# Quadratic Homogeneous ODE Systems of Jordan-Rigid Body Type

*Dedicated to Acad. Radu Miron on the occasion of his 75 birthday*

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**Abstract**

Trying to yield a Lax representation with the Lax pair of $so(3)$-type for some quadratic differential systems but with the Jordan bracket instead of usual Lie bracket, a class of homogeneous systems with remarkable properties is obtained.

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The class of polynomial homogeneous ODE systems has special features, important from the point of view of applications ([3]). In this class, the subset of quadratic homogeneous systems presents interesting geometric aspects, cf. [4]. Such a quadratic system reads:

$$\frac{dx^i}{dt} = a^i_{jk} x^j x^k$$

and important examples are the Euler-Poincaré equations on Lie algebras ([10, p. 19]) and homogeneous sprays ([12]) if $x^i = \dfrac{dy^i}{dt}$.

For differential systems a useful method of study is provided by the so-called *Lax representation* which yields, in a systematic manner, a lot of first integrals, with mechanical significance, like energy or momentum ([10]).

Unfortunately, for some remarkable quadratic systems, one cannot find a Lax pair. Our paper gives two examples, namely the Nahm system and the system of elliptic Jacobi functions, treated in the second section. But we try to obtain a Lax pair for these systems, using another bracket instead of classical Lie bracket, more precisely the Jordan bracket. These types of deformation brackets appear in the first section where we add the so-called Brockett perturbation, a notion useful in rigid body control ([2], [5]). Let us note that we use the form of Lax pair given by the rigid body, which admits a classical Lax representation i.e. with respect to the Lie multiplication. For our aim, we must restrict to the upper-triangular part from a matriceal differential

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system and then a quadratic homogeneous ODE system is obtained called Jordan-rigid body type system. In the third section, using the Brockett perturbation for system of elliptic Jacobi functions, we obtain a so-called Brockett perturbation of trigonometric functions and a rational first integral of this last system is derived via Bernoulli type equations. In the following section, a Jordan counterpart of two systems, rigid body and Maxwell-Bloch, with usual Lax representation, is obtained. In the next section to a Jordan-rigid body type system we associate (i) two particular solutions, (ii) two quadratic first integrals and (iii) a numerical integrator preserving these first integrals. A search for $so(4)$ version of our result and an appendix with the Maple computation of some matrices, useful for our models, and the Maple plot of three examples are inserted at the end of paper.

An important remark is that in this new approach the above mentioned physical significance of Lax pairs is lost. Our initial aim was to point an algebraic character of Jordan-Lax type for some important systems but this searching for Lax yields a veritable round through algebra, geometry, differential equations and mechanics. For a survey on other applications of Jordan algebras in physics see [8].

# 1 Deformation algebras and Brockett perturbation

Let $A$ be a real algebra in which the product is denoted by adjacency and let two numbers $h, k \in \mathbb{R}$. Let us consider the bracket $[,]_{h,k}$ given by:

$$[a,b]_{h,k} = h \cdot ab + k \cdot ba \tag{1.1}$$

for every $a, b \in A$. Then the pair $\left(A, [,]_{h,k}\right)$ is a new real algebra, appeared in [15] and [16]. Let us note that in the cited papers the bracket is denoted by "$*$" and $(A, *)$ is called *G-algebra*. We prefer the bracket notations having in mind the following examples and as name *the $(h,k)$-deformation algebra* of $A$.

**Examples** ([16]):

(i) $h \equiv 1, k \equiv -1$ is *the Lie algebra* associated to $A$,

(ii) $h = k \equiv 1$ is *the Jordan algebra* associated to $A$.

Inspired by [5, p. 135] for $a, b \in A$ we consider *the Brockett perturbation* with respect to $[,]_{h,k}$ as the element $[a,b]_{B,h,k}$ defined by

$$[a,b]_{B,h,k} = [a,b]_{h,k} + \left[a, [a,b]_{h,k}\right]_{h,k}. \tag{1.2}$$

A straightforward calculation gives

$$[a,b]_{B,h,k} = h \cdot ab + k \cdot ba + h^2 \cdot a^2 b + 2hk \cdot aba + k^2 \cdot ba^2 \tag{1.3}$$

which for the considered examples becomes

$$[a,b]_{B,1,-1} = ab - ba + a^2 b - 2aba + ba^2 \tag{1.3$i$}$$

$$[a,b]_{B,1,1} = ab + ba + a^2 b + 2aba + ba^2. \tag{1.3$ii$}$$

## 2 Systems of Jordan-rigid body type

It is well-known that some important ordinary differential systems admit *a Lax formulation*, that is, can be put in the equivalent form:

$$\frac{d}{dt}L = [L, B] \tag{2.1}$$

where $L$ and $B$ are matrices and $[L, B] = LB - BL = [L, B]_{1,-1}$ is their commutator. The pair $(L, B)$ is called *the Lax pair* of the given differential system and in Russian usage a Lax formulation is called sometimes *Heisenberg representation* ([10]). The utility of a Lax representation is given by the fact that the eigenvalues of $L$ are first integrals of (2.1)([10, p. 98], [11]).

Examples of dynamical systems admitting a Lax pair: the 1-dimensional harmonic oscillator, the free rigid body, the Maxwell-Bloch equations from laser matter dynamics, the Toda lattice and so on (see the previous citations). In the following we recall in detail the rigid body, for which the equations of motion are ([5, p. 136]):

$$\begin{cases} \dfrac{dx_1}{dt} = a_1 x_2 x_3 \\ \dfrac{dx_2}{dt} = a_2 x_3 x_1 \\ \dfrac{dx_3}{dt} = a_3 x_1 x_2 \end{cases} \tag{2.2}$$

where

$$a_1 = \frac{1}{I_3} - \frac{1}{I_2}, \quad a_2 = \frac{1}{I_1} - \frac{1}{I_3}, \quad a_3 = \frac{1}{I_2} - \frac{1}{I_1} \tag{2.3}$$

with $I_1, I_2, I_3$ the principal moments of inertia and $I_1 > I_2 > I_3 > 0$. Because the matrices of $so(3)$-type appear naturally in equations of motion for orthonormal basis in $\mathbf{R}^3$, we search for a Lax pair of $so(3)$-type:

(2.4)
$$L = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, B = B(\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} 0 & -\alpha_3 x_3 & \alpha_2 x_2 \\ \alpha_3 x_3 & 0 & -\alpha_1 x_1 \\ -\alpha_2 x_2 & \alpha_1 x_1 & 0 \end{pmatrix}$$

with the unknowns $\alpha_1, \alpha_2, \alpha_3$. The eq. (2.1) then becomes:

$$\begin{cases} \dfrac{dx_1}{dt} = (\alpha_3 - \alpha_2) x_2 x_3 \\ \dfrac{dx_2}{dt} = (\alpha_1 - \alpha_3) x_3 x_1 \\ \dfrac{dx_3}{dt} = (\alpha_2 - \alpha_1) x_1 x_2 \end{cases} \tag{2.5}$$

and comparing with (2.2) yields the equations:

$$\begin{cases} \alpha_3 - \alpha_2 = a_1 = \frac{1}{I_3} - \dfrac{1}{I_2} \\ \alpha_1 - \alpha_3 = a_2 = \frac{1}{I_1} - \dfrac{1}{I_3} \\ \alpha_2 - \alpha_1 = a_3 = \frac{1}{I_2} - \dfrac{1}{I_1} \end{cases}$$

with solution

$$\alpha_i = \frac{1}{I_i}, \quad 1 \leq i \leq 3. \tag{2.6}$$

It results that we obtain the Lax pair of the rigid body and then one have the associated Brockett perturbation ([5, p. 137]):

$$\begin{cases} \dfrac{dx_1}{dt} = a_1 x_2 x_3 + x_1 \left( a_3 x_2^2 - a_2 x_3^2 \right) \\ \dfrac{dx_2}{dt} = a_2 x_3 x_1 + x_2 \left( a_1 x_3^2 - a_3 x_1^2 \right) \\ \dfrac{dx_3}{dt} = a_3 x_1 x_2 + x_3 \left( a_2 x_1^2 - a_1 x_2^2 \right) \end{cases} . \tag{2.7}$$

In the following we try the same program for other two remarkable quadratic systems ([6]):

A *the Nahm system of static SU(2)-monopoles* or Tzitzeica gradient flow

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 x_3 \\ \dfrac{dx_2}{dt} = x_3 x_1 \\ \dfrac{dx_3}{dt} = x_1 x_2 \end{cases} \tag{2.8}$$

B *the differential system of elliptic Jacobi functions* Tzitzeica-Lorentz gradient flow

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 x_3 \\ \dfrac{dx_2}{dt} = -x_3 x_1 \\ \dfrac{dx_3}{dt} = -k^2 x_1 x_2 \end{cases} \tag{2.9}$$

with $0 < k^2 < 1$.

Comparing (2.5) with (2.8) and (2.9) we have:

A

$$\begin{cases} \alpha_3 - \alpha_2 = 1 \\ \alpha_1 - \alpha_3 = 1 \\ \alpha_2 - \alpha_1 = 1 \end{cases}$$

and adding all equations it results $0 = 3$, false

B

$$\begin{cases} \alpha_3 - \alpha_2 = 1 \\ \alpha_1 - \alpha_3 = -1 \\ \alpha_2 - \alpha_1 = -k^2 \end{cases}$$

and adding it results $0 = -k^2$, false.

Therefore the Lie bracket is not suitable for systems A and B with the pair (2.4) and then we try with the next $(h, k)$ -deformation bracket, namely Jordan deformation. In this case the system:

$$\frac{d}{dt}L = [L, B]_{1,1} = LB + BL \tag{2.10}$$

for the pair (2.4) reads:

$$\frac{d}{dt}\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -2\left(\alpha_2 x_2^2 + \alpha_3 x_3^2\right) & (\alpha_1 + \alpha_2) x_1 x_2 & (\alpha_3 + \alpha_1) x_3 x_1 \\ (\alpha_1 + \alpha_2) x_1 x_2 & -2\left(\alpha_3 x_3^2 + \alpha_1 x_1^2\right) & (\alpha_2 + \alpha_3) x_2 x_3 \\ (\alpha_3 + \alpha_1) x_3 x_1 & (\alpha_2 + \alpha_3) x_2 x_3 & -2\left(\alpha_1 x_1^2 + \alpha_2 x_2^2\right) \end{pmatrix}. \tag{2.11}$$

The global equality (2.11) is impossible because the LHS is skew-symmetric and the RHS is symmetric. Then we restrict only to the upper-triangular system

$$\begin{cases} \dfrac{dx_1}{dt} = -(\alpha_2 + \alpha_3) x_2 x_3 \\ \dfrac{dx_2}{dt} = (\alpha_3 + \alpha_1) x_3 x_1 \\ \dfrac{dx_3}{dt} = -(\alpha_1 + \alpha_2) x_1 x_2 \end{cases} \tag{2.12}$$

because if $(x_1, x_2, x_3)$ is a solution of this system then $(y_1, y_2, y_3) = (-x_1, -x_2, -x_3)$ is solution for the lower-triangular system:

$$\begin{cases} \dfrac{dy_1}{dt} = (\alpha_2 + \alpha_3) y_2 y_3 \\ \dfrac{dy_2}{dt} = -(\alpha_3 + \alpha_1) y_3 y_1 \\ \dfrac{dy_3}{dt} = (\alpha_1 + \alpha_2) y_1 y_2 \end{cases}.$$

For our examples the system (2.12) becomes:

A

$$\begin{cases} \alpha_2 + \alpha_3 = -1 \\ \alpha_3 + \alpha_1 = 1 \\ \alpha_1 + \alpha_2 = -1 \end{cases}$$

with solution

$$\alpha_1 = \alpha_3 = \frac{1}{2},\ \alpha_2 = -\frac{3}{2}. \tag{2.13}$$

B

$$\begin{cases} \alpha_2 + \alpha_3 = -1 \\ \alpha_3 + \alpha_1 = -1 \\ \alpha_1 + \alpha_2 = k^2 \end{cases}$$

with solution

$$\alpha_1 = \alpha_2 = \frac{k^2}{2},\ \alpha_3 = -\frac{k^2}{2} - 1 \tag{2.14}$$

Using the boundary of parameter $k^2$ it results for this example:

$$-\frac{3}{2} < \alpha_3 < -1 < 0 < \alpha_1 = \alpha_2 < \frac{1}{2}. \tag{2.15}$$

In conclusion, examples A and B admit a Lax pair of so (3)-type with respect to the Jordan bracket. Systems of (2.5)-type we call *Lie-rigid body* and systems of (2.12)-type we call *Jordan-rigid body.*

The systems (2.12) admit another form. Deriving the first eq. it results:

$$\frac{d^2x_1}{dt^2} = -(\alpha_2 + \alpha_3)\, x_1 \left[(\alpha_3 + \alpha_1)\, x_3^2 - (\alpha_1 + \alpha_2)\, x_2^3\right] \tag{2.16}$$

and deriving again one obtains

$$\frac{d^3x_1}{dt^3} = \frac{dx_1}{dt}\frac{\frac{d^2x_1}{dt^2}}{x_1} + 4\,(\alpha_1 + \alpha_2)\,(\alpha_2 + \alpha_3)\,(\alpha_3 + \alpha_1)\, x_1^2\, \frac{dx_1}{dt}$$

or

$$\frac{x_1 \frac{d^3x_1}{dt^3} - \frac{dx_1}{dt}\frac{d^2x_1}{dt^2}}{x_1^2} = 4\,(\alpha_1 + \alpha_2)\,(\alpha_2 + \alpha_3)\,(\alpha_3 + \alpha_1)\, x_1\, \frac{dx_1}{dt}$$

which can be integrated

$$\frac{\frac{d^2x_1}{dt^2}}{x_1} = 2\,(\alpha_1 + \alpha_2)\,(\alpha_2 + \alpha_3)\,(\alpha_3 + \alpha_1)\, x_1^2 + C_1 \tag{2.17}$$

with $C_1$ a real constant. For $C_1 = 0$ two particular solutions are obtained in section 5, see relations (5.3).

Using these Jordan-Lax representations and $(1.3ii)$ we can write the Brockett perturbation

A

$$\begin{cases} \dfrac{dx_1}{dt} = x_2x_3 - 2x_1\left(x_1^2 - 2x_2^2 + x_3^2\right) \\ \dfrac{dx_2}{dt} = x_3x_1 + 6x_2^3 \\ \dfrac{dx_3}{dt} = x_1x_2 - 2x_3\left(x_1^2 - 2x_2^2 + x_3^2\right) \end{cases} \tag{2.18}$$

B

$$(2.19)\qquad \begin{cases} \dfrac{dx_1}{dt} = x_2x_3 + x_1\left[-2k^2x_1^2 - 2k^2x_2^2 + \left(k^2+3\right)x_3^2\right] \\ \dfrac{dx_2}{dt} = -x_3x_1 + x_2\left[-2k^2x_1^2 - 2k^2x_2^2 + \left(k^2+3\right)x_3^2\right] \\ \dfrac{dx_3}{dt} = -k^2x_1x_2 + x_3\left[\left(1-k^2\right)x_1^2 + \left(1-k^2\right)x_2^2 + 2\left(k^2+2\right)x_3^2\right] \end{cases}$$

Another remarkable quadratic homogeneous system is([9]):

$$(2.20)\qquad \begin{cases} \dfrac{dx_1}{dt} = x_2x_3 - (x_1)^2 \\ \dfrac{dx_2}{dt} = x_3x_1 - (x_2)^2 \\ \dfrac{dx_3}{dt} = x_1x_2 - (x_3)^2 \end{cases}$$

and because

$$B\left(x_1, -x_2, x_3\right) = \begin{pmatrix} 0 & -(x_3)^2 & -(x_2)^2 \\ (x_3)^2 & 0 & -(x_1)^2 \\ (x_2)^2 & (x_1)^2 & 0 \end{pmatrix}$$

it results that the Kasner system (2.18) is the upper-triangular part of relation:

$$\frac{dL}{dt} = \left[L, B\left(\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}\right)\right]_{1,1} + B\left(x_1, -x_2, x_3\right).$$

# 3 Brockett perturbation of trigonometric functions

A well-known property for the system of elliptic Jacobi function is that when $k^2 \to 0$ the solution $(x_1, x_2, x_3)$ of (2.9) satisfy:

$$(x_1, x_2, x_3) \to (\sin t, \cos t, 1)$$

and this fact is used, for example, in order to obtain a generalization of Newtonian equations of motion in [19].

As consequence, the limit for $k^2 \to 0$ of system (2.17), namely:

$$(3.1)\qquad \begin{cases} \dfrac{dx_1}{dt} = x_2x_3 + 3x_1x_3^2 \\ \dfrac{dx_2}{dt} = -x_3x_1 + 3x_2x_3^2 \\ \dfrac{dx_3}{dt} = x_3\left(x_1^2 + x_2^2 + 4x_3^2\right) \end{cases}$$

we call *the Brockett perturbation of trigonometric functions.*

The aim of this section is to obtain a first integral of this differential system. On this respect we multiply the first eq. with $x_1$ and the second eq. with $x_2$:

$$\begin{cases} x_1 \dfrac{dx_1}{dt} = x_1x_2x_3 + 3x_1^2x_3^2 \\ x_2 \dfrac{dx_2}{dt} = -x_1x_2x_3 + 3x_2^2x_3^2 \end{cases}$$

and adding these relations it results

$$\frac{d}{dt}\left(x_1^2 + x_2^2\right) = 6x_3^2\left(x_1^2 + x_2^2\right).$$

From the previous eq. and the last eq. (3.1) we have:

$$\frac{dx_3}{d\left(x_1^2 + x_2^2\right)} = \frac{x_3\left(x_1^2 + x_2^2 + 4x_3^2\right)}{6x_3^2\left(x_1^2 + x_2^2\right)} = \frac{2}{3\left(x_1^2 + x_2^2\right)}x_3 + \frac{1}{6x_3}. \tag{3.2}$$

The eq. (3.2) considered in the unknown $x_3$ as function of $x_1^2 + x_2^2$ is a Bernoulli type equation:

$$\frac{dx}{dt} + P(t)\,x = Q(t)\,x^n$$

with: $x = x_3, t = x_1^2 + x_2^2, P(t) = \dfrac{-2}{3t}, Q(t) = \dfrac{1}{6}, n = -1$. The standard transformation $z = x^{1-n}$, in our case $z = x^{1-(-1)} = x^2$, yields the equation:

$$\frac{dz}{dt} = 2x\frac{dx}{dt} = 2x\left(\frac{2}{3t}x + \frac{1}{6x}\right) = \frac{4}{3t}x^2 + \frac{1}{3} = \frac{4}{3t}z + \frac{1}{3}$$

which is a linear one and then has the solution:

$$z = x_3^2 = Ct^{\frac{4}{3}} - t = C\left(x_1^2 + x_2^2\right)^{\frac{4}{3}} - \left(x_1^2 + x_2^2\right) \tag{3.3}$$

with $C$ a real constant. The final relation:

$$C^3 = \frac{\left(x_1^2 + x_2^2 + x_3^2\right)^3}{\left(x_1^2 + x_2^2\right)^4} \tag{3.4}$$

is, in conclusion, a first integral for the system (3.1) of Brockett perturbation of trigonometric functions.

# 4 Jordan version of some classical Lax systems

At the beginning of section 2 there are given some examples of systems with usual Lax representation: the rigid body and Maxwell-Bloch equations. Is amazing that these systems appear also in the Jordan setting:

C From the rigid body system (2.2) and the system (2.12) it results:

$$\begin{cases} \alpha_2 + \alpha_3 = -a_1 \\ \alpha_3 + \alpha_1 = a_2 \\ \alpha_1 + \alpha_2 = -a_3 \end{cases} \tag{4.1}$$

with solution

(4.2) $$\alpha_1 = -a_3, \alpha_2 = 0, \alpha_3 = -a_1$$

and the Brockett perturbation

(4.3) $$\begin{cases} \dfrac{dx_1}{dt} = a_1 x_2 x_3 + x_1 \left[4a_3 x_1^2 + a_3 x_2^2 + (3a_1 + a_3) x_3^2\right] \\ \dfrac{dx_2}{dt} = a_2 x_3 x_1 + 3x_2 \left(a_3 x_1^2 + a_1 x_3^2\right) \\ \dfrac{dx_3}{dt} = a_3 x_1 x_2 + x_3 \left[(a_1 + 3a_3) x_1^2 + a_1 x_2^2 + 4a_1 x_3\right] \end{cases}.$$

D The system of Maxwell-Bloch equations is ([13], [14])

(4.4) $$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 x_1 \\ \dfrac{dx_3}{dt} = -x_1 x_2 \end{cases}$$

and comparing with (2.12) it follows

(4.5) $$\begin{cases} \alpha_2 + \alpha_3 = -\dfrac{1}{x_3} \\ \alpha_3 + \alpha_1 = 1 \\ \alpha_1 + \alpha_2 = 1 \end{cases}$$

with solution

(4.6) $$\alpha_1 = 1 + \frac{1}{2x_3}, \alpha_2 = \alpha_3 = -\frac{1}{2x_3}$$

and Brockett perturbation

(4.7) $$\begin{cases} \dfrac{dx_1}{dt} = x_2 - x_1 \left[\left(4 + \dfrac{2}{x_3}\right) x_1^2 + \left(1 - \dfrac{1}{x_3}\right) x_2^2 + \left(1 - \dfrac{1}{x_3}\right) x_3^2\right] \\ \dfrac{dx_2}{dt} = x_3 x_1 - x_2 \left[\left(3 + \dfrac{1}{x_3}\right) x_1^2 - \dfrac{2}{x_3} x_2^2 - 2x_3\right] \\ \dfrac{dx_3}{dt} = -x_1 x_2 - x_3 \left[\left(3 + \dfrac{1}{x_3}\right) x_1^2 - \dfrac{2}{x_3} x_2^2 - 2x_3\right] \end{cases}$$

# 5 More about Jordan-rigid body systems

Recall the property of Lax representation with respect to the eigenvalues of $L$ as first integrals. Unfortunately, this useful fact is not true in Jordan representation. So, the only non-zero eigenvalue of $L$ is:

(5.1) $$E = x_1^2 + x_2^2 + x_3^2$$

and then along the solutions of A and B we have:

$$\frac{dE}{dt} = \frac{\partial E}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial E}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial E}{\partial x_3}\frac{dx_3}{dt} =$$

$$= \begin{cases} \overset{A}{=} 2x_1 (x_2 x_3) + 2x_2 (x_3 x_1) + 2x_3 (x_1 x_2) = 6x_1 x_2 x_3 \neq 0 \\ \overset{B}{=} 2x_1 (x_2 x_3) + 2x_2 (-x_3 x_1) + 2x_3 \left(-k^2 x_1 x_2\right) = -2k^2 x_1 x_2 x_3 \neq 0. \end{cases}$$

The Jordan-rigid body systems are particular cases of homogeneous systems treated in detail in [10, p. 311]:

$$\begin{cases} \dot{z}_1 = \varepsilon_1 z_2 z_3 \\ \dot{z}_2 = \varepsilon_2 z_3 z_1 \\ \dot{z}_3 = \varepsilon_3 z_1 z_2 \end{cases}. \tag{5.2}$$

Following the conclusion of [10, p. 311-312] and [17] about the above systems it results the existence of two particular solutions of the form $z_k(t) = \frac{c_k}{t}$ with $c_k$ real constants given by:

$$\begin{cases} c_1 = \pm\dfrac{1}{\sqrt{-(\alpha_3+\alpha_1)(\alpha_1+\alpha_2)}} \\ c_2 = \pm\dfrac{1}{\sqrt{(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)}} \\ c_3 = \pm\dfrac{1}{\sqrt{-(\alpha_2+\alpha_3)(\alpha_3+\alpha_1)}}. \end{cases} \tag{5.3}$$

For our examples

A

$$\begin{cases} (x_1, x_2, x_3)(t) = \left(\dfrac{1}{t}, -\dfrac{1}{t}, \dfrac{1}{t}\right) \\ (x_1, x_2, x_3)(t) = \left(\dfrac{1}{t}, \dfrac{1}{t}, -\dfrac{1}{t}\right) \end{cases} \tag{5.4}$$

B

$$\begin{cases} (x_1, x_2, x_3)(t) = \left(\dfrac{1}{kt}, \dfrac{i}{kt}, \dfrac{i}{t}\right) \\ (x_1, x_2, x_3)(t) = \left(\dfrac{1}{kt}, -\dfrac{i}{kt}, -\dfrac{i}{t}\right). \end{cases} \tag{5.5}$$

Also, from the above citations, it results that systems (2.12) have two independent quadratic first integrals and then

$$H = \frac{1}{2}\left[\sigma_1 (x_1)^2 + \sigma_2 (x_2)^2 + \sigma_3 (x_3)^2\right]$$

as first integral if

$$\sigma_1 (\alpha_2 + \alpha_3) - \sigma_2 (\alpha_3 + \alpha_1) + \sigma_3 (\alpha_1 + \alpha_2) = 0 \tag{5.6}$$

which for $\sigma_1 = 1$ becomes

$$(\alpha_2+\alpha_3)-\sigma_2(\alpha_3+\alpha_1)+\sigma_3(\alpha_1+\alpha_2)=0$$

with solutions

$$\text{(5.7)}\qquad \begin{cases} \sigma_2=0\Rightarrow\sigma_3=-\dfrac{\alpha_2+\alpha_3}{\alpha_1+\alpha_2} \\ \sigma_3=0\Rightarrow\sigma_2=\dfrac{\alpha_2+\alpha_3}{\alpha_3+\alpha_1}. \end{cases}$$

Then a Jordan-rigid body system has the first integrals

$$\text{(5.8)}\qquad \begin{cases} H_1=\dfrac{1}{2}\left[(\alpha_3+\alpha_1)(x_1)^2+(\alpha_2+\alpha_3)(x_2)^2\right] \\ H_2=\dfrac{1}{2}\left[(\alpha_1+\alpha_2)(x_1)^2-(\alpha_2+\alpha_3)(x_3)^2\right] \end{cases}$$

which for our examples are ([6]):

A

$$\text{(5.9)}\qquad H_1=\frac{1}{2}\left(x_1^2-x_2^2\right),\ H_2=\frac{1}{2}\left(x_1^2-x_3^2\right)$$

B

$$\text{(5.10)}\qquad H_1=\frac{1}{2}\left(x_1^2+x_2^2\right),\ H_2=\frac{1}{2}\left(k^2x_1^2+x_3^2\right).$$

Unfortunately, these functions are not first integrals of associated Brockett perturbation. Indeed, from (2.16):

A

$$\begin{aligned} \frac{dH_1}{dt}&=-2x_1^2\left(x_1^2-2x_2^2+x_3^2\right)-6x_2^4 \\ \frac{dH_2}{dt}&=2\left(x_3^2-x_1^2\right)\left(x_1^2-2x_2^2+x_3^2\right)=-4H_2\left(x_1^2-2x_2^2+x_3^2\right) \end{aligned}$$

and from (2.17)

$$\begin{aligned} \frac{dH_1}{dt}&=\left(x_1^2+x_2^2\right)\left[-2k^2x_1^2-2k^2x_2^2+\left(k^2+3\right)x_3^2\right]= \\ &=2H_1\left[-4H_1+\left(k^2+3\right)x_3^2\right] \end{aligned}$$

$$\begin{aligned} \frac{dH_2}{dt}&=k^2x_1^2\left[-2k^2x_1^2-2k^2x_2^2+\left(k^2+3\right)x_3^2\right]+ \\ &+x_3^2\left[\left(1-k^2\right)x_1^2+\left(1-k^2\right)x_2^2+2\left(k^2+2\right)x_3^2\right]. \end{aligned}$$

The first integrals (5.8) gives for $x_1(t)$ the interpretation of inverse for an elliptic integral. So, from (5.8)

$$\text{(5.11)}\qquad \begin{cases} x_2^2=\dfrac{1}{\alpha_2+\alpha_3}\left[2H_1-(\alpha_3+\alpha_1)x_1^2\right] \\ x_3^2=\dfrac{1}{\alpha_3+\alpha_2}\left[(\alpha_1+\alpha_2)x_1^2-2H_2\right] \end{cases}$$

and then, the first eq. (2.12) reads

$$\frac{dt}{dx_1} = \frac{\alpha_2 + \alpha_3}{\sqrt{\left[2H_1 - (\alpha_3 + \alpha_1)\, x_1^2\right]\left[(\alpha_1 + \alpha_2)\, x_1^2 - 2H_2\right]}}$$

which yields

$$t = t\,(x_1) = \int_0^{x_1} \frac{(\alpha_2 + \alpha_3)\, du}{\sqrt{\left[2H_1 - (\alpha_3 + \alpha_1)\, u^2\right]\left[(\alpha_1 + \alpha_2)\, u^2 - 2H_2\right]}} \tag{5.12}$$

and also, the algebraic curve

$$(\alpha_2 + \alpha_3)^2\, y^2 = \left[2H_1 - (\alpha_3 + \alpha_1)\, x^2\right]\left[(\alpha_1 + \alpha_2)\, x^2 - 2H_2\right]. \tag{5.13}$$

For the system B this computation appears in [7, p. 138].

Again associated with the first integrals (5.8) we obtain a numerical integrator which preserves these integrals. Recall that for an ODE system $\dot{x}_i = \phi_i\,(x)$ the mid-point rule with step $h$ is giving by ([1]):

$$\frac{x_i^{n+1} - x_i^n}{h} = \phi_i\left(\frac{x^{n+1} + x^n}{2}\right).$$

For a Jordan-rigid body system this rule is

$$\begin{cases} x_1^{n+1} - x_1^n = -\dfrac{h}{4}\,(\alpha_2 + \alpha_3)\left(x_2^{n+1} + x_2^n\right)\left(x_3^{n+1} + x_3^n\right) \\ x_2^{n+1} - x_2^2 = \dfrac{h}{4}\,(\alpha_3 + \alpha_1)\left(x_3^{n+1} + x_3^n\right)\left(x_1^{n+1} + x_1^n\right) \\ x_3^{n+1} - x_3^n = -\dfrac{h}{4}\,(\alpha_1 + \alpha_2)\left(x_1^{n+1} + x_1^n\right)\left(x_2^{n+1} + x_2^n\right) \end{cases} \tag{5.14}$$

and let us prove that preserves the first integrals (5.8). Indeed

$$\begin{aligned} H_1^{n+1} - H_1^n &= \frac{1}{2}\,(\alpha_3 + \alpha_1)\left[\left(x_1^{n+1}\right)^2 - \left(x_1^n\right)^2\right] + \frac{1}{2}\left[\left(x_2^{n+1}\right)^2 - \left(x_2^n\right)^2\right] = \\ &= -\frac{h}{8}\,(\alpha_3 + \alpha_1)\,(\alpha_2 + \alpha_3)\left(x_1^{n+1} + x_1^n\right)\left(x_2^{n+1} + x_2^n\right)\left(x_3^{n+1} + x_3^n\right) + \\ &+ \frac{h}{8}\,(\alpha_2 + \alpha_3)\,(\alpha_3 + \alpha_1)\left(x_1^{n+1} + x_1^n\right)\left(x_2^{n+1} + x_2^n\right)\left(x_3^{n+1} + x_3^n\right) = 0 \end{aligned}$$

$$\begin{aligned} H_2^{n+1} - H_2^n &= \frac{1}{2}\,(\alpha_1 + \alpha_2)\left[\left(x_1^{n+1}\right)^2 - \left(x_1^n\right)^2\right] - \frac{1}{2}\,(\alpha_2 + \alpha_3)\left[\left(x_2^{n+1}\right)^2 - \left(x_2^n\right)^2\right] = \\ &= -\frac{h}{8}\,(\alpha_1 + \alpha_2)\,(\alpha_2 + \alpha_3)\left(x_1^{n+1} + x_1^n\right)\left(x_2^{n+1} + x_2^n\right)\left(x_3^{n+1} + x_3^n\right) + \\ &+ \frac{h}{8}\,(\alpha_2 + \alpha_3)\,(\alpha_1 + \alpha_2)\left(x_1^{n+1} + x_1^n\right)\left(x_2^{n+1} + x_2^n\right)\left(x_3^{n+1} + x_3^n\right) = 0. \end{aligned}$$

# 6 The $so\,(4)$-version

In this section we sketch the $so\,(4)$ version of Jordan-rigid body type systems. For

$$L=\begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{pmatrix}$$

and

$$B=B\,(\alpha_1,\ldots,\alpha_6)=\begin{pmatrix} 0 & \alpha_1 x_1 & \alpha_2 x_2 & \alpha_3 x_3 \\ -\alpha_1 x_1 & 0 & \alpha_4 x_4 & \alpha_5 x_5 \\ -\alpha_2 x_2 & -\alpha_4 x_4 & 0 & \alpha_6 x_6 \\ -\alpha_3 x_3 & -\alpha_5 x_5 & -\alpha_6 x_6 & 0 \end{pmatrix}$$

one have, using Maple:

$$LB=$$

$$\begin{pmatrix} -x_1^2\alpha_1-x_2^2\alpha_2-x_3^2\alpha_3 & -x_2\alpha_4x_4-x_3\alpha_5x_5 & x_1\alpha_4x_4-x_3\alpha_6x_6 & x_1\alpha_5x_5+x_2\alpha_6x_6 \\ -x_4\alpha_2x_2-x_5\alpha_3x_3 & -x_1^2\alpha_1-x_4^2\alpha_4-x_5^2\alpha_5 & -x_1\alpha_2x_2-x_5\alpha_6x_6 & -x_1\alpha_3x_3+x_4\alpha_6x_6 \\ x_4\alpha_1x_1-x_6\alpha_3x_3 & -x_2\alpha_1x_1-x_6\alpha_5x_5 & -x_2^2\alpha_2-x_4^2\alpha_4-x_6^2\alpha_6 & -x_2\alpha_3x_3-x_4\alpha_5x_5 \\ x_5\alpha_1x_1+x_6\alpha_2x_2 & -x_3\alpha_1x_1+x_6\alpha_4x_4 & -x_3\alpha_2x_2-x_5\alpha_4x_4 & -x_3^2\alpha_3-x_5^2\alpha_5-x_6^2\alpha_6 \end{pmatrix}$$

$$BL=$$

$$\begin{pmatrix} -x_1^2\alpha_1-x_2^2\alpha_2-x_3^2\alpha_3 & -x_4\alpha_2x_2-x_5\alpha_3x_3 & x_4\alpha_1x_1-x_6\alpha_3x_3 & x_5\alpha_1x_1+x_6\alpha_2x_2 \\ -x_2\alpha_4x_4-x_3\alpha_5x_5 & -x_1^2\alpha_1-x_4^2\alpha_4-x_5^2\alpha_5 & -x_2\alpha_1x_1-x_6\alpha_5x_5 & -x_3\alpha_1x_1+x_6\alpha_4x_4 \\ x_1\alpha_4x_4-x_3\alpha_6x_6 & -x_1\alpha_2x_2-x_5\alpha_6x_6 & -x_2^2\alpha_2-x_4^2\alpha_4-x_6^2\alpha_6 & -x_3\alpha_2x_2-x_5\alpha_4x_4 \\ x_1\alpha_5x_5+x_2\alpha_6x_6 & -x_1\alpha_3x_3+x_4\alpha_6x_6 & -x_2\alpha_3x_3-x_4\alpha_5x_5 & -x_3^2\alpha_3-x_5^2\alpha_5-x_6^2\alpha_6 \end{pmatrix}$$

which means that a $so\,(4)$ Jordan-rigid body system reads:

$$(6.1)\qquad \begin{cases} \dfrac{dx_1}{dt}=-\left(\alpha_2+\alpha_4\right)x_2x_4-\left(\alpha_3+\alpha_5\right)x_3x_5 \\ \dfrac{dx_2}{dt}=\left(\alpha_1+\alpha_4\right)x_1x_4-\left(\alpha_3+\alpha_5\right)x_3x_6 \\ \dfrac{dx_3}{dt}=\left(\alpha_1+\alpha_5\right)x_1x_5+\left(\alpha_2+\alpha_6\right)x_2x_6 \\ \dfrac{dx_4}{dt}=-\left(\alpha_1+\alpha_2\right)x_1x_2-\left(\alpha_5+\alpha_6\right)x_5x_6 \\ \dfrac{dx_5}{dt}=-\left(\alpha_1+\alpha_3\right)x_1x_3+\left(\alpha_4+\alpha_6\right)x_4x_6 \\ \dfrac{dx_6}{dt}=-\left(\alpha_2+\alpha_3\right)x_2x_3-\left(\alpha_4+\alpha_5\right)x_4x_5. \end{cases}$$

We treat only the main subject of equations, namely first integrals. Searching for

$$H=\frac{1}{2}\left(\rho_1x_1^2+\ldots+\rho_6x_6^2\right)$$

it results the system

$$(6.2)\qquad \begin{cases} \rho_1\left(\alpha_2+\alpha_4\right)-\rho_2\left(\alpha_1+\alpha_4\right)+\rho_4\left(\alpha_1+\alpha_2\right)=0 \\ \rho_1\left(\alpha_3+\alpha_5\right)-\rho_3\left(\alpha_1+\alpha_5\right)+\rho_5\left(\alpha_1+\alpha_3\right)=0 \\ \rho_2\left(\alpha_3+\alpha_6\right)-\rho_3\left(\alpha_2+\alpha_6\right)+\rho_6\left(\alpha_2+\alpha_3\right)=0 \\ \rho_4\left(\alpha_4+\alpha_6\right)-\rho_5\left(\alpha_4+\alpha_6\right)+\rho_6\left(\alpha_4+\alpha_5\right)=0. \end{cases}$$

# Appendix

Using Maple, for matrices (2.4) the following formulae hold:

$$L^2 = \left( \begin{array}{ccc} -x_2^2 - x_3^2 & x_1 x_2 & x_1 x_3 \\ x_1 x_2 & -x_3^2 - x_1^2 & x_2 x_3 \\ x_1 x_3 & x_2 x_3 & -x_1^2 - x_2^2, \end{array} \right)$$

$$L^2 B + B L^2 = \left( \begin{array}{ccc} 0 & \omega_3 x_3 & -\omega_2 x_2 \\ -\omega_3 x_3 & 0 & \omega_1 x_1 \\ \omega_2 x_2 & -\omega_1 x_1 & 0 \end{array} \right)$$

where

$$\begin{cases} \omega_1 = 2\alpha_1 x_1^2 + (\alpha_1 + \alpha_2)\, x_2^2 + (\alpha_1 + \alpha_3)\, x_3^2 \\ \omega_2 = (\alpha_1 + \alpha_2)\, x_1^2 + 2\alpha_2 x_2^2 + (\alpha_2 + \alpha_3)\, x_3^2 \\ \omega_3 = (\alpha_1 + \alpha_3)\, x_1^2 + (\alpha_2 + \alpha_3)\, x_2^2 + 2\alpha_3 x_3^2, \end{cases}$$

$$LBL = \left( \begin{array}{ccc} 0 & \rho x_3 & -\rho x_2 \\ -\rho x_3 & 0 & \rho x_1 \\ \rho x_2 & -\rho x_1 & 0, \end{array} \right)$$

where

$$\rho = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2.$$

$$B^2 = \left( \begin{array}{ccc} -\left(\alpha_2^2 x_2^2 + \alpha_3^2 x_3^2\right) & \alpha_1 \alpha_2 x_1 x_2 & \alpha_1 \alpha_3 x_1 x_3 \\ \alpha_1 \alpha_2 x_1 x_2 & -\left(\alpha_3^2 x_3^2 + \alpha_1^2 x_1^2\right) & \alpha_2 \alpha_3 x_2 x_3 \\ \alpha_1 \alpha_3 x_1 x_3 & \alpha_2 \alpha_3 x_2 x_3 & -\left(\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2\right) \end{array} \right).$$

The Maple plot of our systems:

A B

for $k^2 = \dfrac{1}{2}$ :

The Kasner system

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