On the Exponential Representation of $GL_{p,q}(1|1)$

Sultan A. Çelik and Dilek Gülten

Abstract

Quantum supergroups have some special properties in two dimensions. We give an exponential representation of a supermatrix in $GL_{p,q}(1|1)$ using a different method of the first author and co-worker [*Balkan Phys. Lett.* **5**, 32 (1997)] and we obtain the commutation relations satisfied by the matrix elements of the exponent. We show that these relations can be expressed in terms of an r-matrix. In the Appendix, we get the matrix representations of the generators of a supermatrix in $GL_{p,q}(1|1)$.

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1 Introduction

In fact quantum groups are, in mathematical sense, not groups they are quantum deformations of the well-known classical groups. However the quantum groups smoothly go over to the classical groups if the limit of deformation parameter has to be 1. The theory and applications of quantum groups have attracted a lot of attention among mathematicians and physicists. It should be added that the quantum deformations have been subsequently extended to supergroups and superalgebras [1-6].

For convenience of the reader, we recall the basic notations concerning the quantum supergroup $\mathrm{GL}_{p,q}(1|1)$. Manin [1] have considered quantum deformations of the concepts of a super-vector space (or the quantum superplane in two dimensions) and its dual which are defined in terms of variables satisfying certain quadratic algebra. Quantum supergroups as matrix supergroups can be viewed as linear transformations on the quantum coordinates which preserve the defining relations for quantum superplanes and their dual. In this point of view, the quantum supergroup $\mathrm{GL}_{p,q}(1|1)$ consists of all matrices in the form

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$$

where the matrix elements a, β, γ and d obey the following commutation relations

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$$
\begin{aligned}
a\beta &= q\beta a, & d\beta &= q\beta d,\\
a\gamma &= p\gamma a, & d\gamma &= p\gamma d,\\
q\beta\gamma + p\gamma\beta &= 0, & \beta^2 &= 0 = \gamma^2,
\end{aligned}
\tag{1}
$$

$$[a,d] = (p - q^{-1})\gamma\beta,$$

as usual, latin and greek letters denote even and odd matrix elements, respectively and $[\cdot,\cdot]$ stants for Lie bracket. Here the deformation parameters p and q are non-zero complex numbers and $pq \pm 1$ is assumed to be nonzero.

These commutation relations are equivalent to the equation

$$RT_1T_2 = T_2T_1R \tag{2}$$

where

$$T_1 = T \otimes_s I, \quad T_2 = I \otimes_s T, \tag{3}$$

and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & q & 1-pq & 0\\ 0 & 0 & p & 0\\ 0 & 0 & 0 & pq \end{pmatrix}. \tag{4}$$

Here we employ the convenient grading notation

$$(T_1)^{ij}{}_{kl} = T^i{}_k\delta^j{}_l, \quad (T_2)^{ij}{}_{kl} = (-1)^{\pi_i(\pi_j+\pi_l)}T^j{}_l\delta^i{}_k, \tag{5}$$

where the z_2-grade of the entries T^i_j of the matrix T is defined as $\pi_i + \pi_j$. The nonation $\otimes_s$ denotes the super-tensor product. (Note that the product of the supermatrices, i.e. the z_2-graded matrices, is the same as the non-graded case but for the tensor product of two graded matrices, we have (5). For this reason, we substitute $\otimes_s$ for the tensor notation $\otimes$.)

The inverse supermatrix is

$$T^{-1} = \begin{pmatrix} a^{-1} + a^{-1}\beta d^{-1}\gamma a^{-1} & -a^{-1}\beta d^{-1}\\ -d^{-1}\gamma a^{-1} & d^{-1} + d^{-1}\gamma a^{-1}\beta d^{-1} \end{pmatrix}, \tag{6}$$

provided that the matrix elements a and d of T are invertible. Note that the matrix T^{-1} does not belong to $\mathrm{GL}_{p,q}(1|1)$ but it belongs to $\mathrm{GL}_{p^{-1},q^{-1}}(1|1)$.

The quantum superdeterminant of T is defined as

$$\mathcal{D} = ad^{-1} - \beta d^{-1}\gamma d^{-1} \tag{7}$$

and it is easy to show that $\mathcal{D}$ is central.

In [6] it was shown that any element of $\mathrm{GL}_{p,q}(1|1)$ can be written as the exponential of a matrix. The entries of the matrix of the exponent belong to algebras depending to p, q.

In this work we will give an exponential representation of a supermatrix $T \in \mathrm{GL}_{p,q}(1|1)$ which is not the same with the work in [6] and with the help of the methods of finite differences for equations we will obtain the commutation relations satisfied by the matrix elements of the exponent. This will be presented in section 2. In section 3, we will show that these relations can be expressed in terms of an r-matrix which is the exponential of quantum R-matrix in (4). In the Appendix we will give the matrix representations of the generators of T and we will check that the relations (1) are satisfied.

2 Exponential expansion of T

In this section, we will give the exponential representation of a supermatrix $T \in \mathrm{GL}_{p,q}(1|1)$ which is not the same with [6]. At first we note that the matrix T^n, the n-th power of T, belongs to the quantum supergroup $\mathrm{GL}_{p^n,q^n}(1|1)$ if T is in $\mathrm{GL}_{p,q}(1|1)$. This is proved in [6]. This result suggests that a quantum supermatrix can be expressed as an exponential of a matrix whose entries are non-commutative.

Let T be a quantum supermatrix, i.e, the entries of T obey the (p,q)-commutation relations given by (1). Then one can write

$$T = e^M, \tag{8}$$

where

$$M = \begin{pmatrix} x & \mu \\ \nu & y \end{pmatrix}. \tag{9}$$

We want to obtain the commutation relations satisfied by the matrix elements of M. Because of this, we write the equation (8) in the form

$$M = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(T-I)^n \tag{10}$$

where I denotes the 2x2 unit matrix.

First we obtain explicit formulas for the matrix elements of the n-th power of $T - I$ in order to find the matrix elements of M. The following transformation plays an important role in our calculations. If f is any function of the matrix elements of T and of the deformation parameters p, q we define

$$f^\tau(a,\beta,\gamma,d,p,q) = f(d,\gamma,\beta,a,q,p). \tag{11}$$

Then the relations (1) are invariant under τ. So the function f^τ is well-defined.

Let

$$T - I = \begin{pmatrix} \hat{a} & \beta \\ \gamma & \hat{d} \end{pmatrix} \tag{12}$$

where

$$\hat{a} = a - 1, \quad \text{and} \quad \hat{d} = d - 1.$$

Then the matrix $(T-I)^n$ has the form

$$(T-I)^n = \begin{pmatrix} (a-1)^n + F_n\beta\gamma & G_n\beta \\ G_n^\tau\gamma & (d-1)^n + F_n^\tau\gamma\beta \end{pmatrix} \tag{13}$$

with some F_n and G_n.

The proof can be done by induction on n. It is obviously true for $n=1$. Let the formula (13) be true for $n=k$. Using the fact that

$$X^{k+1} = X^k X$$

for any square matrix X, we have

$$(T-I)^{n+1} = \begin{pmatrix} \hat{a}^{k+1} + F_k\beta\gamma\hat{a} + G_k\beta\gamma & \hat{a}^k\beta + G_k\beta\hat{d} \\ G_k^\tau\gamma\hat{a} + \hat{d}^k\gamma & \hat{d}^{k+1} + F_k^\tau\gamma\beta\hat{d} + G_k^\tau\gamma\beta \end{pmatrix}. \tag{14}$$

Thus it is obvious that if $(T-I)^n$ satisfies the reguired property, $(T-I)^{n+1}$ will satisfy it too.

The formula (14) gives the following equation for G_n:

$$G_{n+1} = G_n(q^{-1}d - 1) + (a-1)^n. \tag{15}$$

This equation may be solved using the methods of finite differences for equations as in [7]. In this way, one finds

$$(G_n)_h = C_1(q^{-1}d-1)^n \tag{16}$$

as the homegenous solution of (15). The particular solution of (15) is in the form

$$(G_n)_p = C_2(a-1)^n. \tag{17}$$

Since the particular solution must satisfy the equation (15), with the that one gets

$$C_2 = (a - q^{-1}d)^{-1}. \tag{18}$$

Therefore the solution of (15) is

$$G_n = C_1(q^{-1}d-1)^n + (a-q^{-1}d)^{-1}(a-1)^n. \tag{19}$$

The initial condition according to (13) is $G_1 = 1$. Consequently the solution of (15) formally is

$$G_n = (a-q^{-1}d)^{-1}\{(a-1)^n - (q^{-1}d-1)^n\}. \tag{20}$$

Then

$$G_n\beta = \frac{(a-1)^n - (q^{-1}d-1)^n}{(a-q^{-1}d)}\beta. \tag{21}$$

Note that the matrix elements a and d of T behave as commuting quantities when are in a product case by β or γ. Thus it is not necessary to order the arguments in (21).

It is verify that the formula (21) is the same with

$$G_n\beta = \sum_{j=1}^{n-1}(a-1)^{n-j-1}(q^{-1}d-1)^j\beta. \tag{22}$$

This formula was given in [6]. Thus, using the transformation (11) we can write

$$G_n^\tau \gamma = \frac{(d-1)^n - (p^{-1}a-1)^n}{d-p^{-1}a}\gamma. \tag{23}$$

Now we will obtain the F_n from (14). By (13) one can write

$$F_{n+1} = F_n(\lambda^{-1}a-1) + G_n, \quad \lambda = pq. \tag{24}$$

The homegenous solution of (24) is

$$(F_n)_h = K_1(\lambda^{-1}a-1)^n. \tag{25}$$

But the particular solution of (24) must be in the form

$$(F_n)_p = (a-q^{-1}d)^{-1}\left[K_2(a-1)^n - K_3(q^{-1}d-1)^n\right], \tag{26}$$

since there is two expressions in G_n. The initial condition for F_n is $F_1 = 0$. Hence the solution of (24) formally is

$$F_n\beta = q^2\left\{\frac{(q^{-1}d-1)^n}{(qa-d)(p^{-1}a-d)} + \frac{1}{(p^{-1}-q)a}\left(\frac{(p^{-1}q^{-1}a-1)^n}{p^{-1}a-d} - \frac{(a-1)^n}{qa-d}\right)\right\}\beta. \tag{27}$$

Therefore with (11) one can find

$$F_n^\tau\gamma = p^2\left\{\frac{(p^{-1}a-1)^n}{(a-pd)(a-q^{-1}d)} + \frac{1}{(q^{-1}-p)d}\left(\frac{(p^{-1}q^{-1}d-1)^n}{q^{-1}d-a} - \frac{(d-1)^n}{pd-a}\right)\right\}\gamma. \tag{28}$$

A careful reader could ask why β appears in the formula (27). The answer is in which to write orderly the formulas and also to use the note in below of (21).

We now easily obtain the expressions for the matrix elements of M in terms of T.

$$x = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\{(a-1)^n + F_n\beta\gamma\} = \ln a + f(X)\beta\gamma \tag{29}$$

where

$$\begin{aligned} f(X) &= f(a,\beta,\gamma,d,p,q) = pq\frac{\ln(da^{-1})}{(a-pd)(a-q^{-1}d)}\beta\gamma \\ &+ \frac{pq^2}{(pq-1)a}\left(\frac{p\ln p}{a-pd} - \frac{q^{-1}\ln q}{a-q^{-1}d}\right)\beta\gamma. \end{aligned} \tag{30}$$

Similarly, one has

$$\mu = \frac{\ln(qad^{-1})}{a-q^{-1}d}\beta. \tag{31}$$

Applying τ in (11) we obtain the other elements of M:

$$y = x^\tau = \ln d + [f(X)]^\tau\gamma\beta, \tag{32}$$

and

$$\nu = \frac{\ln(pda^{-1})}{d - p^{-1}a}\gamma. \tag{33}$$

Now one can verify that the matrix elements of M satisfy the following relations:

$$\begin{aligned} &[x,\mu] = h_1\mu, \quad [y,\mu] = h_1\mu, \quad \mu^2 = 0,\\ &[x,\nu] = h_2\nu, \quad [y,\nu] = h_2\nu, \quad \nu^2 = 0\\ &[x,y] = 0, \qquad \mu\nu + \nu\mu = 0 \end{aligned} \tag{34}$$

where

$$h_1 = \ln q \quad \text{and} \quad h_2 = \ln p. \tag{35}$$

Now we want to show that if the matrix elements of M in (9) satisfy the algebra (34) then $T = e^M \in \mathrm{GL}_{p,q}(1|1)$. First we will derive the explicit form of M^n, the n-th power of M, by methods of equations in finite differences which we used the preceding to find the matrix elements of $(T-I)^n$. It is easy to prove by induction that M^n has the form

$$M^n = \begin{pmatrix} x^n - \mu\nu U_n & \mu V_n \\ \nu V_n^\tau & y^n - \nu\mu U_n^\tau \end{pmatrix} \tag{36}$$

where the transformation τ is given by

$$\tau : (x,\mu,\nu,y,h_1,h_2) \mapsto (y,\nu,\mu,x,h_2,h_1)$$

as just (11). Since $M^{n+1} = MM^n$ we have

$$M^{n+1} = \begin{pmatrix} x^{n+1} - x\mu\nu U_n + \mu\nu V_n^\tau & x\mu V_n + \mu y^n \\ \nu x^n + y\nu V_n^\tau & y^{n+1} - y\nu\mu U_n^\tau + \nu\mu V_n \end{pmatrix}. \tag{37}$$

Hence we have the following recurence relations for U_n and V_n:

$$U_{n+1} = (x + h_1 + h_2)U_n - V_n^\tau, \tag{38}$$

$$V_{n+1} = (x + h_1)V_n + y^n. \tag{39}$$

Using the methods of finite differences for equations, it is easy to verify that the solution of (39) has in the form

$$V_n = \frac{(x+h_1)^n - y^n}{x - y + h_1}. \tag{40}$$

Similarly, one has

$$U_n = \left(\frac{x}{h_1+h_2} - \frac{y+h_2}{w}\right)\frac{(x+h_1+h_2)^{n-1}}{w^\tau} - \frac{x^n}{(h_1+h_2)w^\tau} + \frac{(y+h_2)^n}{ww^\tau} \tag{41}$$

where

$$w = x - y + h_1 \tag{42}$$

or with the required arrangements one gets

$$U_n = -\frac{(x+h_1+h_2)^n}{(h_1+h_2)w} - \frac{x^n}{(h_1+h_2)w^\tau} + \frac{(y+h_2)^n}{ww^\tau}. \tag{43}$$

We easily obtain the expressions for the matrix elements of T in terms of M, using the formula (8):

$$\begin{aligned} a &= e^x + \frac{\mu\nu}{ww^\tau}(\Gamma e^x - pe^y), \\ d &= e^y + \frac{\nu\mu}{ww^\tau}(\Gamma^\tau e^y - qe^x) = a^\tau, \\ \beta &= \frac{\mu}{w}(qe^x - e^y), \\ \gamma &= \tfrac{\nu}{w^\tau}(pe^y - e^x) = \beta^\tau \end{aligned} \tag{44}$$

where

$$\Gamma = \frac{h_1 + pqh_2}{h_1+h_2} + \frac{1-pq}{h_1+h_2}(x-y).$$

Now one can easily prove that the matrix elements a, β, γ and d of T satisfy the relations (1). For simplicity, we prove the invariance of one of the relations in (1) here. Proofs of the remaining relations are similar. Since $xy = yx$, one get

$$\begin{aligned} a\beta &= e^x \frac{\mu}{w}(qe^x - e^y) \\ &= \frac{q}{w}\mu(qe^x - e^y)e^x = q\beta a. \end{aligned}$$

Of course, using the formula (8), one can obtain the matrix elements of T^{-1}, too. To this end we will use the following simple trick. First we note that the matrix elements of $(-M)$ satisfy the commutation relations (34) with $h_1' = -h_1$ and $h_2' = -h_2$. By definition, $T^{-1} = e^{-M}$. Hence, if $T^{-1} = \begin{pmatrix} \tilde{a} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{d} \end{pmatrix}$, we have the following equations

$$\begin{aligned} \tilde{a} &= e^{-x} + \frac{\mu\nu}{pqww^\tau}(\Gamma^\tau e^{-x} - qe^{-y}), \\ \tilde{d} &= e^{-y} + \frac{\nu\mu}{pqww^\tau}(\Gamma e^{-y} - pe^{-x}), \\ \tilde{\beta} &= \frac{\mu}{w}(q^{-1}e^{-x} - e^{-y}), \\ \tilde{\gamma} &= \frac{\nu}{w^\tau}(p^{-1}e^{-y} - e^{-x}). \end{aligned} \tag{45}$$

We also find the identity

$$\mathcal{D} = e^{x-y} \tag{46}$$

using the relations (44) into (7). Note that it is easy verify that the super-trace of M which is defined by

$$strM = x - y$$

is central element of the algebra (34).

We finish this section the following observations. Let us consider the associative algebra (34). One can prove that for $h_1 \neq h_2$ they are all invariant under the commutation relations in (34). Indeed, in this case one can make the following change of basis:

$$X = h_1 x - h_2 y, \tag{47}$$

$$Y = h_2 x - h_1 y. \tag{48}$$

In the new basis the commutation relations are:

$$\begin{aligned} &[X, \mu] = h_1(h_1 - h_2)\mu, \quad &&[X, \nu] = h_2(h_1 - h_2)\nu, \\ &[Y, \mu] = h_1(h_2 - h_2)\mu, \quad &&[Y, \nu] = h_2(h_2 - h_1)\nu, \\ &[X, Y] = 0, &&[\mu, \nu]_+ = 0 \end{aligned} \tag{49}$$

where

$$[A, B]_+ = AB + BA.$$

We rescale

$$U = \frac{1}{h_1 - h_2} X, \quad V = \frac{1}{h_2 - h_1} Y, \tag{50}$$

and find

$$\begin{aligned} &[U, \mu] = h_1 \mu, \quad &&[U, \nu] = h_2 \nu, \\ &[V, \mu] = h_1 \mu, \quad &&[V, \nu] = h_2 \nu, \\ &[U, V] = 0, &&[\mu, \nu]_+ = 0 \end{aligned} \tag{51}$$

These relations are the same with (34). This proves that all algebras with (34) have the same structure if $h_1 \neq h_2$. For $h_1 = h_2$, the quantity X in (47) equal to $strM$ (and also Y) so that the relations (49) become trivial because it is central element for the algebra (34).

If we make the following change of basis

$$X' = h_1 x + h_2 y, \quad Y' = h_2 x + h_1 y, \tag{52}$$

in the new basis the commutation relations are:

$$\begin{aligned} &[X', \mu] = h_1(h_1 + h_2)\mu, \quad &&[X', \nu] = h_2(h_1 + h_2)\nu, \\ &[Y', \mu] = h_1(h_2 + h_2)\mu, \quad &&[Y', \nu] = h_2(h_2 + h_1)\nu, \\ &[X', Y'] = 0, &&[\mu, \nu]_+ = 0. \end{aligned} \tag{53}$$

Again, we rescale the X' and Y' in the form

$$U' = \frac{2}{(h_1 + h_2)^2} X', \quad V' = \frac{2}{(h_1 + h_2)^2} Y', \tag{54}$$

respectively. Then one obtains

$$
\begin{aligned}
&[U', \mu] = \frac{2h_1}{h_1+h_2}\mu, \quad && [U', \nu] = \frac{2h_2}{h_1+h_2}\nu, \\
&[V', \mu] = \frac{2h_1}{h_1+h_2}\mu, \quad && [V', \nu] = \frac{2h_2}{h_1+h_2}\nu, \\
&[U', V'] = 0, && [\mu, \nu]_+ = 0.
\end{aligned}
\tag{55}
$$

This algebra is the same with in [6].

3 The r-matrix for exponential form

In this section we will show that the (h_1, h_2)-commutation relations satisfied by the matrix elements of the exponent matrix M in (9) can be also obtained by an r-matrix which is the exponential of quantum R-matrix in (4). As recall from sect. 1, the (p,q)-commutation relations are equivalent to the equation $RT_1T_2 = T_2T_1R$.

Now we write, for the R-matrix in (4),

$$R = e^r \tag{56}$$

where r is a 4x4 matrix. To find the matrix elements of r, if one write the exponent as

$$r = \ln R,$$

then we obtain

$$
r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_1 & -(h_1+h_2) & 0 \\ 0 & 0 & h_2 & 0 \\ 0 & 0 & 0 & h_1+h_2 \end{pmatrix},
\tag{57}
$$

which is the term of the first order of smallness in the R-matrix. Decomposing the equation (2) to the second order, we find formula

$$[M_1, M_2] = [M_1 + M_2, r] \tag{58}$$

where the matrices M_1 and M_2 have in the form (5), respectively. To obtain this formula, we assumed that $T = e^M$ for the term of the first order in T and that the elements of M are of the same order of smallness as h_1 and h_2 (as recall $q = e^{h_1}$ and $p = e^{h_2}$). The formula (58) written explicitly, reproduce (34).

4 Conclusion

In this work, one has written a supermatrix is in $\mathrm{GL}_{p,q}(1|1)$ as the exponential form of a matrix whose entries are noncommutative and to obtain the commutation relations satisfied by the matrix elements of the exponent matrix, using the finite differences methods for equations one has explicitly gets the matrix elements of the n-th power of the required matrices.

The exponential form which is considered in this work is not the same in Ref. 6. Therefore the commutation relations satisfied by the matrix elements of the exponent matrix are not the same, too. However, using a special transformation the algebra which is obtained in this paper identified with the algebra in [6] for the exponent matrix.

Finally, the relations satisfied by the matrix elements of the exponent matrix are expressed in terms of an r-matrix.

In this work will be finished the following observation: In section 1 it is noted that the relations satisfied by the matrix elements of a quantum supermatrix can be expressed in terms of an R-matrix, using the convention super-tensor product (see, equ.s (2-5)). Of course, the relations (1) can be also obtained with the following way, again with an $\hat{R}$-matrix. Recall that for the tensor product of two supermatrices we have

$$(A \otimes B)(C \otimes D) = (-1)^{\pi_B \pi_C}(AC \otimes BD).$$

Let us consider the matrix

$$\hat{R} = \begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & p^{-1} - q & 1 & 0 \\ 0 & qp^{-1} & 0 & 0 \\ 0 & 0 & 0 & p^{-1} \end{pmatrix}.$$

Then one can easily verify that the relations (1) equivalent to the equation

$$\hat{R}(T \otimes T) = (T \otimes T)\hat{R}.$$

Here, the explicit form of $T \otimes T$ is

$$T \otimes T = \begin{pmatrix} a^2 & a\beta & -\beta a & \beta^2 \\ a\gamma & ad & \beta\gamma & -\beta d \\ -\gamma a & -\gamma\beta & da & -d\beta \\ -\gamma^2 & -\gamma d & -d\gamma & d^2 \end{pmatrix}.$$

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Appendix. Here we will give the matrix representations of the generators of the supermatrix T is in $\mathrm{GL}_{p,q}(1|1)$ and we will check that the relations (1) are satisfied.

Let M be the set of all 4x4 matrices with complex entries such that only in the first row and on the diagonal of their have non-zero elements. Consider the following elements of M:

$$a = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & p^{-1} & 0 & 0 \\ 0 & 0 & pq^{-1} & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$d = \begin{pmatrix} 1 & 0 & -qp^{-1} & 0 \\ 0 & p^{-1} & 0 & 0 \\ 0 & 0 & qp^{-1} & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A1}$$

Then one can easily check that these matrices satisfy the relations (1). In this case, the superdeterminant in (7) takes the form

$$\mathcal{D} = p^2 q^{-2}. \tag{A2}$$

Hence one finds

$$\ln \mathcal{D} = 2(h_2 - h_1). \tag{A3}$$

In other hand, by (29, 31-33) we obtain

$$x = \begin{pmatrix} 0 & 0 & 1+h_1 & 0 \\ 0 & -h_2 & 0 & 0 \\ 0 & 0 & h_2 - h_1 & 0 \\ 0 & 0 & 0 & -h_1 \end{pmatrix}, \qquad \mu = \begin{pmatrix} 0 & 0 & 0 & 1+2h_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$y = \begin{pmatrix} 0 & 0 & -(1+h_1) & 0 \\ 0 & -h_2 & 0 & 0 \\ 0 & 0 & h_1 - h_2 & 0 \\ 0 & 0 & 0 & -h_1 \end{pmatrix}, \qquad \nu = \begin{pmatrix} 0 & 1+2h_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A4}$$

We can easily verify that these matrices satisfy the algebra (34). In this case, the super-trace is

$$str = 2(h_2 - h_1). \tag{A5}$$

Comparing (A4) with (A5), we see that

$$\ln \mathcal{D} = str. \tag{A6}$$

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Yildiz Technical University,
Department of Mathematics
34210 Davutpasa-Esenler, Istanbul, TURKEY.
E-mail address: celik@yildiz.edu.tr