of metric. Consequently (2.8) gives us

$$b_i N^i = 0. \tag{2.9}$$

Therefore (2.7) is rewritten as

$$\bar{N}^i = N^i/\sqrt{p} \qquad (p > 0), \tag{2.10}$$

and then it is clear $\bar{N}^i$ satistifies (2.1). Summarizing the above, we obtain

Theorem 2.1. *For a field of linear frame $(B_1^i, \dots, B_{n-1}^i, N^i)$ of F^n, there exists a field of linear frame $(B_1^i, \dots, B_{n-1}^i, \bar{N}^i = N^i/\sqrt{p})$ of the $\bar{F}^n$ given by the β-change such that (2.1) is satisfied along $\bar{F}^{n-1}$, and then we get (2.9).*

The quantities $\bar{B}_i^\alpha$ are uniquely defined along $\bar{F}^{n-1}$ by

$$\bar{B}^\alpha{}_i = \bar{g}^{\alpha\beta}\bar{g}_{ij}B_\beta{}^j, \tag{2.11}$$

where $(\bar{g}^{\alpha\beta})$ is the inverse matrix of $(\bar{g}_{\alpha\beta})$.

Let $(\bar{B}_i^\alpha, \bar{N}_i)$ be the inverse matrix of $(B_\alpha^i, \bar{N}^i)$, and then we have

$$B_\alpha^i\bar{B}_i^\beta = \delta_\alpha^\beta,\ B_\alpha^i\bar{N}_i = 0,\ \bar{N}^i\bar{B}_i^\alpha = 0,\ \bar{N}^i\bar{N}_i = 1, \tag{2.12}$$

and further

$$B_\alpha^i\bar{B}_j^\alpha + \bar{N}^i\bar{N}_j = \delta^i{}_j. \tag{2.13}$$

We also get $\bar{N}_i = \bar{g}_{ij}\bar{N}^j$, that is,

$$\bar{N}_i = \sqrt{p}N_i. \tag{2.14}$$

If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of the ambient space F^n, then F^{n-1} is called a hyperplane of the first kind. A hyperplane of the 1st kind is characterized by $H_\alpha = 0$.

From (1.5), (2.14) and Lemma 2, we have $\bar{H}_\alpha = \sqrt{p}H_\alpha$. Thus we obtain

Theorem 2.2. *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then a hypersurface F^{n-1} is a hyperplane of the 1st kind, if and only if the hypersurface $\bar{F}^{n-1}$ is a hyperplane of the 1st kind.*

If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also an h-path of the ambient space F^n, then F^{n-1} is called a hyperplane of the second kind. A hyperplane of the 2nd kind is characterlized by $H_{\alpha\beta} = 0$.

From (1.5), (1.6), (2.14) and Lemma 2, we obtain

Theorem 2.3 *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n.Then a hypersurface F^{n-1} is a hyperplane of the 2nd kind, if and only if the hypersurface $\bar{F}^{n-1}$ is a hyperplane of the 2nd kind.*

As to the torsion tensor $\bar{C}_{ijk}$ of $\bar{F}^n$, Shibata [12] gave:

$$\bar{C}_{ijk} = pC_{ijk} + p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)/2 + p_{0\beta}m_im_jm_k/2, \tag{2.15}$$

where we put

$$m_i = b_i - \beta y_i/L^2. \tag{2.16}$$

Using (2.4) and (2.9), we easily get

$$m_iN^i = 0. \tag{2.17}$$

As for the angular metric tensor $h_{ij} = g_{ij} - l_i l_j$, (1.2) and (2.4) yield

$$h_{ij} B^i_\alpha N^j = 0. \tag{2.18}$$

Contracting (2.15) by $B^{ij}_{\alpha\beta} N^k$, (2.17) and (2.18) lead to

$$\bar{C}_{ijk} B^{ij}_{\alpha\beta} N^k = p C_{ijk} B^{ij}_{\alpha\beta} N^k. \tag{2.19}$$

On using (1.7) and (2.10), (2.19) is rewritten as

$$\bar{M}_{\alpha\beta} = \sqrt{p} M_{\alpha\beta}. \tag{2.19}$$

If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1}, then F^{n-1} is called a hyperplane of the third kind. A hyperplane of the 3rd kind is characterized by $H_{\alpha\beta} = M_{\alpha\beta} = 0$.

Thus, from Theorem 2.3 and (2.20), we obtain

Theorem 2.4. *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then a hypersurface F^{n-1} is a hyperplane of the 3rd kind, if and only if the hypersurface $\bar{F}^{n-1}$ is a hyperplane of the 3rd kind.*

Taking account of Lemma 1, as to $B\Gamma$ we have [6]

$$G_\beta{}^\alpha{}_\gamma = B^\alpha_i A^i_{\beta\gamma} \tag{2.21}$$

where $A^i_{\beta\gamma} := B^i_{\beta\gamma} + G_j{}^i{}_k B^{jk}_{\beta\gamma}$. Now using (1.4), then (2.21) becomes

$$A^i_{\beta\gamma} = B^i_\delta G_\beta{}^\delta{}_\gamma + N^i N_h A^h_{\beta\gamma} \tag{2.22}$$

Since Lemma 1 leads to $\bar{A}^i_{\beta\gamma} = A^i_{\beta\gamma}$, we immediately get

$$\bar{G}^\alpha_{\beta\gamma} = \bar{B}^\alpha_i A^i_{\beta\gamma}. \tag{2.23}$$

On substituting (2.22) in (2.23) and paying attention to (2.10) and (2.12), we find $\bar{G}_\beta{}^\alpha{}_\gamma = G_\beta{}^\alpha{}_\gamma$. Thus we obtain

Theorem 2.5. *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then a hyperplane F^{n-1} of the 1st kind is a Berwald space, if and only if the hyperplane $\bar{F}^{n-1}$ of the 1st kind is a Berwald space.*

Paying attention to Lemma 1, as to $C\Gamma$ the $(v)hv$-torsion tensor is written as

$$P^\alpha{}_{\beta\gamma} = B^\alpha_i K^i_{\beta\gamma}, \tag{2.24}$$

where $K^i_{\beta\gamma} := P^i_{jk} B^{jk}_{\beta\gamma}$. On using (1.4), then (2.24) becomes

$$K^i_{\beta\gamma} = B^i_\delta P^\delta{}_{\beta\gamma} + N^i N_h K^h_{\beta\gamma}. \tag{2.25}$$

Lemma 2 gives us $\bar{K}^i_{\beta\gamma} = K^i_{\beta\gamma}$, and then we immediately obtain

$$\bar{P}^\alpha_{\beta\gamma} = \bar{B}^\alpha_i K^i_{\beta\gamma}. \tag{2.26}$$

On substituting (2.25) in (2.26) and taking account of (2.10) and (2.12), we find $\bar{P}^\alpha_{\beta\gamma} = P^\alpha{}_{\beta\gamma}$. Thus we obtain

Theorem 2.6. *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then a hyperplane F^{n-1} of the 1st kind is Landsberg, if and only if the hyperplane $\bar{F}^{n-1}$ of the 1st kind is Landsberg.*

From (1.8) the Gauss equation of hyperplane of the 1st kind is rewritten as

$$R_{\alpha\beta\gamma\delta} = R_{ijkh} B^{ijkh}_{\alpha\beta\gamma\delta} + (H_{\alpha\gamma} H_{\beta\delta} - H_{\alpha\delta} H_{\beta\gamma}). \tag{2.27}$$

Then Lemma 2 and $\bar{H}_{\alpha\beta} = \sqrt{p} H_{\alpha\beta}$ give us the following.

Theorem 2.7. *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then the curvature tensor $R_{\alpha\beta\gamma\delta}$ of a hyperplane F^{n-1} of the 1st kind of F^n with $R_{ijkh} = 0$ vanishes, if and only if the curvature tensor $\bar{R}_{\alpha\beta\gamma\delta}$ of the hyperplane $\bar{F}^{n-1}$ of the 1st kind of $\bar{F}^n$ with $\bar{R}_{ijkh} = 0$ vanishes.*

Further Theorem 2.5 and Theorem 2.7 immediately give

Theorem 2.8. *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then a hyperplane F^{n-1} of the 1st kind of F^n with $R_{ijkh} = 0$ is a locally Minkowski space, if and only if the hyperplane $\bar{F}^{n-1}$ of the 1st kind of $\bar{F}^n$ with $\bar{R}_{ijkh} = 0$ is locally Minkowskian.*

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