Minimal Surfaces PDE as a Monge–Ampère Type Equation

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Abstract

In the recent Bîlă's paper [1] it was determined the symmetry group of the minimal surfaces PDE (using classical methods). The aim of this paper is to find the Lie algebra of contact symmetries of the minimal surfaces PDE using the correspondence, established by V. V. Lychagin [2], between the second order non-linear differential operators and differential forms which are given on the manifold of 1-jets.

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Introduction

A surface is called *minimal* if the mean curvature of this surface is equal to zero. In what follows we show that the minimal surfaces PDE is a Monge–Ampère type equation. So, in order to find symmetries of this equation we can use the relation, established by V. V. Lychagin [2], between Monge–Ampère equations and differential forms which are given on the manifold of 1-jets.

In Sections 1 and 2 we recall basic definitions and constructions and then we investigate symmetries of the minimal surfaces PDE.

1 Monge–Ampère Operators and Equations

Let M be a smooth manifold, $\dim M = n$. Let also $J^1(M)$ be the manifold of 1-jets of smooth functions which are given on M and $\omega \in \Lambda^1(J^1(M))$ the Cartan form on $J^1(M)$. The Cartan distribution generated by the Cartan form we denote by $\mathcal{C}$.

On the manifold $J^1(M)$ we have the following *natural coordinates*:

$$(1) \qquad \big(t^1, t^2, \dots, t^n, u, u_1, u_2, \dots, u_n\big),$$

see [7], [8], [9]. Here coordinates $(t^1, t^2, \dots, t^n)$ correspond to the local coordinates $(x^1, x^2, \dots, x^n)$ on M, u corresponds to a function given on M and $(u_1, u_2, \dots, u_n)$ correspond to its first order partial derivatives.

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In such local coordinates (which we also denote by (t, u, u')) the Cartan form can be written as $\omega = du - u_1 dt^1 - u_2 dt^2 - \cdots - u_n dt^n$ (or briefly $\omega = du - u'dt$).

As we can see, any differential n-form $\theta \in \Lambda^n(J^1(M))$ defines a second order non-linear differential operator $\Delta_\theta : C^\infty(M) \longrightarrow \Lambda^n(M)$ which acts on functions as follows:

$$\Delta_\theta(h) = j_1(h)^*(\theta), \quad \forall h \in C^\infty(M), \tag{2}$$

where $j_1(h)$ is the 1-jet of the function h, see [2], [3].

In the local coordinates we get that

$$\Delta_\theta(h) = F_\theta(h)(x) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \tag{3}$$

where $F_\theta : C^\infty(\mathbf{R}^n) \longrightarrow C^\infty(\mathbf{R}^n)$ is a second order non-linear scalar differential operator.

Operators Δ_θ are called *Monge-Ampère operators* and corresponding equations are called *Monge-Ampère equations.* A *multivalued (or generalized) solution* of the Monge-Ampère equation defined by the n-form θ is an n-dimensional integral manifold L of the Cartan distribution such that $\theta|_L = 0$, see [2], [3].

Correspondence $\theta \longmapsto \Delta_\theta$ is not bijective but Monge-Ampère operators are uniquely determined by the elements of the quotient-modul $\Lambda^n(J^1(M))/C$, where $C = \{\theta \in \Lambda^n(J^1(M)) \mid \Delta_\theta = 0\}$.

At each point $x \in J^1(M)$ the restriction of the exterior differential of the Cartan form $d\omega_x$ onto Cartan space $\mathcal{C}_x$ determines a sympletic structure on $\mathcal{C}_x$ and it allows to describe the elements of $\Lambda^n(J^1(M))/C$ by the effective forms, see [2], [3].

Differential s-forms on $J^1(M)$ can be expressed in the following way:

$$\Lambda^s(J^1(M)) = \Lambda^s(\mathcal{C}^*) \oplus \left(\omega \wedge \Lambda^{s-1}(\mathcal{C}^*)\right), \tag{4}$$

where by $\Lambda^s(\mathcal{C}^*)$ are denoted differential s-forms that vanish along X_1 and X_1 is the contact vector field with generating function 1, see [3], [4]. Therefore we can consider the natural projection

$$p : \Lambda^s(J^1(M)) \longrightarrow \Lambda^s(\mathcal{C}^*), \quad p(\theta) = \theta - \omega \wedge (X_1 \lrcorner \theta) \tag{5}$$

and the operator

$$d_p : \Lambda^s(\mathcal{C}^*) \longrightarrow \Lambda^{s+1}(\mathcal{C}^*), \quad d_p = p \circ d. \tag{6}$$

Using the Hodge–Lepage decomposition we obtain, that any differential s-form θ has the following unique representation:

$$\theta = \theta_{ef} + \omega \wedge \theta_1 + d\omega \wedge \theta_2, \tag{7}$$

where $\theta_1 \in \Lambda^{s-1}(J^1(M))$, $\theta_2 \in \Lambda^{s-2}(J^1(M))$ and $\theta_{ef} \in \Lambda^s_{ef}(J^1(M)) \text{def=} \Lambda^s_{ef}(\mathcal{C}^*)$ is an effective s-form, see [3].

Effective differential n-forms can be described as follows: θ is effective iff $X_1 \lrcorner \theta = 0$ and $\theta \wedge d\omega = 0$, see [2].

Example 1.0.1 *Let $n = 3$. Let us consider the following 3-form given on the manifold of 1-jets in the natural coordinates:*

$$\theta = du \wedge \left(\begin{matrix} du_1 & u_1 \\ du_2 & u_2 \end{matrix} \wedge dt^3 - \begin{matrix} du_1 & u_1 \\ du_3 & u_3 \end{matrix} \wedge dt^2 + \begin{matrix} du_2 & u_2 \\ du_3 & u_3 \end{matrix} \wedge dt^1 \right). \tag{8}$$

This form as we can see is decomposible.

Differential operator, that corresponds to θ, has the following form:

$$\Delta_\theta(h) = \left(\begin{matrix} h_{11} & h_{12} & h_1 \\ h_{12} & h_{22} & h_2 \\ h_1 & h_2 & 0 \end{matrix} + \begin{matrix} h_{11} & h_{13} & h_1 \\ h_{13} & h_{33} & h_3 \\ h_1 & h_3 & 0 \end{matrix} + \begin{matrix} h_{22} & h_{23} & h_2 \\ h_{23} & h_{33} & h_3 \\ h_2 & h_3 & 0 \end{matrix} \right) dx^1 \wedge dx^2 \wedge dx^3, \tag{9}$$

where $h_i = \dfrac{\partial h}{\partial x^i}$, $i = 1,\, 2$ *and* $h_{jk} = \dfrac{\partial^2 h}{\partial x^j \partial x^k}$, $j,\, k = 1,\, 2,\, 3$. *In this case we get:*

$$F_\theta(h) = \begin{matrix} h_{11} & h_{12} & h_1 \\ h_{12} & h_{22} & h_2 \\ h_1 & h_2 & 0 \end{matrix} + \begin{matrix} h_{11} & h_{13} & h_1 \\ h_{13} & h_{33} & h_3 \\ h_1 & h_3 & 0 \end{matrix} + \begin{matrix} h_{22} & h_{23} & h_2 \\ h_{23} & h_{33} & h_3 \\ h_2 & h_3 & 0 \end{matrix}. \tag{10}$$

If we now consider 3-form

$$\theta_H = \frac{1}{(u_1^2 + u_2^2 + u_3^2)^{3/2}} \theta, \tag{11}$$

then we get the following operator

$$H(h) = \frac{\begin{matrix} h_{11} & h_{12} & h_1 \\ h_{12} & h_{22} & h_2 \\ h_1 & h_2 & 0 \end{matrix} + \begin{matrix} h_{11} & h_{13} & h_1 \\ h_{13} & h_{33} & h_3 \\ h_1 & h_3 & 0 \end{matrix} + \begin{matrix} h_{22} & h_{23} & h_2 \\ h_{23} & h_{33} & h_3 \\ h_2 & h_3 & 0 \end{matrix}}{(h_1^2 + h_2^2 + h_3^2)^{3/2}}, \tag{12}$$

which corresponds to the mean curvature of the surface given in the space $x^1x^2x^3$ by equation $h(x^1,\, x^2,\, x^3) = const.$

If the surface is given by equation $x^3 = \varphi(x^1,\, x^2)$, i.e. by $h(x^1,\, x^2,\, x^3) = 0$, where $h(x^1,\, x^2,\, x^3) = x^3 - \varphi(x^1,\, x^2)$, then the mean curvature of this surface has the folowing form:

$$\tilde{H}(\varphi) = \frac{\varphi_{11}(1 + \varphi_2^2) + \varphi_{22}(1 + \varphi_1^2) - 2\varphi_1\varphi_2\varphi_{12}}{(\varphi_1^2 + \varphi_2^2 + 1)^{3/2}}. \tag{13}$$

Corresponding effective 2-form can be expressed as follows:

$$\theta_{\tilde{H}} = \frac{(1 + u_1^2)dt^1 \wedge du_2 - (1 + u_2^2)dt^2 \wedge du_1 - u_1u_2dt^1 \wedge du_1 + u_1u_2dt^2 \wedge du_2}{(u_1^2 + u_2^2 + 1)^{3/2}}. \tag{14}$$

2 Symmetries of Monge–Ampère Operators and Equations

Lie group $Ct(J^1(M))$ of contact diffeomorphisms acts on Monge–Ampère operators in the following way:

$$F(\Delta_\theta) \text{def=} \Delta_{F^*(\theta)}, \quad F \in Ct(J^1(M)). \tag{15}$$

Lie algebra $ct(J^1(M))$ of contact vector fields acts similarly:

$$X_f(\Delta_\theta) \text{def=} \Delta_{\mathcal{L}_{X_f}(\theta)}. \tag{16}$$

Here $X_f \in ct(J^1(M))$ is the contact vector field with generating function $f \in C^\infty(J^1(M))$, see [4], and $\mathcal{L}_{X_f}$ is an operator of Lie derivation along X_f.

A contact transformation $F \in Ct(J^1(M))$ is called a *symmetry of Monge–Ampère operator* Δ_θ if $F(\Delta_\theta) = \Delta_\theta$. A contact vector field $X_f \in ct(J^1(M))$ is called an *infinitesimal symmetry of Monge–Ampère operator* Δ_θ if $X_f(\Delta_\theta) = 0$.

Finite and infinitesimal symmetries of Monge–Ampère equations are defined similarly: A contact transformation $F \in Ct(J^1(M))$ is called a *symmetry of Monge–Ampère equation* defined by Δ_θ if $F(\Delta_\theta) = \mu\Delta_\theta$ for some function $\mu \in C^\infty(J^1(M))$. A contact vector field $X_f \in ct(J^1(M))$ is called an *infinitesimal symmetry of Monge–Ampère equation* defined by Δ_θ if $X_f(\Delta_\theta) = \lambda\Delta_\theta$ for some function $\lambda \in C^\infty(J^1(M))$.

If θ is an effective n-form, then X_f is an infinitesimal symmetry of Monge–Ampère equation if the following condition holds:

$$p(\mathcal{L}_{X_f}(\theta)) = \lambda\theta \tag{17}$$

for some smooth function $\lambda \in C^\infty(J^1(M))$, see [3].

Moreover, X_f is an infinitesimal symmetry if and only if the following Lie equation holds:

$$(i_f \circ d_p)(\theta) + (d_p \circ i_f)(\theta) + f\mathcal{L}_{X_1}(\theta) = \lambda\theta, \tag{18}$$

for some function $\lambda \in C^\infty(J^1(M))$, see [2], [3]. Here i_f is an operator of inner multiplication by X_f.

3 Minimal Surfaces PDE

A surface is called *minimal* if the mean curvature of this surface is equal to zero. As we can see from (13), the minimal surface PDE of the surface given by equation $x^3 = \varphi(x^1, x^2)$ has the following form:

$$\varphi_{11}(1 + \varphi_2^2) + \varphi_{22}(1 + \varphi_1^2) - 2\varphi_1\varphi_2\varphi_{12} = 0. \tag{19}$$

An effective differential 2-form in the space $J^1(\mathbf{R}^2)$, which corresponds to the minimal surfaces PDE is

$$\theta = (1 + u_1^2)dt^1 \wedge du_2 - (1 + u_2^2)dt^2 \wedge du_1 - u_1u_2dt^1 \wedge du_1 + u_1u_2dt^2 \wedge du_2. \tag{20}$$

Further we will study the infinitesimal symmetries of the minimal surfaces PDE using the method described above. The investigation of the infinitesimal symmetries of this PDE by the classical method which is based on regarding the manifold of 2-jets, see [5], [6], can be found in [1].

We will find only symmetries that are prolongations of the vector fields given on $J^0(\mathbf{R}^2)$. The generating functions of such symmetries must have the form $f(t, u, u') = \varphi - \xi u_1 - \eta u_2$, where functions ξ, η and φ depend only on t and u. Corresponding contact vector field is

$$X_f = \xi\frac{\partial}{\partial t^1} + \eta\frac{\partial}{\partial t^2} + \varphi\frac{\partial}{\partial u} + \Phi^1\frac{\partial}{\partial u_1} + \Phi^2\frac{\partial}{\partial u_2}. \tag{21}$$

where

$$\begin{aligned}\Phi^1 &= \varphi_{t^1} + (\varphi_u - \xi_{t^1})u_1 - \eta_{t^1}u_2 - \xi_u u_1^2 - \eta_u u_1 u_2,\\ \Phi^2 &= \varphi_{t^2} + (\varphi_u - \eta_{t^2})u_2 - \xi_{t^2}u_1 - \eta_u u_2^2 - \xi_u u_1 u_2.\end{aligned}$$

Considering that $\mathcal{L}_{X_f}(t^1) = \xi$, $\mathcal{L}_{X_f}(t^2) = \eta$, $\mathcal{L}_{X_f}(u_1) = \Phi^1$, $\mathcal{L}_{X_f}(u_2) = \Phi^2$, we get:

$$\begin{aligned}&\mathcal{L}_{X_f}(\theta) = 2u_1\Phi^1 dt^1\wedge du_2 + (1+u_1^2)d\xi\wedge du_2 + (1+u_1^2)dt^1\wedge d\Phi^2-\\ &-2u_2\Phi^2 dt^2\wedge du_1 - (1+u_2^2)d\eta\wedge du_1 - (1+u_2^2)dt^2\wedge d\Phi^1-\\ &-(\Phi^1 u_2 + u_1\Phi^2)dt^1\wedge du_1 - u_1u_2 d\xi\wedge du_1 - u_1u_2 dt^1\wedge d\Phi^1+\\ &+(\Phi^1u_2 + u_1\Phi^2)dt^2\wedge du_2 + u_1u_2 d\eta\wedge du_2 + u_1u_2 dt^2\wedge d\Phi^2.\end{aligned}$$

And further we find:

$$\begin{aligned}&\mathcal{L}_{X_f}(\theta) = (2\varphi_{t^1}u_1 + 2(\varphi_u - \xi_{t^1})u_1^2 - 2\eta_{t^1}u_1u_2 - 2\xi_u u_1^3 - 2\eta_u u_1^2 u_2)dt^1\wedge du_2+\\ &+(1+u_1^2)\xi_{t^1}dt^1\wedge du_2 + (1+u_1^2)\xi_{t^2}dt^2\wedge du_2 + (1+u_1^2)\xi_u du\wedge du_2+\\ &+(1+u_1^2)(\varphi_{t^2t^2} + (\varphi_{t^2u} - \eta_{t^2t^2})u_2 - \xi_{t^2t^2}u_1 - \eta_{t^2u}u_2^2 - \xi_{t^2u}u_1u_2)dt^1\wedge dt^2+\\ &+(1+u_1^2)(\varphi_{t^2u} + (\varphi_{uu} - \eta_{t^2u})u_2 - \xi_{t^2u}u_1 - \eta_{uu}u_2^2 - \xi_{uu}u_1u_2)dt^1\wedge du-\\ &-(1+u_1^2)(\xi_{t^2} + \xi_u u_2)dt^1\wedge du_1 + (1+u_1^2)(\varphi_u - \eta_{t^2} - 2\eta_u u_2 - \xi_u u_1)dt^1\wedge du_2-\end{aligned}$$

$$\begin{aligned}&-(2\varphi_{t^2}u_2 + 2(\varphi_u - \eta_{t^2})u_2^2 - 2\xi_{t^2}u_1u_2 - 2\eta_u u_2^3 - 2\xi_u u_1 u_2^2)dt^2\wedge du_1-\\ &-(1+u_2^2)\eta_{t^1}dt^1\wedge du_1 - (1+u_2^2)\eta_{t^2}dt^2\wedge du_1 - (1+u_2^2)\eta_u du\wedge du_1+\\ &+(1+u_2^2)(\varphi_{t^1t^1} + (\varphi_{t^1u} - \xi_{t^1t^1})u_1 - \eta_{t^1t^1}u_2 - \xi_{t^1u}u_1^2 - \eta_{t^1u}u_1u_2)dt^1\wedge dt^2-\\ &-(1+u_2^2)(\varphi_{t^1u} + (\varphi_{uu} - \xi_{t^1u})u_1 - \eta_{t^1u}u_2 - \xi_{uu}u_1^2 - \eta_{uu}u_1u_2)dt^1\wedge du-\\ &-(1+u_2^2)(\varphi_u - \xi_{t^1} - 2\xi_u u_1 - \eta_u u_2)dt^1\wedge du_1 + (1+u_2^2)(\eta_{t^1} + \eta_u u_1)dt^1\wedge du_2-\\ &-(\varphi_{t^2}u_1 + \varphi_{t^1}u_2 - \xi_{t^2}u_1^2 - \eta_{t^1}u_2^2 + (2\varphi_u - \xi_{t^1} - \eta_{t^2})u_1u_2 - 2\xi_u u_1^2u_2 - 2\eta_u u_1u_2^2)\times\\ &\times dt^1\wedge du_1 - u_1u_2\xi_{t^1}dt^1\wedge du_1 - u_1u_2\xi_{t^2}dt^2\wedge du_1 - u_1u_2\xi_u du\wedge du_1-\\ &-u_1u_2(\varphi_{t^1t^2} + (\varphi_{t^2u} - \xi_{t^1t^2})u_1 - \eta_{t^1t^2}u_2 - \xi_{t^2u}u_1^2 - \eta_{t^2u}u_1u_2)dt^1\wedge dt^2-\\ &-u_1u_2(\varphi_{t^1u} + (\varphi_{uu} - \xi_{t^1u})u_1 - \eta_{t^1u}u_2 - \xi_{uu}u_1^2 - \eta_{uu}u_1u_2)dt^1\wedge du-\\ &-u_1u_2(\varphi_u - \xi_{t^1} - 2\xi_u u_1 - \eta_u u_2)dt^1\wedge dt^2 + u_1u_2(\eta_{t^1} + \eta_{t^2u}u_1)dt^1\wedge dt^2+\\ &+(\varphi_{t^2}u_1 + \varphi_{t^1}u_2 - \xi_{t^2}u_1^2 - \eta_{t^1}u_2^2 + (2\varphi_u - \xi_{t^1} - \eta_{t^2})u_1u_2 - 2\xi_u u_1^2u_2 - 2\eta_u u_1u_2^2)\times\\ &\times dt^2\wedge du_2 + u_1u_2\eta_{t^1}dt^1\wedge du_2 + u_1u_2\eta_{t^2}dt^2\wedge du_2 + u_1u_2\eta_u du\wedge du_2-\\ &-u_1u_2(\varphi_{t^1t^2} + (\varphi_{t^1u} - \eta_{t^1t^2})u_2 - \xi_{t^1t^2}u_1 - \eta_{t^1u}u_2^2 - \xi_{t^1u}u_1u_2)dt^1\wedge dt^2+\\ &+u_1u_2(\varphi_{t^2u} + (\varphi_{uu} - \eta_{t^2u})u_2 - \xi_{t^2u}u_1 - \eta_{uu}u_2^2 - \xi_{uu}u_1u_2)dt^2\wedge du-\\ &-u_1u_2(\xi_{t^2} + \xi_u u_2)dt^2\wedge du + u_1u_2(\varphi_u - \eta_{t^2} - 2\eta_u u_2 - \xi_u u_1)dt^2\wedge du.\end{aligned}$$

Substituting $du = \omega + u_1dt^1 + u_2dt^2$ and considering that after the natural projection $p : \Lambda^s(J^1(\mathbf{R}^2)) \longrightarrow \Lambda^s(\mathcal{C}^*)$ the part which is proportional to ω will eliminate we obtain:

$$
\begin{aligned}
&p(\mathcal{L}_{X_f}(\theta)) = \big[\varphi_{t^1t^1} + \varphi_{t^2t^2} + (2\varphi_{t^1u} - \xi_{t^1t^1} - \xi_{t^2t^2})u_1 + (2\varphi_{t^2u} - \eta_{t^1t^1} - \eta_{t^2t^2})u_2 + \\
&+(\varphi_{t^2t^2} + \varphi_{uu} - 2\xi_{t^1u})u_1^2 + (\varphi_{t^1t^1} + \varphi_{uu} - 2\eta_{t^2u})u_2^2 - \\
&-2(\varphi_{t^1t^2} + \xi_{t^2u} + \eta_{t^1u})u_1u_2 - (\xi_{t^2t^2} + \xi_{uu})u_1^3 - (\eta_{t^1t^1} + \eta_{uu})u_2^3 + \\
&+(2\xi_{t^1t^2} - \eta_{t^2t^2} - \eta_{uu})u_1^2u_2 + (2\eta_{t^1t^2} - \xi_{t^1t^1} - \xi_{uu})u_1u_2^2\big]dt^1 \wedge dt^2 + \\
&+\big[\varphi_u + \xi_{t^1} - \eta_{t^2} + 2\varphi_{t^1}u_1 - 2\eta_u u_2 + \\
&+(3\varphi_u - \xi_{t^1} - \eta_{t^2})u_1^2 - 2\xi_u u_1^3 - 2\eta_u u_1^2 u_2\big]dt^1 \wedge du_2 - \\
&-\big[\varphi_u - \xi_{t^1} + \eta_{t^2} - 2\xi_u u_1 + 2\varphi_{t^2}u_2 + \\
&+(3\varphi_u - \xi_{t^1} - \eta_{t^2})u_2^2 - 2\eta_u u_2^3 - 2\xi_u u_1 u_2^2\big]dt^2 \wedge du_1 - \\
&-\big[\eta_{t^1} + \xi_{t^2} + (\varphi_{t^2} + \eta_u)u_1 + (\varphi_{t^1} + \xi_u)u_2 + \\
&+(3\varphi_u - \xi_{t^1} - \eta_{t^2})u_1u_2 - 2\xi_u u_1^2 u_2 - 2\eta_u u_1 u_2^2\big](dt^1 \wedge du_1 - dt^2 \wedge du_2).
\end{aligned}
$$

Field X_f is an infinitesimal symmetry of the minimal surfaces PDE if the condition (17) (or equivalent condition (18)) holds. In this case this condition can be expressed in the following way:

$$
\begin{aligned}
&\varphi_{t^1t^1} + \varphi_{t^2t^2} + (2\varphi_{t^1u} - \xi_{t^1t^1} - \xi_{t^2t^2})u_1 + (2\varphi_{t^2u} - \eta_{t^1t^1} - \eta_{t^2t^2})u_2 + \\
&+(\varphi_{t^2t^2} + \varphi_{uu} - 2\xi_{t^1u})u_1^2 + (\varphi_{t^1t^1} + \varphi_{uu} - 2\eta_{t^2u})u_2^2 - \\
&-2(\varphi_{t^1t^2} + \xi_{t^2u} + \eta_{t^1u})u_1u_2 - (\xi_{t^2t^2} + \xi_{uu})u_1^3 - (\eta_{t^1t^1} + \eta_{uu})u_2^3 + \\
&+(2\xi_{t^1t^2} - \eta_{t^2t^2} - \eta_{uu})u_1^2u_2 + (2\eta_{t^1t^2} - \xi_{t^1t^1} - \xi_{uu})u_1u_2^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_u + \xi_{t^1} - \eta_{t^2} + 2\varphi_{t^1}u_1 - 2\eta_u u_2 + \\
&+(3\varphi_u - \xi_{t^1} - \eta_{t^2})u_1^2 - 2\xi_u u_1^3 - 2\eta_u u_1^2 u_2 = \lambda(1 + u_1^2)
\end{aligned}
$$

$$
\begin{aligned}
&\varphi_u - \xi_{t^1} + \eta_{t^2} - 2\xi_u u_1 + 2\varphi_{t^2}u_2 + \\
&+(3\varphi_u - \xi_{t^1} - \eta_{t^2})u_2^2 - 2\eta_u u_2^3 - 2\xi_u u_1 u_2^2 = \lambda(1 + u_2^2)
\end{aligned}
$$

$$
\begin{aligned}
&\eta_{t^1} + \xi_{t^2} + (\varphi_{t^2} + \eta_u)u_1 + (\varphi_{t^1} + \xi_u)u_2 + \\
&+(3\varphi_u - \xi_{t^1} - \eta_{t^2})u_1u_2 - 2\xi_u u_1^2 u_2 - 2\eta_u u_1 u_2^2 = \lambda u_1 u_2
\end{aligned}
$$

Eliminating λ, we get the following system of partial differential equations:

$$
\begin{aligned}
&\varphi_{t^1t^1} + \varphi_{t^2t^2} + (2\varphi_{t^1u} - \xi_{t^1t^1} - \xi_{t^2t^2})u_1 + (2\varphi_{t^2u} - \eta_{t^1t^1} - \eta_{t^2t^2})u_2 + \\
&+(\varphi_{t^2t^2} + \varphi_{uu} - 2\xi_{t^1u})u_1^2 + (\varphi_{t^1t^1} + \varphi_{uu} - 2\eta_{t^2u})u_2^2 - \\
&-2(\varphi_{t^1t^2} + \xi_{t^2u} + \eta_{t^1u})u_1u_2 - (\xi_{t^2t^2} + \xi_{uu})u_1^3 - (\eta_{t^1t^1} + \eta_{uu})u_2^3 + \\
&+(2\xi_{t^1t^2} - \eta_{t^2t^2} - \eta_{uu})u_1^2u_2 + (2\eta_{t^1t^2} - \xi_{t^1t^1} - \xi_{uu})u_1u_2^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&\xi_{t^2} + \eta_{t^1} + (\varphi_{t^2} + \eta_u)u_1 + (\varphi_{t^1} + \xi_u)u_2 + (\xi_{t^2} + \eta_{t^1})u_1^2 + \\
&+2(\varphi_u - \xi_{t^1})u_1u_2 + (\varphi_{t^2} + \eta_u)u_1^3 - (\varphi_{t^1} + \xi_u)u_1^2u_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&\xi_{t^2} + \eta_{t^1} + (\varphi_{t^2} + \eta_u)u_1 + (\varphi_{t^1} + \xi_u)u_2 + (\xi_{t^2} + \eta_{t^1})u_2^2 + \\
&+2(\varphi_u - \eta_{t^2})u_1u_2 + (\varphi_{t^1} + \xi_u)u_2^3 - (\varphi_{t^2} + \eta_u)u_1u_2^2 = 0
\end{aligned}
$$

Since functions ξ, η and φ do not depend on u_1 and u_2, then from the last system it follows that the coefficients of the monomials depending on u_1 and u_2 must be equal to zero. So, we obtain the following system of PDE's:

$$\begin{array}{ll}
\varphi_{t^1t^1} + \varphi_{t^2t^2} = 0 & \xi_{t^1t^1} + \xi_{t^2t^2} = 2\varphi_{t^1u} \\
\eta_{t^1t^1} + \eta_{t^2t^2} = 2\varphi_{t^2u} & \varphi_{t^2t^2} + \varphi_{uu} = 2\xi_{t^1u} \\
\varphi_{t^1t^1} + \varphi_{uu} = 2\eta_{t^2u} & \varphi_{t^1t^2} + \xi_{t^2u} + \eta_{t^1u} = 0 \\
\xi_{t^2t^2} + \xi_{uu} = 0 & \eta_{t^1t^1} + \eta_{uu} = 0 \\
\eta_{t^2t^2} + \eta_{uu} = 2\xi_{t^1t^2} & \xi_{t^1t^1} + \xi_{uu} = 2\eta_{t^1t^2} \\
\xi_{t^2} + \eta_{t^1} = 0 & \varphi_{t^2} + \eta_u = 0 \\
\varphi_{t^1} + \xi_u = 0 & \varphi_u - \xi_{t^1} = 0 \\
\varphi_u - \eta_{t^2} = 0 &
\end{array}$$

Solutions of this system have the following form:

$$\begin{aligned}
\xi(t^1, t^2, u) &= C_1 + C_7 t^1 + C_4 t^2 + C_6 u \\
\eta(t^1, t^2, u) &= C_2 - C_4 t^1 + C_7 t^2 + C_5 u \\
\varphi(t^1, t^2, u) &= C_3 - C_6 t^1 - C_5 t^2 + C_7 u
\end{aligned}$$

for any $C_1, C_2, \dots, C_7 \in \mathbf{R}$.

We get the following theorem:

Theorem 3.0.1 *The functions*

$$\begin{aligned}
&f_1 = -u_1, \quad f_2 = -u_2, \quad f_3 = 1, \quad f_4 = t^1 u_2 - t^2 u_1, \\
&f_5 = -t^1 - u u_1, \quad f_6 = -t^2 - u u_2, \quad f_7 = u - t^1 u_1 - t^2 u_2
\end{aligned}$$

form a basis of the Lie algebra of generating functions of the minimal surfaces PDE (19) which have the form $f = \varphi - \xi u_1 - \eta u_2$. *Corresponding vector fields*

$$\begin{aligned}
X_{f_1} &= \frac{\partial}{\partial t^1}, \quad X_{f_2} = \frac{\partial}{\partial t^2}, \quad X_{f_3} = \frac{\partial}{\partial u}, \\
X_{f_4} &= t^2 \frac{\partial}{\partial t^1} - t^1 \frac{\partial}{\partial t^2} + u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2}, \\
X_{f_5} &= u \frac{\partial}{\partial t^1} - t^1 \frac{\partial}{\partial u} - (1 + u_1^2) \frac{\partial}{\partial u_1} - u_1 u_2 \frac{\partial}{\partial u_2}, \\
X_{f_6} &= u \frac{\partial}{\partial t^2} - t^2 \frac{\partial}{\partial u} - u_1 u_2 \frac{\partial}{\partial u_1} - (1 + u_2^2) \frac{\partial}{\partial u_2}, \\
X_{f_7} &= t^1 \frac{\partial}{\partial t^1} + t^2 \frac{\partial}{\partial t^2} + u \frac{\partial}{\partial u},
\end{aligned}$$

form a basis of the Lie algebra of symmetries of the minimal surfaces PDE (19) which are prolognations of the vector fields given on $J^0(\boldsymbol{R}^2)$.

Similar result obtained by the classical method can be found in the [1].

We note that the method described above allows to find more general symmetries, not only the prolognations of the vector fields given on $J^0(\mathbf{R}^2)$.

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