An Almost Paracontact Structure on the Indicatrix Bundle of a Finsler Space

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Abstract

In a paper by I. Hasegawa, K. Yamaguchi and H. Shimada, [2], it was proved that the indicatrix bundle of a Finsler space $F^n = (M, L)$ has a natural almost contact structure. On a different way, the same structure was found by M. Anastasiei in [1]. Adopting the approach from [1] we prove that the indicatrix bundle of F^n carries also an almost paracontact structure.

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1 Introduction

Let $F^n = (M, L)$ be a Finsler space. Here M is a real C^∞ manifold of dimension n with local coordinates (x^i), $i, j, k... = 1, ..., n$. For the tangent manifold TM with the projection τ over M we take the local coordinates $(x^i \circ \tau, y^i)$, where y^i are the components of a vector from T_pM, in the natural basis $\partial_i = \dfrac{\partial}{\partial x^i}$.

The function $L : T_0M : TM \setminus \{0\} \to \mathbf{R}_+$ is smooth, positively homogeneous of degree 1 with respect to y^i and the matrix $\left(g_{ij}(x, y) = \dfrac{1}{2} \dfrac{\partial^2 L^2}{\partial y^i \partial y^j} \right)$ is of rank n. We set $\dot{\partial}_i = \dfrac{\partial}{\partial y^i}$.

The homogeneity of L implies

$$L^2(x, y) = g_{ij}(x, y) y^i y^j = y^i y_i \quad \text{for} \quad y_i = g_{ij} y^j.$$

The functions $N^i_j(x, y) = \dfrac{1}{2} \dot{\partial}_j (\gamma^i_{00})$, for $\gamma^i_{00} = \gamma^i_{jk}(x, y) y^j y^k$ and $\gamma^i_{jk}(x, y)$ the "generalized" Christoffel symbols, are the local coefficients of the nonlinear Cartan connection. See [Ch. VIII, 4] for details. One considers a new local basis $\{\delta_i, \dot{\partial}_i\}$, with $\delta_i = \partial_i - N^k_i(x, y) \dot{\partial}_k$, on T_0M. Its dual basis is $(dx^i, \delta y^i)$ with $\delta y^i = dy^i + N^i_k(x, y) dx^k$. If we assume that the quadratic form $g_{ij}(x, y) \xi^i \xi^j$, $\xi \in \mathbf{R}^n$ is positive definite, then

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$$G_S = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j$$

is a Riemannian metric on T_0M.

The linear operator P given in the local basis by

$$P(\delta_i) = \delta_i, \quad P(\dot{\partial}_i) = -\dot{\partial}_i, \tag{1.1}$$

defines an almost product structure on T_0M and we have

$$G_S(PX, PY) = G_S(X,Y), \quad X, Y \in \chi(T_0M). \tag{1.2}$$

Here $\chi(T_0M)$ is the module of vector fields on T_0M. The vector field $C = y^i\dot{\partial}_i$ is called the Liouville vector field on T_0M and $S = y^i\delta_i$ is the geodesic spray of F^n.

An almost paracontact structure on a manifold N is a set (φ, ξ, η), where φ is a tensor field of type $(1,1)$, ξ a vector field and η an 1-form such that

$$\eta(\xi) = 1,\ \varphi(\xi) = 0,\ \eta\circ\varphi = 0,\ \varphi^2 = +I - \eta\otimes\xi, \tag{1.3}$$

where I denotes the Kronecker tensor field.

This structure generalizes as follows. One considers on a manifold N of dimension $(2n+s)$ a tensor field f of type $(1,1)$. If there exists on N the vector fields (ξ_α) and the 1- forms (η^α) $(\alpha = 1, 2, ...s)$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, f(\xi_\alpha) = 0, \eta^\alpha\circ f = 0, f^2 = I - \sum\nolimits_\alpha \eta^\alpha\otimes\xi_\alpha, \tag{1.4}$$

then $(f, (\xi_\alpha), (\eta^\alpha))$ is called a framed $f(3,-1)$- structure. The term was suggested by the equation $f^3 - I = 0$. This is in some sense dual to the framed f-structure which generalizes the almost contact structure and which may be called a framed $f(3,+1)$-structure. For an account of such kind of structures we refer to the book [3].

In the following (Section 2) we show that the slit tangent bundle T_0M of a Finsler space carries a natural framed $f(3,-1)$- structure. The set $I(M) = \{(x,y) \mid L(x,y) = 1\}$ is a $(2n-1)-$ dimensional submanifold of T_0M. In Section 3 we prove that the framed $f(3,-1)$- structure on T_0M induces on $I(M)$ an almost paracontact structure. We note that it was known that $I(M)$ carries an almost contact structure [2], [1] but only the approach from [1] allowed us to construct this almost paracontact structure.

2 A framed $f(3,-1)$- structure on T_0M

Let us put $\xi_1 := S = y^i\delta_i$ and $\xi_2 := C = y^i\dot{\partial}_i$. By a direct calculation one finds (P is the almost product structure (1.1)).

Lemma 2.1. $P(\xi_1) = \xi_1,\ P(\xi_2) = -\xi_2$. We consider the 1- forms

$$\eta^1 = \frac{y_i}{L^2}dx^i, \eta^2 = \frac{y_i}{L^2}\delta y^i$$

and we prove

Lemma 2.2. $\eta^1\circ P = \eta^1, \eta^2\circ P = -\eta^2$.

Proof. It is sufficient to check these equalities on the adapted basis $(\delta_i, \dot{\partial}_i)$. We have

$$(\eta^1\circ P)(\delta_i) = \eta^1(P(\delta_i)) = \eta^1(\delta_i) \quad \text{and} \quad (\eta^1\circ P)(\dot{\partial}_i) = -\eta^1(\dot{\partial}_i) = 0.$$

Then
$$(\eta^2 \circ P)(\delta_i) = \eta^2(\delta_i) = 0 \quad \text{and} \quad (\eta^2 \circ P)(\dot{\partial}_i) = -\eta^2(\dot{\partial}_i).$$
□

Let be $G = \frac{1}{L^2} G_S$ a Riemannian metric which is conformal with G_S.

Lemma 2.3. $\eta^1(X) = G(X, \xi_1)$, $\eta^2(X) = G(X, \xi_2), \forall X \in \chi(T_0M)$.

Proof. It is sufficient to check these equalities on the basis $(\delta_i, \dot{\partial}_i)$. We have: $\eta^1(\delta_j) = = \frac{y_j}{L^2} = di\frac{1}{L^2} g_{jk} y^k$ and $G(\delta_j, \xi_1) = \frac{1}{L^2} G_S(\delta_j, y^k \delta_k) = \frac{1}{L^2} y^k G_S(\delta_j, \delta_k) = \frac{1}{L^2} y^k g_{jk}$. Further, $\eta^1(\dot{\partial}_i) = 0$ and $G(\dot{\partial}_j, \xi_1) = \frac{1}{L^2} G_S(\dot{\partial}_i, y^k \delta_k) = 0$. Similarly, one checks the equation $\eta^2(X) = G(X, \xi_2)$.

□

Now we define a tensor field p of type $(1,1)$ on T_0M by

$$p(X) = P(X) - \eta^1(X)\xi_1 + \eta^2(X)\xi_2, X \in \chi(T_0M). \tag{2.1}$$

This can be written in a more compact form as $p = P - \eta^1 \otimes \xi_1 + \eta^2 \otimes \xi_2$.

Theorem 2.1. *For the data $(p, (\xi_a), (\eta^a))$, $a = 1, 2$ the following hold*

(i) $\eta^a(\xi_b) = \delta^a_b$, $p(\xi_a) = 0$, $\eta^a \circ p = 0$,

(ii) $p^2 = I - \eta^1 \otimes \xi_1 - \eta^2 \otimes \xi_2, X \in \chi(T_0M)$,

(iii) *p is of rank $2n-2$ and $p^3 - p = 0$.*

Proof. (i) follows easily from Lemmas 2.1, 2.2 and the formula (2.1). For (ii) we have

$$p^2(X) = p(p(X)) = P(P(X) - \eta^1(X)\xi_1 + \eta^2(X)\xi_2) - \eta^1(P(X) - \eta^1(X)\xi_1 +$$
$$+\eta^2(X)\xi_2) + \eta^2(P(X) - \eta^1(X)\xi_1 + \eta^2(X)\xi_2) = +X - \eta^1(X)\xi_1 - \eta^2(X)\xi_2,$$

the other terms vanish or cancel because of Lemmas 2.1, 2.2 and (i). Applying p to the equality (ii) and using again the Lemmas 2.1, 2.2 and (i) one gets $p^3 - p = 0$. From the second equation in (i) we see that the subspace $span(\xi_1, \xi_2)$ is contained in $Ker\ p$. Let now $X = X^i\delta_i + Y^i\dot{\partial}_i \in Kerp$. On using (2.1),

$$p(X) = X^i\delta_i - Y^i\dot{\partial}_i - (X^i\frac{y_i}{L^2})\xi_1 + Y^i\frac{y_i}{L^2}\xi_2 = (X^i - \frac{(X^k y_k)}{L^2}y^i)\delta_i - (Y^i - (Y^k\frac{y_k}{L^2})y^i)\dot{\partial}_i = 0$$

equivalent to
$$X^i = \frac{X^k y_k}{L^2}y^i, \quad Y^i = \frac{(Y^k y_k)}{L^2}y^i.$$

Hence $X = \frac{X^k y_k}{L^2}\xi_1 + \frac{Y^k y_k}{L^2}\xi_2$ that is X belongs to $span(\xi_1, \xi_2)$. In other words, Ker $p = span(\xi_1, \xi_2)$. Thus rank $p = 2n-2$.

□

Theorem 2.2. *The Riemannian metric $G = \frac{1}{L^2} G_S$ satisfies*

(2.2) $$G(pX, pY) = G(X, Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y), X, Y \in \chi(T_0M).$$

Proof. Use (2.1) and Lemma 2.3 and Lemma 2.1 as well as $G(\xi_1, \xi_1) = 1$, $G(\xi_2, \xi_2) = 1$, $G(\xi_1, \xi_2) = 0$ to obtain

$$\begin{aligned}
&G(pX, pY) = G(PX, PY) - \eta^1(Y)G(PX, \xi_1) + \eta^2(Y)G(PX, \xi_2)- \\
&-\eta^1(X)G(\xi_1, PY) + \eta^1(X)\eta^1(Y) + \eta^2(X)G(\xi_2, PY) + \eta^2(X)\eta^2(Y) = \\
&= G(X, Y) - \eta^1(Y)\eta^1(P(X)) + \eta^2(Y)\eta^2(PX) - \eta^1(X)\eta^1(PY) + \eta^2(X)\eta^2(PY)+ \\
&+\eta^1(X)\eta^1(Y) + \eta^2(X)\eta^2(Y) = G(X, Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y)
\end{aligned}$$

□

Remark. In the local basis $(\delta_i, \dot\partial_i)$, we get

(2.3) $$\begin{aligned}
G(p(\delta_i), p(\delta_j)) &= \frac{1}{L^2}(g_{ij} - \frac{y_i y_j}{L^2}),\ G(p(\delta_i), p(\dot\partial_i)) = 0, \\
G(p(\dot\partial_i),\ p(\dot\partial_j)) &= \frac{1}{L^2}(g_{ij} - \frac{y_i y_j}{L^2}).
\end{aligned}$$

Let us put

(2.4) $$h(X, Y) = G(pX, Y), X, Y \in \chi(T_0M).$$

We have

Theorem 2.3. *The map h is a symmetric bilinear form on T_0M of rank $2n-2$, with the null space $span(\xi_1, \xi_2)$.*

Proof. h is bilinear since G is so. As for the symmetry we have

$$\begin{aligned}
&h(Y, X) = G(pY, X) = G(pY, p^2X + \eta^1(X)\xi_1 + \eta^2(X)\xi_2) = \\
&= G(pY, p(pX)) + \eta^1(X)G(pY, \xi_1) + \eta^2(X)G(pY, \xi_2) = \\
&= G(pY, p(pX)) + \eta^1(X)\eta^1(PY) + \eta^2(X)\eta^2(PY) = \\
&= G(Y, pX) - \eta^1(Y)\eta^1(pX) - \eta^2(Y)\eta^2(pX) = G(Y, pX) = h(X, Y).
\end{aligned}$$

Then we have $h(\xi_1, \xi_1) = h(\xi_2, \xi_2) = 0$. Thus $span(\xi_1, \xi_2)$ is contained in the null space of h. Conversely, if $X = X^i\delta_i$ is such that $h(X, X) = 0 \Longleftrightarrow G(pX, X) = 0$ it results $X = \frac{X^k y_k}{L^2}\xi_1$ and similarly, if $X = Y^i\dot\partial_i$ is such that $h(X, X) = 0$, it results $X = \frac{Y^k y_k}{L^2}\xi_2$. Thus the null space of h is just $span(\xi_1, \xi_2)$ and the proof is finished.

Remark. The map h is a singular pseudo-Riemannian metric on T_0M. Locally it looks as follows

$$h = \frac{1}{L^2}(g_{ij} - \frac{y_i y_j}{L^2})dx^i \otimes dx^j - \frac{1}{L^2}(g_{ij} - \frac{y_i y_j}{L^2})\delta y^i \otimes \delta y^j,$$

with

$$\text{rank } (g_{ij} - \frac{y_i y_j}{L^2}) = n - 1$$

since

$$(g_{ij} - \frac{y_i y_j}{L^2})y^j = y_i - y_i = 0 \quad (y_j y^j = L^2).$$

3 An almost paracontact structure on the indicatrix bundle of the Finsler space $F^n = (M, L)$

The indicatrix bundle of F^n is the submanifold

$$I(M) = \{(x, y) \in T_0M \mid L(x, y) = 1\}$$

of T_0M projected over M. It is well-known that $\xi_2 = y^i \dot{\partial}_i$ is normal to $I(M)$ and this is unitary with respect to G since

$$G(\xi_1, \xi_2) = \frac{1}{L^2} y^i y^j g_{ij} = 1.$$

We consider T_0M with the Riemannian metric G and then $I(M)$ appears as a hypersurface of T_0M with normal vector field ξ_2. We restrict to $I(M)$ all the objects introduced above and indicate this fact by putting a bar over the letters denoting those objects. We have:

- $\overline{\xi}_1 = \xi_1$ since ξ_1 is tangent to $I(M)$,
- $\overline{\eta}^2 = 0$ on $I(M)$ since $\eta^2(X) = G(X, \xi_2) = 0$ for $X \in \chi(I(M))$,
- $\overline{G} = G_S \mid_{I(M)}$ because $L^2 = 1$ on $I(M)$,
- $\overline{p}(X) = P(X) - \overline{\eta}^1(X)\xi_1$ for $X \in \chi(I(M))$.
- The map $\overline{p}$ is an endomorphism of the tangent bundle to $I(M)$ since $G(\overline{p}X, \xi_2) = 0$.

We put $\overline{\xi}_1 = \overline{\xi}, \overline{\eta}^1 = \overline{\eta}$ and as a consequence of the Theorem 2.1 we get
Theorem 3.1. *The triple $(\overline{p}, \overline{\xi}, \overline{\eta})$ defines an almost paracontact structure on $I(M)$, that is,*

(i) $\overline{\eta}(\overline{\xi}) = 1, \overline{p}(\overline{\xi}) = 0, \overline{\eta} \circ \overline{p} = 0,$

(ii) $\overline{p}^2(X) = X - \overline{\eta}(X)\overline{\xi}, X \in \chi(I(M))),$

(iii) $\overline{p}^3 - p = 0,$ *rank* $\overline{p} = 2n - 2 = (2n - 1) - 1.$

Using the restriction to $I(M)$ and the Theorem 2.2 one infers
Theorem 3.2. *The Riemannian metric $\overline{G}$ satisfies*

$$\overline{G}(\overline{p}X, \overline{p}Y) = \overline{G}(X, Y) - \overline{\eta}(X)\overline{\eta}(Y), X, Y \in \chi(I(M)). \tag{3.1}$$

From the last two theorems we see that the ensemble $(\overline{p}, \overline{\xi}, \overline{\eta}, \overline{G})$ defines an almost metrical paracontact structure on $I(M)$.

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