Kähler Submanifolds with Lower Bounded Totally Real Bisectional Curvature Tensor

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Abstract

In this paper, we prove that if every totally real bisectional curvature of an $n(\geq 3)$-dimensional complete Kähler submanifold of a complex projective space of constant holomorphic sectional curvature c is greater than $\frac{c}{4(n^2-1)}n(2n-1)$, then it is totally geodesic.

Mathematics Subject Classifications: 53C50, 53C55, 53C56.
Key words: Kähler manifold, complex space form, sectional curvature, holomorphic sectional curvature, totally real bisectional curvature, totally geodesic submanifolds.

1 Introduction

For the curvatures of a Kähler manifold M, we can consider two kinds of sectional curvature which are related to almost the complex structure J and different then the usual sectional curvatures (i.e., the holomorphic sectional curvatures and the totally real bisectional curvatures). The pinching problem for these three kinds of curvatures, the sectional curvature, the holomorphic sectional curvature and the totally real bisectional curvature, is an interesting topic +in Kähler geometry.

For a complex submanifold $M = M^n$ of a complex space form $M' = M^{n+p}(c)$, the set $B(M)$ of the totally real bisectional curvatures satisfies $B(M) \leq \frac{c}{2}$ by the Gauss equation. It is easily seen that a totally geodesic complex submanifold $M = M^n(c)$ of $M' = M^{n+p}(c)$ satisfies $B(M) = \frac{c}{2}$ again by the Gauss equation. On the other hand, a complex quadric $M = Q^n$ of $M' = M^{n+p}(c)$, $c > 0$, satisfies $0 \leq B(M) \leq \frac{c}{2}$ [6]. By paying attention to this fact, and concerning the following theorem by Ros [9] for holomorphic sectional curvatures, the purpose of this paper is to consider the similar problem for totally real bisectional curvatures.

Theorem A. *Let $M = M^n$ be an n-dimensional complete Kähler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. If every holomorphic sectional curvature of M is greater than $\frac{c}{2}$, then M is totally geodesic.*

Ogiue [7] gave also the following

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Theorem B. *Let $M = M^n$ be an n-dimensional complete Kähler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(>0)$. If every Ricci curvature of M is greater than $\frac{c}{2}n$, then M is totally geodesic.*

2 Kähler manifolds

This section is concerned with recalling basic formulas on Kähler manifolds. Let M be a complex $n(\geq 2)$-dimensional Kähler manifold equipped with Kähler metric tensor g and almost complex structure J. We can choose a local field $\{E_j, E_{j^*}\} = \{E_1, \cdots, E_n, E_{1^*}, \cdots, E_{n^*}\}$ of orthonormal frames on a neighborhood of M, where $E_{j^*} = JE_j$ and $j^* = n + j$. Here and in the sequel, the Latin small indices $j, k, \cdots$ run from 1 to n. We set $U_j = \frac{1}{\sqrt{2}}(E_j - iE_{j^*})$ and $\bar{U}_j = \frac{1}{\sqrt{2}}(E_j + iE_{j^*})$, where i denotes the imaginary unit. Then $\{U_j\}$ constitutes a local field of unitary frames on the neighborhood of M. With respect to the Kähler metric, we have $g(U_j, \bar{U}_k) = \delta_{jk}$.

Now let $\{\omega_j\}$ be the canonical form with respect to the local field $\{U_j\}$ of unitary frames on the neighborhood of M. Then $\{\omega_j\} = \{\omega_1, \cdots, \omega_n\}$ consists of complex valued 1-forms of type (1,0) on M such that $\omega_j(U_k) = \delta_{jk}$ and $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent. The Kähler metric g of M can be expressed as $g = 2\sum_j \omega_j \otimes \bar{\omega}_j$. Associated with the frame field $\{U_j\}$, there exist complex-valued 1-forms ω_{jk}, which are usually called *complex connection forms* on M such which satisfy the structure equations of M

$$
\begin{aligned}
&d\omega_i + \sum_k \omega_{ik} \wedge \omega_k = 0, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0, \\
&d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \quad \Omega_{ij} = \sum_k K_{\bar{i}jk\bar{l}}\, \omega_k \wedge \bar{\omega}_l,
\end{aligned} \tag{2.1}
$$

where Ω_{ij} (resp. $K_{\bar{i}jk\bar{l}}$) are the components of the curvature form (resp. of the Riemannian curvature tensor R) of M. From the structure equations, the components of the curvature tensor satisfy

$$K_{\bar{i}jk\bar{l}} = \bar{K}_{\bar{j}il\bar{k}}, \tag{2.2}$$

$$K_{\bar{i}jk\bar{l}} = K_{\bar{i}kj\bar{l}} = K_{\bar{l}jk\bar{i}} = K_{\bar{l}kj\bar{i}}. \tag{2.3}$$

Next, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows :

$$S = \sum_{i,j} (S_{i\bar{j}}\omega_i \otimes \bar{\omega}_j + S_{\bar{i}j}\bar{\omega}_i \otimes \omega_j), \tag{2.4}$$

where $S_{i\bar{j}} = \sum_k K_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \bar{S}_{\bar{i}j}$. The scalar curvature r of M is also given by

$$r = 2\sum_j S_{j\bar{j}}. \tag{2.5}$$

An n-dimensional Kähler manifold M is said to be *Einstein*, if the Ricci tensor S satisfies the condition

$$S_{i\bar{j}} = \frac{r}{2n}\delta_{ij}. \tag{2.6}$$

The components $K_{\bar{i}jk\bar{l}m}$ and $K_{\bar{i}jk\bar{l}\bar{m}}$ (resp. $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor R (resp. the Ricci tensor S) are given by

$$\begin{aligned}&\sum_m (K_{\bar{i}jk\bar{l}m}\omega_m + K_{\bar{i}jk\bar{l}\bar{m}}\bar{\omega}_m) = dK_{\bar{i}jk\bar{l}}\\ &-\sum_m (K_{\bar{m}jk\bar{l}}\bar{\omega}_{mi} + K_{\bar{i}mk\bar{l}}\omega_{mj} + K_{\bar{i}jm\bar{l}}\omega_{mk} + K_{\bar{i}jk\bar{m}}\bar{\omega}_{ml}),\end{aligned} \tag{2.7}$$

$$\sum_k (S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k (S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}). \tag{2.8}$$

The second Bianchi identity is given as follows :

$$K_{\bar{i}jk\bar{l}m} = K_{\bar{i}jm\bar{l}k}. \tag{2.9}$$

And hence we have

$$S_{i\bar{j}k} = S_{k\bar{j}i} = \sum_m K_{\bar{j}ik\bar{m}m}. \tag{2.10}$$

Lastly, a Kähler manifold of constant holomorphic sectional curvature is called a *complex space form*. The components $K_{\bar{i}jk\bar{l}}$ of the Riemannian curvature tensor R of an n-dimensional complex space form of constant holomorphic sectional curvature c are given by

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{2.11}$$

3 Complex submanifolds

This section recalls basics of complex submanifolds of a Kähler manifold. First of all, the main formulas for the theory of complex submanifolds are prepared.

Let $M' = M^{n+p}$ be an $(n+p)$-dimensional Kähler manifold with Kähler structure (g', J'). Let M be an n-dimensional complex submanifold of M' and let g be the induced Kähler metric tensor on M from g'. We can choose a local field $\{U_A\} = \{U_i, U_x\} = \{U_1, \cdots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that, restricted to M, $U_1, \cdots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated :

$$\begin{aligned}&A, B, \cdots = 1, \cdots, n, n+1, \cdots, n+p,\\ &i, j, \cdots = 1, \cdots, n, \ x, y, \cdots = n+1, \cdots, n+p.\end{aligned}$$

With respect to the frame field, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame fields. Then the Kähler metric tensor g' of M' is given $g' = 2\sum_A \omega_A \otimes \bar{\omega}_A$. The canonical forms ω_A, the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$
\begin{aligned}
&d\omega_A + \sum \epsilon_C \omega_{AC} \wedge \omega_C = 0, \qquad \omega_{AB} + \bar{\omega}_{BA} = 0, \\
&rd\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB,} \quad \Omega'_{AB} = \sum_{C,D} K'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D,
\end{aligned} \tag{3.1}
$$

where Ω'_{AB} (resp. $K'_{\bar{A}BC\bar{D}}$) denotes the components of the curvature form (resp. of the Riemannian curvature tensor R') of M'.

Restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{3.2}$$

and the induced Kähler metric tensor g of M is given by $g = 2\sum_j \omega_j \otimes \bar{\omega}_j$. Then $\{U_j\}$ is a local unitary frame field with respect to the induced metric and $\{\omega_j\}$ is a local dual frame filed due to $\{U_j\}$, which consists of complex-valued 1-forms of type (1,0) on M. Moreover, $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent, and $\{\omega_j\}$ are the canonical forms on M. It follows from (3.2) and Cartan's lemma that the exterior derivatives of (3.2) give rise to

$$\omega_{xi} = \sum_j h^x_{ij} \omega_j, \qquad h^x_{ij} = h^x_{ji}. \tag{3.3}$$

The quadratic form $\alpha = \sum_{i,j,x} h^x_{ij} \omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle on M in M' is called the *second fundamental form* of the submanifold M. From the structure equations for M', it follows that the structure equations for M are similarly given by

$$
\begin{aligned}
&d\omega_i + \sum_k \omega_{ik} \wedge \omega_k = 0, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0, \\
&d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_k = \Omega_{ij,} \quad \Omega_{ij} = \sum_{k,l} K_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l.
\end{aligned} \tag{3.4}
$$

For the Riemannian curvature tensors R and R' of M and M', respectively, it follows from (3.1), (3.3) and (3.4) that

$$K_{\bar{i}jk\bar{l}} = K'_{\bar{i}jk\bar{l}} - \sum_x h^x_{jk} \bar{h}^x_{il}. \tag{3.5}$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r on M are given by

$$S_{i\bar{j}} = \sum_k K'_{\bar{k}ki\bar{j}} - {h_{i\bar{j}}}^2, \tag{3.6}$$

$$r = 2(\sum_{j,k} K'_{\bar{k}kj\bar{j}} - h_2), \tag{3.7}$$

where ${h_{i\bar{j}}}^2 = {h_{\bar{j}i}}^2 = \sum_{m,x} h^x_{im} \bar{h}^x_{mj}$ and $h_2 = \sum_j {h_{j\bar{j}}}^2$.

Now the components h^x_{ijk} and $h^x_{ij\bar{k}}$ of the covariant derivative of the second fundamental form on M are given by

$$\sum_k (h^x_{ijk}\omega_k + h^x_{ij\bar{k}}\bar{\omega}_k) = dh^x_{ij} - \sum_k (h^x_{jk}\omega_{ki} + h^x_{ik}\omega_{kj}) + \sum_y h^y_{ij}\omega_{xy}. \tag{3.8}$$

Then, substituting dh^x_{ij} from this definition into the exterior derivative

$$d\omega_{xi} = \sum_j (dh^x_{ij} \wedge \omega_j + h^x_{ij} d\omega_j)$$

of (3.3) and using (3.1) $\sim$ (3.4) and (3.6), we have

$$h^x_{ijk} = h^x_{ikj}, \qquad h^x_{ij\bar{k}} = -K'_{\bar{x}ij\bar{k}}. \tag{3.9}$$

In particular, let the ambient space $M' = M^{n+p}(c)$ be an $(n+p)$-dimensional complex space form of constant holomorphic sectional curvature c. Then, by (2.11) and (3.5) - (3.7), we get

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x h^x_{jk}\bar{h}^x_{il}, \tag{3.10}$$

$$S_{i\bar{j}} = \frac{c}{2}(n+1)\delta_{ij} - {h_{i\bar{j}}}^2 , \tag{3.11}$$

$$r = cn(n+1) - 2h_2 , \tag{3.12}$$

$$h^x_{ij\bar{k}} = 0. \tag{3.13}$$

4 Totally real bisectional curvatures

In this section, we are concerned with the totally real bisectional curvature of a semi-definite Kähler manifold. Let (M, g) be an n-dimensional semi-definite Kähler manifold with almost complex structure J. In their paper [3], Bishop and Goldberg introduced the notion for totally real bisectional curvature $B(X, Y)$ on a Kähler manifold.

A plane section P in the tangent space T_pM at any point p in M is said to be *totally real* or *anti-holomorphic* if P is orthogonal to JP. For an orthonormal basis $\{X, Y\}$ of the totally real plane section P, any vectors X, JX, Y and JY are mutually orthogonal. This implies that for orthogonal vectors X and Y in P, it is totally real if and only if two holomorphic plane sections spanned by X, JX and Y, JY are orthogonal.

Houh [5] showed that an $n(\geq 3)$-dimensional Kähler manifold has constant totally real bisectional curvature c if and only if it has constant holomorphic sectional curvature $2c$. On the other hand, Goldberg and Kobayashi [4] introduced the notion of holomorphic bisectional curvature $H(X, Y)$ which is determined by two holomorphic planes $\mathrm{Span}\{X, JX\}$ and $\mathrm{Span}\{Y, JY\}$, and asserted that the complex projective space $CP^n(c)$ is the only compact Kähler manifold with positive holomorphic bisectional curvature and constant scalar curvature. If we compare $B(X, Y)$ with the holomorphic bisectional curvature $H(X, Y)$ and the holomorphic sectional curvature

$H(X)$, then the holomorphic bisectional curvature $H(X,Y)$ turns out to be totally real bisectional curvature $B(X,Y)$ (resp. holomorphic sectional curvature $H(X)$), when two holomorphic planes Span$\{X, JX\}$ and Span$\{Y, JY\}$ are orthogonal to each other (resp. coincides with each other). From this, it follows that the positiveness of $B(X,Y)$ is weaker than the positiveness of $H(X,Y)$, because $H(X,Y) > 0$ implies that both of $B(X,Y)$ and $H(X)$ are positive but we don't know whether $B(X,Y) > 0$ implies $H(X,Y) > 0$.

Furthermore, Goldberg and Kobayashi [4] showed that a complete Kähler manifold M with constant scalar curvature and positive holomorphic bisectional curvature is Einstein. In order to get this result, they should have verified that the Ricci tensor is positive definite. In that proof, they used that the fact that the holomorphic sectional curvature $H(X)$ is positive, which follows necessarily from the condition $H(X,Y) > 0$. But the condition $B(X,Y) > 0$ carries less information than the condition of $H(X,Y) > 0$, and it gives us no reason for using Goldberg and Kobayashi's method to derive the fact that M is Einstein (that is, we can not use the condition $H(X,Y) > 0$). The totally real bisectional curvature $B(X,Y)$ can be also consider for non-degenerate totally real planes Span$\{X,Y\}$ in any indefinite Kähler manifold. In their paper [2], Barros and Romero asserted that above mentioned Houh′s result can be extended to indefinite Kähler manifolds. Aiyama, Kwon and Nakagawa [1] have also studied the classification problem of space-like complex submanifolds of indefinite complex hyperbolic space $CH^{n+p}_{0+p}(c)$ with bounded scalar curvature.

Motivated by these results, we present in the followinf the classification problems with bounded totally real bisectional curvature.

Let (M,g) be an n-dimensional semi-definite Kähler manifold with almost complex structure J. In the sequel, we only consider the definite totally real planes, unless otherwise stated.

Definition 4.1. For a totally real plane section P spanned by orthonormal vectors X and Y, the *totally real bisectional curvature* $B(X,Y)$ is defined by

$$B(X,Y) = g(R(X,JX)JY,Y). \tag{4.1}$$

Then, using the first Bianchi identity to (4.1) and the fundamental properties of the Riemannian curvature tensor of semi-definite Kähler manifolds, we get

$$\begin{aligned} B(X,Y) &= g(R(X,Y)Y,X) + g(R(X,JY)JY,X) \\ &= K(X,Y) + K(X,JY), \end{aligned} \tag{4.2}$$

where $K(X,Y)$ means the sectional curvature of the plane spanned by X and Y.

Example 4.1. Let $M^n_s(c)$ be an n-dimensional semi-definite complex space form of constant holomorphic sectional curvature c and of index $2s$, $0 \le s \le n$. Then, $M^n_s(c)$ has constant totally real bisectional curvature $\frac{c}{2}$. In fact, if a plane Span$\{X,Y\}$ is totally real, then we have

$$B(X,Y) = \frac{g(R(X,JX)JY,Y)}{g(X,X)g(Y,Y)} = \frac{c}{2}\,,$$

which follows easily from the form of the curvature tensor of $M^n_s(c)$.

Example 4.2. Let Q^n be a complex quadric in a complex projective space $CP^{n+1}(c)$ of constant holomorphic sectional curvature c. In $CP^{n+1}(c)$ with homogeneous coordinates $z^0, z^1, \cdots, z^{n+1}$, the complex quadric Q^n is complex hypersurface defined by

the equation

$$(z^0)^2 + (z^1)^2 + \cdots + (z^{n+1})^2 = 0.$$

Let g be the Fubini-Study metric on $CP^{n+1}(c)$ of constant holomorphic sectional curvature c. Its restriction g to Q^n is a Kähler metric. Then, it is seen [6] that Q^n is an Einstein hypersurface whose Ricci tensor S satisfies

$$S = \frac{c}{2}ng,$$

and its totally real bisectional curvature B satisfies

$$0 \leq B(M) \leq \frac{c}{2}.$$

In the rest of this section, we suppose that X and Y are orthonormal vectors in a non-degenerate totally real plane section such that $g(X,X) = g(Y,Y) = \pm 1$. If we put $X' = \frac{1}{\sqrt{2}}(X+Y)$ and $Y' = \frac{1}{\sqrt{2}}(X-Y)$, then it is easily seen that

$$g(X',X') = g(Y',Y') = \pm 1, \quad g(X',Y') = 0.$$

Thus we get

$$\begin{aligned} B(X',Y') &= g(R(X',JX')JY',Y') \\ &= \frac{1}{4}\{H(X) + H(Y) + 2B(X,Y) - 4K(X,JY)\}, \end{aligned}$$

where $H(X) = K(X,JX)$ means the holomorphic sectional curvature of the holomorphic plane spanned by X and JX. Hence we have

$$4B(X',Y') - 2B(X,Y) = H(X) + H(Y) - 4K(X,JY). \tag{4.3}$$

If we put $X'' = \frac{1}{\sqrt{2}}(X+JY)$ and $Y'' = \frac{1}{\sqrt{2}}(JX+Y)$, then we get from the definiteness of the plane $\mathrm{Span}\{X,Y\}$

$$g(X'',X'') = g(Y'',Y'') = \pm 1, \quad g(X'',Y'') = 0.$$

Using the similar method as in (4.3), we have

$$4B(X'',Y'') - 2B(X,Y) = H(X) + H(Y) - 4K(X,Y). \tag{4.4}$$

Summing up (4.3) and (4.4) and taking account of (4.2), we obtain

$$2B(X',Y') + 2B(X'',Y'') = H(X) + H(Y). \tag{4.5}$$

Now let $M = M_0^n$ be an $n(\geq 3)$-dimensional space-like complex submanifold of an $(n+p)$-dimensional semi-definite Kähler manifold $M' = M_{0+p}^{n+p}(c)$ of index $2p$ and of constant holomorphic sectional curvature c.

Assume that the totally real bisectional curvatures on M is bounded from below (resp. above) by a constant a (resp. b), and let $a(M)$ and $b(M)$ be the infimum and the supremum of the set $B(M)$ of the totally real bisectional curvatures on M, respectively. By definition, we see $a \leq a(M)$ (resp. $b \geq b(M)$). From (4.5), we have

$$H(X) + H(Y) \geq 4a \text{ (resp. } \leq 4b). \tag{4.6}$$

For an orthonormal frame field $\{E_1, \cdots, E_n\}$ on a neighborhood of M, the holomorphic sectional curvature $H(E_j)$ of the holomorphic plane spanned by E_j can be expressed as

$$H(E_j) = g(R(E_j, JE_j)JE_j, E_j) = R_{jj^*j^*j} = K_{\bar{j}jj\bar{j}}. \tag{4.7}$$

On the other hand, it is easily seen that the plane sections $\mathrm{Span}\{E_j, JE_j\}$, and $\mathrm{Span}\{E_k, JE_k\}$, $j \neq k$, are orthogonal and the totally real bisectional curvature $B(E_j, E_k)$ is given by

$$B(E_j, E_k) = g(R(E_j, JE_j)JE_k, E_k) = K_{\bar{j}jk\bar{k}}, \quad j \neq k. \tag{4.8}$$

From the inequality (4.6) for $X = E_j$ and $Y = E_k$, we have

$$K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}} \geq 4a \text{ (resp. } \leq 4b), \quad j \neq k. \tag{4.9}$$

Thus we have

$$\sum_{j<k} (K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}}) \geq 2an(n-1) \quad (\text{resp. } \leq 2bn(n-1)), \tag{4.10}$$

which implies that

$$\sum_{j} K_{\bar{j}jj\bar{j}} \geq 2an \quad (\text{resp. } \leq 2bn), \tag{4.11}$$

where the equality holds if and only if $K_{\bar{j}jj\bar{j}} = 2a$ (resp. $= 2b$) for any index j.

Since the scalar curvature r is given by

$$r = 2\sum_{j,k} K_{\bar{j}jk\bar{k}} = 2(\sum_{j} K_{\bar{j}jj\bar{j}} + \sum_{j\neq k} K_{\bar{j}jk\bar{k}}),$$

we have by (4.10)

$$r \geq 2\sum_{j} K_{\bar{j}jj\bar{j}} + 2an(n-1) \quad (\text{resp. } \leq 2\sum_{j} K_{\bar{j}jj\bar{j}} + 2bn(n-1)),$$

from which it follows that

$$\sum_{j} K_{\bar{j}jj\bar{j}} \leq \frac{r}{2} - an(n-1) \text{ (resp. } \geq \frac{r}{2} - bn(n-1)), \tag{4.12}$$

where the equality holds if and only if $K_{\bar{j}jk\bar{k}} = a$ (resp. $= b$) for any distinct indices j and k. In this case, M is locally congruent to $M^n(a)$ (resp. $M^n(b)$) due to Houh [5]. Also (4.9) gives us

$$\sum_{k(\neq j)} (K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}}) \geq 4a(n-1) \quad (\text{resp. } \leq 4b(n-1))$$

for each j, so that

$$(n-2)K_{\bar{j}jj\bar{j}} + \sum_k K_{\bar{k}kk\bar{k}} \geq 4a(n-1) \quad (\text{resp. } \leq 4b(n-1)).$$

From this inequality together with (4.12), it follows that

$$(4.13) \qquad \begin{array}{lll} (n-2)K_{\bar{j}jj\bar{j}} & \geq & a(n-1)(n+4) - \dfrac{r}{2} \\ (\text{resp.} & \leq & b(n-1)(n+4) - \frac{r}{2}) \end{array}$$

for any index j, so that the holomorphic sectional curvature $K_{\bar{j}jj\bar{j}}$ is bounded from below (resp. above) for $n \geq 3$. Moreover, the equality holds for some index j if and only if M is locally congruent to $M^n(2a)$ (resp. $M^n(2b)$).

By applying Theorem A we infer

Theorem 4.1. *Let $M = M^n$ be an $n(\geq 3)$-dimensional complete Kähler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. If every totally real bisectional curvature of M is greater than $\frac{c}{4(n^2-1)}n(2n-1)$, then M is totally geodesic.*

Proof. By the assumption $B(M) \geq a$ and (4.13), we have

$$(n-2)H(M) \geq a(n-1)(n+4) - \frac{r}{2}.$$

Since we see $r = cn(n+1) - 2h_2$ by (3.12), we obtain

$$H(M) \geq \frac{1}{2(n-2)}\{2a(n-1)(n+4) - cn(n+1)\} \equiv a_0.$$

Thus we have by (3.10)

$$(4.14) \qquad K_{\bar{j}jj\bar{j}} = c - \sum_x h^x_{jj}\bar{h}^x_{jj} \geq a_0, \qquad K_{\bar{i}ij\bar{j}} = \frac{c}{2} - \sum_x h^x_{ij}\bar{h}^x_{ij} \geq a$$

for any distinct indices i and j. Since the Ricci curvature $S_{j\bar{j}}$ of M is given by (3.11)

$$S_{j\bar{j}} = \frac{c}{2}(n+1) - \lambda_j, \qquad \lambda_j = \sum_{m,x} h^x_{jm}\bar{h}^x_{jm}$$

and

$$\lambda_j = \sum_x h^x_{jj}\bar{h}^x_{jj} + \sum_{m(\neq j),x} h^x_{jm}\bar{h}^x_{jm} \leq (c - a_0) + (\frac{c}{2} - a)(n-1)$$

from (4.14) and using the Ricci curvatures it follows that

$$S_{j\bar{j}} \geq a_0 + a(n-1).$$

Given the constants a and a_0, we obtain

$$S_{j\bar{j}} > \frac{c}{2}n$$

for any index j. By Theorem B, this completes the proof. □

Remark 4.1. We should here remark that $\frac{c}{4(n^2-1)}n(2n-1) < \frac{c}{2}$ for $n \geq 3$ and $c > 0$. Hence Theorem 4.1 is a generalization of Theorem A in the case where $n \geq 3$.

As a direct consequence of Theorem 4.1 combined with the equation (4.2), we can prove

Corollary 4.2. *Let $M = M^n$ be an $n(\geq 3)$-dimensional complete Kähler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. If every sectional curvature of M is greater than $\frac{c}{8(n^2-1)}n(2n-1)$, then M is totally geodesic.*

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