Slant Surfaces with Prescribed Gaussian Curvature

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Abstract

A slant immersion was introduced in [2] as an isometric immersion of a Riemannian manifold into an almost Hermitian manifold $(\tilde{M}, g, J)$ with constant Wirtinger angle. From J-action point of view, the most natural surfaces in an almost Hermitian manifold are slant surfaces. Flat slant surfaces in complex space forms have been studied in [3, 4]. In this article, we study slant surfaces in complex space forms with arbitrary Gauss curvature. In particular, we prove that, for any $\theta \in (0, \frac{\pi}{2}]$, there exist infinitely many θ-slant surfaces in complex projective plane and in complex hyperbolic plane with prescribed Gaussian curvature.

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1 Introduction

Let M be a Riemannian n-manifold and $\tilde{M}$ an almost Hermitian manifold with almost complex structure J and almost Hermitian metric g. We denote by $\langle\ ,\ \rangle$ the inner product for M as well as for $\tilde{M}$. For any vector X tangent to M, we put

$$JX = PX + FX, \tag{1.1}$$

where PX and FX denote the tangential and normal components of JX, respectively. For each nonzero vector X tangent to M at x, the angle $\theta(X)$ between JX and T_xM is called the Wirtinger angle of X. The immersion $f : M \to \tilde{M}$ is said to be *slant* if the Wirtinger angle $\theta(X)$ is a constant (cf. [2]). The Wirtinger angle θ of a slant immersion is called the *slant angle*. A slant submanifold with slant angle θ is also called a θ-slant submanifold. Holomorphic and totally real immersions are slant immersions with slant angle 0 and $\frac{\pi}{2}$, respectively. A slant immersion is said to be *proper slant* if it is neither holomorphic nor totally real.

From J-action point of view, slant submanifolds are the simplest and the most natural submanifolds of an almost Hermitian manifold. Slant submanifolds have been studied by many geometers in the last two decades (see, for examples, [1]-[12]). Slant submanifolds arise naturally and play some important roles in the studies of submanifolds of Kählerian manifolds. For example, it was proved in [8] that every surface in

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a complex space form $\tilde{M}^2(4c)$ is proper slant if it has constant curvature and nonzero parallel mean curvature vector. Flat slant surfaces in complex space forms have been studied in [3, 4]. There exist ample minimal proper slant surfaces in C^2 (see, [2]). In contrast, it was proved in [5] that there do not exist minimal proper slant surfaces in complex projective and in complex hyperbolic planes. Furthermore, it was proved in [7] that, for any $\theta \in (0, \frac{\pi}{2}]$ and any given function $G = G(x)$, there exists a θ-slant surface in the complex Euclidean plane $\mathbf{C}^2$ with G as the prescribed Gaussian curvature function.

In this article, we investigate slant surfaces in complex projective and complex hyerbolic planes with arbitrary Gauss curvature. More precisely, we prove the following existence theorem for slant surfaces prescribed Gauss curvature.

Theorem 1. *Locally, for any given $\theta \in (0, \frac{\pi}{2}]$ and any given function $G = G(x)$, there exist infinitely many θ-slant surfaces in the complex projective plane CP^2 and in the complex hyperbolic plane CH^2 with G as the prescribed Gaussian curvature.*

Remark. The Theorem is false in general if Gaussian curvature were replaced by mean curvature. For example, for any $\theta \in (0, \frac{\pi}{2})$, there does not exist θ-slant surfaces in CP^2 and in CH^2 with zero as the prescribed mean curvature (see [5]).

2 Preliminaries

Let $x : M \to \tilde{M}^m$ be an isometric immersion of a Riemannian n-manifold into a Kählerian m-manifold. Denote by R and $\tilde{R}$ the Riemann curvature tensors of M and $\tilde{M}^m$, respectively. Denote by h and A the second fundamental form and the shape operator of the immersion x; and by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^m$, respectively. The second fundamental form h and the shape operator A are related by

$$\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle \tag{2.1}$$

for ξ normal to M.

The well-known *equation of Gauss* is given by

$$\tilde{R}(X,Y;Z,W) = R(X,Y;Z,W) + \langle h(X,Z), h(Y,W)\rangle - \langle h(X,W), h(Y,Z)\rangle \tag{2.2}$$

for X, Y, Z, W tangent to M.

For the second fundamental form h, we define its covariant derivative $\bar{\nabla} h$ with respect to the connection on $TM \oplus T^\perp M$ by

$$(\bar{\nabla}_X h)(Y,Z) = D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.3}$$

The *equation of Codazzi* is

$$(\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{2.4}$$

where $(\tilde{R}(X,Y)Z)^\perp$ denotes the normal component of $\tilde{R}(X,Y)Z$.

For an endomorphism Q on the tangent bundle of the submanifold, we define its covariant derivative ∇Q by $(\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y)$.

For a θ-slant submanifold M in a Kählerian n-manifold $\tilde{M}^n$, we have [2]

$$P^2 = -(\cos^2\theta)I, \quad \langle PX, Y\rangle + \langle X, PY\rangle = 0, \tag{2.5}$$

$$(\nabla_X P)Y = th(X,Y) + A_{FY}X, \tag{2.6}$$

$$D_X(FY) - F(\nabla_X Y) = fh(X,Y) - h(X,PY), \tag{2.7}$$

where I denotes the identity map and $fh(X,Y)$ is the normal part of $Jh(X,Y)$.

If we define a symmetric bilinear TM-valued form α on M by

$$\alpha(X,Y) = th(X,Y), \tag{2.8}$$

then we obtain

$$h(X,Y) = \csc^2\theta(P\alpha(X,Y) - J\alpha(X,Y)). \tag{2.9}$$

Denote by $\tilde{M}^m(4c)$ the complete simply-connected Kählerian m-manifold with constant holomorphic sectional curvature $4c$. Hence, $\tilde{M}^m(4c)$ is holomorphically isometric to $CP^m(4c)$, $\mathbf{C}^m$, or $CH^m(4c)$, according to $c > 0$, $c = 0$, or $c < 0$.

For an n-dimensional θ-slant submanifold with $\theta \neq 0$ in $\tilde{M}^n(4c)$, the equations of Gauss and Codazzi become

$$\begin{aligned} R(X,Y;Z,W) &= \csc^2\theta\{\langle\alpha(X,W),\alpha(Y,Z)\rangle - \langle\alpha(X,Z),\alpha(Y,W)\rangle\}+ \\ &+ c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle PX,W\rangle\langle PY,Z\rangle \\ &- \langle PX,Z\rangle\langle PY,W\rangle + 2\langle X,PY\rangle\langle PZ,W\rangle\}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &(\nabla_X\alpha)(Y,Z) + \csc^2\theta\{P\alpha(X,\alpha(Y,Z)) + \alpha(X,P\alpha(Y,Z))\}+ \\ &+(\sin^2\theta)c\{\langle X,PY\rangle Z + \langle X,PZ\rangle Y\} = \\ &= (\nabla_Y\alpha)(X,Z) + \csc^2\theta\{P\alpha(Y,\alpha(X,Z)) + \alpha(Y,P\alpha(X,Z))\}+ \\ &+(\sin^2\theta)c\{\langle Y,PX\rangle Z + \langle Y,PZ\rangle X\}. \end{aligned} \tag{2.11}$$

We recall the following Existence and Uniqueness Theorems from [7].

Theorem A (Existence Theorem). *Let c,θ be two constants with $0 < \theta \leq \frac{\pi}{2}$ and M a simply-connected Riemannian n-manifold with inner product $\langle\ ,\ \rangle$. Suppose there exist an endomorphism P of the tangent bundle TM and a symmetric bilinear TM-valued form α on M such that for $X,Y,Z,W \in TM$, we have*

$$P^2 = -(\cos^2\theta)I, \tag{2.12}$$

$$\langle PX,Y\rangle + \langle X,PY\rangle = 0, \tag{2.13}$$

$$\langle(\nabla_X P)Y,Z\rangle = \langle\alpha(X,Y),Z\rangle - \langle\alpha(X,Z),Y\rangle, \tag{2.14}$$

$$R(X,Y;Z,W) = \csc^2\theta\{\langle\alpha(X,W),\alpha(Y,Z)\rangle - \langle\alpha(X,Z),\alpha(Y,W)\rangle\}+ \tag{2.15}$$
$$+c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle PX,W\rangle\langle PY,Z\rangle -$$
$$-\langle PX,Z\rangle\langle PY,W\rangle + 2\langle X,PY\rangle\langle PZ,W\rangle\},$$

and

$$\begin{aligned}(\nabla_X\alpha)(Y,Z) \quad &+ \quad \csc^2\theta\{P\alpha(X,\alpha(Y,Z)) + \alpha(X,P\alpha(Y,Z))\}+ \\ &+ \quad (\sin^2\theta)c\{\langle X,PZ\rangle Y + \langle X,PY\rangle Z\}\end{aligned} \tag{2.16}$$

is totally symmetric. Then there exists a θ-slant isometric immersion from M into $\tilde{M}^n(4c)$ whose second fundamental form h is given by

$$h(X,Y) = \csc^2\theta(P\alpha(X,Y) - J\alpha(X,Y)). \tag{2.17}$$

Theorem B (Uniqueness Theorem). *Let $x^1, x^2 : M \to \tilde{M}^n(4c)$ be two θ-slant $(0 < \theta \le \frac{\pi}{2})$ isometric immersions of a connected Riemannian n-manifold M into the complex space form $\tilde{M}^n(4c)$ with second fundamental form h^1 and h^2. If $c \neq 0$ and*

$$\left\langle h^1(X,Y), Jx^1_\star Z\right\rangle = \left\langle h^2(X,Y), Jx^2_\star Z\right\rangle \tag{2.18}$$

for all vector fields X, Y, Z tangent to M, then $P_1 = P_2$ and there exists an isometry ϕ of $\tilde{M}^n(4c)$ such that $x^1 = \phi(x^2)$.

3 Proof of Theorem

Let $c \in \{1,-1\}$, $\theta \in (0, \frac{\pi}{2}]$, and let $G = G(x)$ be a differentiable function defined on an open interval I. Consider the following Riccati's differential equation:

$$\psi'(x) + \psi^2(x) + G(x) = 0. \tag{3.1}$$

The well-known existence theorem of first order differential equations implies that equation (3.1) does have many solutions on some neighborhoods of 0.

For each given solution $\psi = \psi(x)$ of (3.1) on an open interval containing 0, we consider the following system of first order differential equations:

(3.2)
$$\begin{aligned}
y_1'(x) \quad &= \quad \left\{2y_3^2 - 2y_1y_2 + (G(x) - c - 3c\cos^2\theta)\sin^2\theta\right\}\csc\theta\cot\theta - \\
&- \quad 3\psi y_1 + 3c\sin^2\theta\cos\theta, \\
y_2'(x) \quad &= \quad \left\{2y_1y_2 - 2y_3^2 - (G(x) - c - 3c\cos^2\theta)\sin^2\theta\right\}\csc\theta\cot\theta + \\
&+ \quad (2y_1 - y_2)\psi + 3c\sin^2\theta\cos\theta, \\
y_3'(x) \quad &= \quad \frac{\psi}{y_3}\{y_1^2 + y_3^2 - y_1y_2 + (G(x) - c - 3c\cos^2\theta)\sin^2\theta\} - 2\psi y_3 + \\
&+ \quad \left\{\tfrac{y_2}{y_3}\big(y_1^2 + y_3^2 - y_1y_2 + (G(x) - c - 3c\cos^2\theta)\sin^2\theta\big) - y_1y_3\right\}\csc\theta\cot\theta
\end{aligned}$$

with the initial conditions:

$$y_1(0) = c_1, \quad y_2(0) = c_2, \quad y_3 = c_3 \neq 0. \tag{3.3}$$

Although the equations in (3.2) can be written in slightly easier form, however, (3.2) is the most suitable for checking that condition (2.16) will be satisfied.

It is well-known that the system (3.2) with the initial conditions (3.3) has a unique solution $y_1 = \mu(x), y_2 = \delta(x), y_3 = \varphi(x)$ on some open interval I containing 0 on which $\varphi(x)$ is nowhere zero.

We put

$$\lambda = \frac{1}{\varphi}\{\mu^2 + \varphi^2 - \mu\delta + (G(x) - c - 3c\cos^2\theta)\sin^2\theta\}, \tag{3.4}$$

$$f(x) = \exp\Big(\int^x \psi\, dx\Big), \tag{3.5}$$

where $\displaystyle\int^x \psi dx$ is an anti-derivative of ψ.

Let M be a simply-connected domain containing the origin $(0,0)$ of the Euclidean plane $\mathbf{E}^2$. Assume that the metric on M is the warped product metric:

$$g = dx \otimes dx + f^2(x) dy \otimes dy. \tag{3.6}$$

Let $e_1 = \partial/\partial x, e_2 = f^{-1}\partial/\partial y$. Then $\{e_1, e_2\}$ is an orthonormal frame field of the tangent bundle TM of M. Then, by a direct computation, we have

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_1 = \psi e_2, \quad \nabla_{e_2} e_2 = -\psi e_1. \tag{3.7}$$

We define a symmetric bilinear TM-valued form α on M by

$$\alpha(e_1, e_1) = \lambda e_1 + \mu e_2, \;\; \alpha(e_1, e_2) = \mu e_1 + \varphi e_2, \;\; \alpha(e_2, e_2) = \varphi e_1 + \delta e_2. \tag{3.8}$$

Then

$$\langle \alpha(X, Y), Z\rangle = \langle \alpha(X, Z), Y\rangle \tag{3.9}$$

for X, Y, Z tangent to M.

It is well-known that the oriented Riemannian 2-manifold (M, g) admits a canonical Kählerian structure J, that is, (g, J) is a Hermitian structure on M with $\nabla J = 0$. If we put $P = \cos\theta J$, then we have

$$P^2 = -(\cos^2\theta) I, \quad \nabla P = 0, \quad \langle PX, Y\rangle + \langle X, PY\rangle = 0. \tag{3.10}$$

Hence, (M, P, α) satisfies conditions (2.12), (2.13) and (2.14).

On the other hand, by using (3.1)-(3.2), (3.4)-(3.7), and a straightforward long computation, we know that (M, P, α) also satisfies the remaining two conditions stated in the Existence Theorem. Thus, by applying the Existence and Uniqueness Theorem, we know that, up to rigid motions, there exists a unique θ-slant isometric immersion

$$\phi_{\psi, c_1, c_2, c_3} : M \to \tilde{M}^2(4c) \tag{3.11}$$

whose second fundamental form h is given by

$$h(X,Y) = \csc^2\theta(P\alpha(X,Y) - J\alpha(X,Y)). \tag{3.12}$$

From (3.1) and (3.5), we know that the Gaussian curvature of the slant surface is given by G.

Since, for any prescribed Gaussian curvature $G = G(x)$, the function ψ can be chosen to be any of the solutions of the Riccati equation (3.1) and c_1, c_2, c_3 to be any three real numbers with $c_3 \neq 0$, we conclude that there exist infinitely many θ-slant surfaces in $CP^2(4)$ and in $CH^2(-4)$ with G as the prescribed Gaussian curvature.

□

4 Some explicit solutions of the differential system

Here, we provide some explicit solutions of the differential system (3.1)-(3.2) with $c = \pm 1$. For simplicity, we assume that $G = 0$ and we choose $\psi = 0$ which is the trivial solution of the Riccati equation (3.1). Then the differential system (3.2) reduces to

$$y_1'(x) = 2(y_3^2 - y_1 y_2)\csc\theta\cot\theta - c(1 + 3\cos 2\theta)\cos\theta, \tag{4.1}$$

$$y_2'(x) = 2(y_1 y_2 - y_3^2)\csc\theta\cot\theta + 4c\cos\theta, \tag{4.2}$$

$$y_3 y_3' = \big[y_2\big(y_1^2 - y_1 y_2 - (c + 3c\cos^2\theta)\sin^2\theta\big) + (y_2 - y_1)y_3^2\big]\csc\theta\cot\theta. \tag{4.3}$$

Summing up (4.1) and (4.2) gives

$$(y_1 + y_2)'(x) = 6c\cos\theta\sin^2\theta. \tag{4.4}$$

Thus, by solving (4.4), we have

$$y_2 = 6cx\cos\theta\sin^2\theta - y_1 - b \tag{4.5}$$

for some constant b. Substituting (4.5) into (4.1) yields

$$y_1'(x) = 2(y_1^2 + y_3^2 + by_1)\csc\theta\cot\theta - 12cxy_1\cos^2\theta - c(1 + 3\cos 2\theta)\cos\theta. \tag{4.6}$$

If we denote y_3^2 by Φ, then (4.6) becomes

$$\Phi = \frac{y_1'}{2}\sin\theta\tan\theta - (b + y_1)y_1 + \frac{c}{2}(1 + 12xy_1\cos\theta + 3\cos 2\theta)\sin^2\theta. \tag{4.7}$$

By differentiating (4.6) we find

$$y_1''(x) = 2((b + 2y_1)y_1' + 2y_3y_3')\csc\theta\cot\theta - 12cy_1\cos^2\theta - 12cxy_1'\cos^2\theta. \tag{4.8}$$

Hence, by substituting (4.3), (4.5) and (4.7) into (4.8), we obtain

$$y_1''(x) = 4c(2b - 3cx\cos\theta + 3cx\cos 3\theta)\cot^2\theta. \tag{4.9}$$

After solving the differential equation (4.9), we have

$$y_1(x) = c_1 + c_2 x + 4bcx^2\cot^2\theta - 8x^3\cos^3\theta \tag{4.10}$$

for some integration constants c_1, c_2.

From (4.5) and (4.10) we obtain

$$y_2 = 6cx\cos\theta\sin^2\theta - 4bcx^2\cot^2\theta + 8x^3\cos^3\theta - b - c_1 - c_2x. \tag{4.11}$$

Hence, if we substituting (4.10) and (4.11) into (4.7), we have

$$\begin{aligned} y_3^2(x) &= -(c_1 + c_2x + 4bcx^2\cot^2\theta - 8x^3\cos^3\theta)\times \\ &\times(b + c_1 + c_2x + 4bcx^2\cot^2\theta - 8x^3\cos^3\theta) \\ &+\frac{c}{2}\big[1 + 3\cos 2\theta + 12x\cos\theta(c_1 + c_2x + 4bcx^2\cot^2\theta - 8x^3\cos^3\theta)\big]\sin^2\theta \\ &\quad+\frac{1}{2}(c_2 - 24x^2\cos^3\theta + 8bcx\cot^2\theta)\sin\theta\tan\theta. \end{aligned} \tag{4.12}$$

In particular, if $c = 1$ and $b = c_1 = c_2 = 0$, then we obtain the explicit solution of (3.1)-(3.2):

$$y_1 = -8x^3\cos^3\theta, \tag{4.13}$$

$$y_2 = 6x\sin^2\theta\cos\theta + 8x^3\cos^3\theta, \tag{4.14}$$

$$y_3^2 = \frac{1}{2}\big(1 + 3\cos 2\theta - 96x^4\cos^4\theta\big)\sin^2\theta - 3x^2\sin^2 2\theta - 64x^6\cos^6\theta. \tag{4.15}$$

Conversely, it is straightforward to verify that (4.10)-(4.12) satisfies the differential system (4.1)-(4.3).

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