The Minimum Number of Zeros of Lipschitz-Killing Curvature

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Abstract

In the previous paper [11] some sufficient topological conditions, expressed in terms of homotopy groups, of two given differentiable manifolds M, N in order that the φ-category of the pair (M, N) be infinite, are given. In the more recent paper [12] it is proved that, in the same topological conditions, the φ-category is actually infinite uncountable. As an application we observe in this paper that a lower bound for the minimum number of points of zeros of the Lipschitz-Killing curvature of some m-dimensional manifold immersible in $\mathbf{R}^{m+k}$, with respect to all this kind of immersions, can be given in terms of φ-category.

Mathematics Subject Classification: 53Q99, 53Q05, 55P99, 53P65, 55Q05
Key words: Lipschitz-Killing curvature, homotopy group, critical set, immersiability category

1 Introduction

Recall that the φ-category of the pair (M, N) of closed manifolds is defined as

$$\varphi(M, N) = \min\{\#C(f) \mid f \in C^{\infty}(M, N)\},$$

where $C(f)$ is the critical set of f. A survey containing general results concerning φ-category is contained in the paper [1] (see also [2] and [3]).

Let M^m be a differentiable manifold immersible in $\mathbf{R}^{m+k}$; let $f : M \to \mathbf{R}^{m+k}$ be an immersion and

$$v \in (df)_p(T_p(M))^{\perp} \cap S^{m+k-1},$$

where $p \in M$ is a given point. Consider the second fundamental form $\phi^v_{f,p}$ of f and the projections $N_f : \mathcal{N}^{f,1} \to S^{m+k-1}$, $\pi : \mathcal{N}^{f,1} \to M$, where

$$\mathcal{N}^{f,1} = \{(x, v) \in M \times S^{m+k-1} \ : \ v \perp (df)_p(T_p(M))\} =$$

$$= \bigcup_{p \in M} \Big(\{p\} \times \big(S^{m+k-1} \cap (df)_p(T_p(M))^{\perp}\big)\Big).$$

Balkan Journal of Geometry and Its Applications, Vol.7, No.2, 2002, pp. 1-6.
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Observe that π is a fibration with fibre S^{k-1}. Recall that the *Lipschitz-Killing curvature* of f at the point $p \in M$ in the direction v is defined as

$$L_{f,v} = \frac{\det(\phi^v_{f,p})}{\det(g_f(p))}$$

and that $\phi^{v_0}_{f,p_0}$ is degenerated if and only if (p_0, v_0) is a critical point of N_f (for more details we refer to the books [4] and [6]). Therefore

$$C(N_f) = \{(p, v) \in \mathcal{N}^{f,1} \ : \ L_{f,v}(p) = 0\},$$

that is the φ-category of the pair $(\mathcal{N}^{f,1}, S^{m+k-1})$ is a lower bound for the minimum number of points of zeros of the Lipschitz-Killing curvature with respect to all immersions of M in $\mathbf{R}^{m+k}$. This fact can be shortly written by means of the inequality

$$L_k(M) \geq \varphi(\mathcal{N}^{f,1}, S^{m+k-1}),$$

where

$$L_k(M) = \min\{\#C(N_f) \ : \ f \in Imm^k(M)\}$$

is the so called L_k-*category* or the *immersiability category* of M, and $Imm^k(M)$ is the set of all immersions of M into $\mathbf{R}^{m+k}$. The immersiability category $L_k(M)$ is studied in the case of surfaces and $k = 1$ in the recent second author's paper [12]. To prove that the $L_k(M)$ is infinite uncountable in certain situations, we will use the following theorem.

Theorem 1.1 *([12]) Let M, N be compact connected differentiable manifolds of the same dimension m.*

(i) If $m \geq 3$ and $\pi_1(M)$ cannot be embedded as a subgroup in $\pi_1(N)$, then $\varphi(M, N) = \aleph_1$;

(ii) If $m \geq 4$ and $\pi_q(M) \ncong \pi_q(N)$ for some $q \in \{2, \ldots, m-2\}$, then $\varphi(M, N) = \aleph_1$.

Let us mention that Theorem 1.1 is improving ([11], Theorem 3.1) which states that, under the same hypothesis, the φ-category of the pair (M, N) is just $\geq \aleph_0$.

2 The implications of Theorem 1.1 on the L_k-category

Because the conditions of Theorem 1.1 are expressed in terms of homotopy groups of the given manifolds and our lower bound for the minimum number of points of zeros of the Lipschitz-Killing curvature is $\varphi(\mathcal{N}^{f,1}, S^{m+k-1})$, we will try to relate in what follows the homotopy groups of $\mathcal{N}^{f,1}$ with those of M. We denote by $k(M)$ the smallest natural number such that M is immersible in $\mathbf{R}^{m+k(M)}$. By means of Whitney's theorem, observe that $k(M) \leq 2m + 1$ and if M^m is a compact manifold, then $k(M) \leq 2m - 1$. Using [9], Theorem 1 and the proof of [9], Theorem 2 one can see some lower bounds for $k(G_{2,n})$ and $k(G_{3,n})$. For some lower bounds of $k(P^n(\mathbf{R}))$ and of $k(P^n(\mathbf{C}))$, $k(P^n(\mathbf{H}))$ see [8] and [7] respectively.

Theorem 2.1 *Let M^m be a connected closed manifold.*

(i) If $m \geq 1$ and $\pi_1(M)$ is not trivial, then $L_k(M) = \aleph_1$, $\forall k \geq \max\{3, k(M)\}$;

(ii) If $m \geq 2$ and $\pi_q(M)$ is not trivial for some for some $q \in \{2, \ldots, k-1\}$, then $L_k(M) \geq \aleph_1$, $\forall k \geq \max\{3, k(M)\}$.

Proof. The homotopy sequence of the fibration $S^{k-1} \hookrightarrow \mathcal{N}^{f,1} \stackrel{\pi}{\to} M$ is

$$\cdots \to \pi_r(S^{k-1}) \to \pi_r(\mathcal{N}^{f,1}) \to \pi_r(M) \to \pi_{r-1}(S^{k-1}) \to \cdots.$$

Because the homotopy groups $\pi_q(S^{k-1}), \pi_{q-1}(S^{k-1})$ of the sphere are trivial for $q \in \{1, \ldots, k-2\}$ and $\pi_{k-1}(S^{k-1}) \simeq \mathbf{Z}$ it follows that $\pi_q(\mathcal{N}^{f,1}) \simeq \pi_q(M)$ for all $q \in \{1, \ldots, k-2\}$ and the morphism $\pi_{k-1}(\mathcal{N}^{f,1}) \to \pi_{k-1}(M)$, induced by the projection $\pi : \mathcal{N}^{f,1}) \to M$ is surjective. Therefore, to prove (i) apply Theorem 1.1 (i), taking into accout that $\pi_1(M) \ncong 0 \simeq \pi_1(S^{m+k-1})$. Annalogously, to prove (ii) apply theorem 1.1 (ii), taking into accout that for some for some $q \in \{2, \ldots, k-1\}$ we have $\pi_q(M) \ncong 0 \simeq \pi_q(S^{m+k-1})$.

Theorem 2.2 *Let M^m be a connected closed manifold immersible in $\mathbf{R}^{m+2}$.*

(i) If $m \geq 2$ and $\pi_1(M)$ is not trivial, then $L_2(M) = \aleph_1$;

(ii) If $m \geq 3$ and $\pi_2(M) \ncong \mathbf{Z}$, then $L_2(M) = \aleph_1$;

(iii) If $m \geq 4$ and $\pi_q(M)$ is not trivial for some for some $q \in \{3, \ldots, m-1\}$, then $L_2(M) = \aleph_1$.

Proof. The homotopy sequence of the fibration $S^1 \hookrightarrow \mathcal{N}^{f,1} \stackrel{\pi}{\to} M$ is

$$\cdots \to \pi_r(S^1) \to \pi_r(\mathcal{N}^{f,1}) \to \pi_r(M) \to \pi_{r-1}(S^1) \to \cdots.$$

Because the homotopy groups $\pi_q(S^1)$ of the circle are trivial for $q = 0$ and $q \geq 2$ and $\pi_1(S^1) \simeq \mathbf{Z}$, it follows that the morphism $\pi_1(\mathcal{N}^{f,1}) \to \pi_1(M)$ induced by the projection $\pi : \mathcal{N}^{f,1} \to M$ is surjective, meaning that $\pi_1(M) \ncong 0$ implies $\pi_1(\mathcal{N}^{f,1}) \ncong 0 \simeq \pi_1(S^{m+1})$. Therefore using Theorem 1.1 (i), it follows that $L_2(M) = \aleph_1$ and the statement (i) is proved. To prove the second statement, observe that the morphism

$$\pi_2(\mathcal{N}^{f,1}) \to \pi_2(M)$$

induced by the projection $\pi : \mathcal{N}^{f,1} \to M$ is injective and

$$\pi_2(M)/\pi_2(\mathcal{N}^{f,1}) \simeq \mathbf{Z}.$$

Combining this observation with the hypothesis of (ii) it follows that

$$\pi_2(\mathcal{N}^{f,1}) \neq 0 \simeq \pi_2(S^{m+1}).$$

Therefore using Theorem 1.1 (ii) conclude that $L_2(M) = \aleph_1$.

The statement (iii) follows immediately from Theorem 1.1 (ii) taking into account that for $q \in \{3, \ldots, m-1\}$ we have $\pi_q(\mathcal{N}^{f,1}) \simeq \pi_r(M) \ncong 0 \simeq \pi_q(S^{m+1})$.

3 Some manifolds with infinite uncountable L_k-category

In this section we are going to use the results of the previous section to obtain some manifolds with infinite L_k-category.

Proposition 3.1 *(i) For $m \geq 1$ we have that*

$$L_k(P^m(\mathbf{R})) = \aleph_1,\ \forall k \geq \max\{3, k(P^m(\mathbf{R}))\};$$

(ii) For $m \geq 1$ we have that

$$L_k(L^{2m-1}(q_1, \ldots, q_p)) = \aleph_1,\ \forall k \geq \max\{3, k(L^{2m-1}(q_1, \ldots, q_p))\};$$

(iii) For $m \geq 2$ we have that $L_k(SO_m) \geq \aleph_1,\ \forall k \geq \max\{3, k(SO_m)\}$.

Proof. Indeed, for $m \geq 3$ we have

$$\pi_1(P^2(\mathbf{R})) \simeq \pi_1(P^m(\mathbf{R})) \simeq \pi_1(SO_m) \simeq \mathbf{Z}_2 \neq 0,$$

while $P^1(\mathbf{R}) \sim_{Top} SO_2 \sim_{Top} S^1$ so that

$$\pi_1(P^1(\mathbf{R})) \simeq \pi_1(SO_2) \simeq \mathbf{Z} \neq 0,$$

and

$$\pi_1(L^{2m-1}(q_1, \ldots, q_p)) \simeq \mathbf{Z}_m \neq 0,\ \forall m \geq 1.$$

Therefore using Theorem 2.1 (i), proposition 3.1 is completely proved.

Proposition 3.2 *(i) $L_k(G_{p,l}) = \aleph_1,\ \forall k \geq \max\{3, k(G_{p,l})\}$;*
(ii) If $m \geq 3$, then $L_k(Spin_m) = \aleph_1,\ \forall k \geq \max\{4, k(Spin_m)\}$.

Proof. (i) Let us assume that $p \geq l$. In this case it follows, using [13], Theorem 10.16, pp. 204, that

$$\pi_q(G_{p,l}) \simeq \pi_q(V_{p+l,l}) \oplus \pi_{q-1}(O_l),\ \forall q.$$

On the other hand the inclusion $SO_l \hookrightarrow O_l$ induces isomorphism of homotopy groups $\pi_i(SO_l) \to \pi_i(O_l)$, so that for $l = 2$ we have $\pi_1(O_2) \simeq \pi_1(SO_2) \simeq \mathbf{Z}$ and for $l \geq 3$ we have $\pi_1(O_l) \simeq \pi_1(SO_l) \simeq \mathbf{Z}_2$. In any case $\pi_2(G_{p,l})$ has as a subgroup either $\mathbf{Z}$ or $\mathbf{Z}_2$, that is, $\pi_2(G_{p,l}) \ncong 0$, meaning by Theorem 2.1 (ii), that $L_k(G_{p,l}) = \aleph_1,\ \forall k \geq \max\{3, k(G_{p,l})\}$.

(ii) Indeed $\pi_q(Spin_m) \simeq \pi_q(SO_m),\ \forall k \geq 2,\ Spin_m$ being the universal cover of SO_m. Hence

$$\pi_3(Spin_m) \simeq \pi_3(SO_m) \simeq \begin{cases} \mathbf{Z} & \text{if } m = 3 \text{ and } m \geq 5 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } m = 4. \end{cases}$$

For the above mentioned homotopy groups, see for instance [5], pp. 224. Therefore in the given conditions, $\pi_3(Spin_m) \ncong 0$, meaning by Theorem 2.1 (ii), that $L_k(Spin_m) = \aleph_1,\ \forall k \geq \max\{4, k(Spin_m)\}$.

Remark 3.3 For $k \geq \max\{3, p+1, k(G_{p,l})\}$ the first point of Proposition 3.2 can be also proved by means of [13], Theorem 10.13, pp. 203, from which it follows that $\pi_p(V_{p+l,l})$ is either infinit cyclic or cyclic of order two.

Acknowledgements. This paper is supported by the European Research and Training Network "GEOMETRIC ANALYSIS". It was written in a period when the second author had a research appointment position at "Paul Sabatier" University, Toulouse, France (November 1, 2000-May 1, 2001)

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