# Cylindrical Tzitzeica Curves Implies Forced Harmonic Oscillators

*Dedicated to Acad. Radu Miron on the occasion of his 75 birthday*

Mircea Crâşmăreanu

**Abstract**

Elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition are obtained via the solution of the forced harmonic equation.

**Mathematics Subject Classification**: 53A04, 34A05.
**Key words**: Tzitzeica curve, cylindrical curve, forced harmonic oscillator.

## Introduction

Gheorghe Ţiţeica (1873-1939), writting in French under the name of Georges Tzitzeica, was a student of Gaston Darboux and thus a member of the second generation of classical differential geometers, after Gauss and Riemann.

There are several *Tzitzeica* notions:
(1) surfaces ([1], [32])
(2) curves ([19])
(3) hypersurfaces ([7], [23])
(4) submanifolds ([6], [8], [13], [14], [21], [27])
(5) algebras ([20], [22])
(6) connections ([9], [10], [11], [16], [18])
(7) equations ([2], [4], [5], [12], [25], [26]).

Tzitzeica has introduced a class of surfaces, nowadays called *Tzitzeica surfaces*, in 1907 ([29]) and a class of curves, called *Tzitzeica curves*, in 1911 ([30]). The relation between these objects is the following: for a Tzitzeica surface with negative Gaussian curvature, the asimptotic lines are Tzitzeica curves. Since their appearence, these notions are a permanent subject of research, suitable for fruitful generalizations.

(Elliptic) cylindrical curves are studied by Gheorghe Vrânceanu in connection with Levi-Civita and Fenchel theorems from surfaces theory in [33].

We consider two types of cylindrical curves: an usual one, which we prefer to call *elliptic*, and another one, called *hyperbolic*, according to the type of circular functions(cosine and sine) used. So, our results are divided in two sections.

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In this paper we are interested in Tzitzeica cylindrical curves, more precisely we ask in what conditions a cylindrical curve is a Tzitzeica one, namely the function $t \to \frac{\tau(t)}{d^2(t)}$ is constant, where $d(t)$ is the distance from origin to the osculating plane of curve. The Tzitzeica condition yields a third-order ODE which in our framework admits a direct integration. Therefore the final answer of main problem is given via a second order ODE which in the elliptic case is exactly the equation of a forced harmonic oscillator! In both cases, elliptic and hyperbolic, the solution depends of four real constants: one defining the Tzitzeica condition and other three obtained by integration.

# 1 Elliptic cylindrical curves

Let in $\mathbf{R}^3$ a curve $C$ given in vectorial form $C : \overline{r} = \overline{r}(t)$. This curve is called *elliptic cylindrical* if has the expression

$$\overline{r}(t) = (\cos t, \sin t, f(t)) \tag{1.1}$$

for some $f \in C^\infty(\mathbf{R})$.

The torsion function is

$$\tau(t) = \frac{(\overline{r}', \overline{r}'', \overline{r}''')}{\|\overline{r}' \times \overline{r}''\|^2} = \frac{f' + f'''}{1 + f'^2 + f''^2}.$$

Then the distance from origin to the osculating plane is

$$d(t) = \frac{\pm(f + f'')}{\sqrt{1 + f'^2 + f''^2}}.$$

Let us suppose that the curve is Tzitzeica with the constant $K \neq 0$, because the curve is not contained in a plane

$$\frac{\tau(t)}{d^2(t)} = K = \frac{f'(t) + f'''(t)}{(f(t) + f''(t))^2}.$$

Integration gives

$$\frac{1}{f(t) + f''(t)} = -(Kt + C)$$

with $C$ a real constant. The last relation reads

$$f''(t) + f(t) = \frac{-1}{Kt + C}. \tag{1.2}$$

But the ODE (1.2) is exacly of forced harmonic oscillator type,

$$\ddot{x} + x = g(t)$$

solved by the formula ([17, p. 60-61])

$$x(t) = x(0)\cos t - \dot{x}(0)\sin t + \int_0^t g(u)\sin(u - t)\, du$$

Therefore we have

**Proposition 1.1.** *An elliptic cylindrical curve* (1.1) *is Tzitzeica if and only if*

$$f(t) = f(0)\cos t - \dot{f}(0)\sin t - \int_0^t \frac{\sin(u-t)}{Ku+C}du$$

*with* $f(0), \dot{f}(0), K \neq 0, C$ *real constants.*

# 2 Hyperbolic cylindrical curves

A space curve is called *hyperbolic cylindrical* if

(2.1) $$\overline{r}(t) = (\cosh t, \sinh t, f(t)).$$

Straightforward computation gives

$$\tau(t) = \frac{f' - f'''}{1 + (f'^2 + f''^2)\left(\cosh^2 t + \sinh^2 t\right) - 4f'f''\cosh t \sinh t}.$$

Also

$$d(t) = \frac{\pm(f - f'')}{\sqrt{1 + (f'^2 + f''^2)\left(\cosh^2 t + \sinh^2 t\right) - 4f'f''\cosh t \sinh t}}.$$

For a Tzitzeica hyperbolic curve

$$\frac{\tau(t)}{d^2(t)} = K = \frac{f' - f'''}{(f - f'')^2}$$

and integration gives

$$\frac{1}{f - f''} = -(Kt + C) \Leftrightarrow f'' - f = \frac{1}{Kt + C}.$$

Using the same formula (3) from [17, p. 60] and the identity

$$e^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

it results

**Proposition 2.1.** *A hyperbolic curve* (2.1) *is Tzitzeica if and only if:*

$$f(t) = f(0)\cosh t + \dot{f}(0)\sinh t + \int_0^t \frac{\sinh(t+u)}{Ku+C}$$

*with* $f(0), \dot{f}(0), K \neq 0$ *and* $C$ *real constants.*

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University "Al. I. Cuza" of Iassy
Faculty of Mathematics
Iaşi, 6600, Romania
email: mcrasm@uaic.ro