Differential-Geometrical Conditions Between Geodesic Curves and Ruled Surfaces in the Lorentz Space

Nihat Ayyildiz, A. Ceylan Çöken, Ahmet Yücesan

Abstract

In this paper, a system of differential equations determining timelike and spacelike ruled surfaces are established in the lines space, using the invariant quantities of a given geodesic curves on the surface in Lorentz space. The solution of the system of differential equations are obtained in spacial cases and as relation of this solutions are given corollaries.

Mathematics Subject Classification: 53C50
Key words: timelike ruled surface, geodesic curve, Blaschke vector, Lorentz space

1 Introduction

Let $\mathbf{R}^3$ be endowed with the pseudo scalar product of x and y is defined by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,\ \forall x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathbf{R}^3.$$

$(R^3, \langle,\rangle)$ is called 3-dimensional Lorentzian space, or Minkowski 3-space denoted by L^3 [6]. Then the Lorentzian vector product is defined by

$$x \times y = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2),\ \forall x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in L^3$$

where $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$ and $e_3 \times e_1 = -e_2$ [1].

A vector x in L^3 is called a spacelike, lightlike, timelike vector if $\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, \rangle x < 0$ accordingly. For $x \in L^3$, the norm of x defined by $\parallel x \parallel = \sqrt{|\langle x, x\rangle|}$, and x is called a unit vector if $\parallel x \parallel = 1$ [8].

Let

$$\alpha : I \subset \mathbf{R} \longrightarrow L^3, \quad s \longrightarrow \alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$$

be a smooth regular curve in L^3 (i.e., $\alpha^{'}(t) > 0$ for any $t \in I$), where I is an open interval. For any $t \in I$, the curve α is called a spacelike, lightlike or timelike curve if $\langle\alpha^{'}, \alpha^{'}\rangle > 0$, $= 0$ or > 0, respectively [8].

Balkan Journal of Geometry and Its Applications, Vol.7, No.1, 2002, pp. 1-12.

Now, we will define the dual Lorentz space under the light of the information given above. The set $D = \{\hat{x} = x + \xi x^* | x, x^* \in \mathbf{R}\}$ of dual numbers is a commutative ring with respect to the operations

i) $(x + \xi x^*) + (y + \xi y^*) = (x + y) + \xi(x^* + y^*)$

ii) $(x + \xi x^*) \cdot (y + \xi y^*) = xy + \xi(xy^* + yx^*)$.

The division $\frac{\hat{x}}{\hat{y}}$ is possible and unambiguous if $y \neq 0$ and it easily see that

$$\frac{\hat{x}}{\hat{y}} = \frac{x + \xi x^*}{y + \xi y^*} = \frac{x}{y} + \xi \frac{x^* y - x y^*}{y^2}.$$

The set

$$\begin{aligned} D^3 = D \times D \times D = \{\hat{x} | \hat{x} &= (x_1 + \xi x_1^*, x_2 + \xi x_2^*, x_3 + \xi x_3^*) \\ &= (x_1, x_2, x_3) + \xi(x_1^*, x_2^*, x_3^*) \\ &= x + \xi x^*, \quad x \in R^3, x^* \in \mathbf{R}^3\} \end{aligned}$$

is a module over the ring D. Let $\hat{x} = x + \xi x^*$, $\widehat{y} = y + \xi y^*$. The Lorentzian inner product of $\hat{x}$ and $\hat{y}$ is defined by

$$\langle \hat{x}, \, \hat{y} \rangle = \langle x, \, y \rangle + \xi(\langle x, \, y^* \rangle + \langle x^*, \, y \rangle).$$

We call the dual space D^3 together with this Lorentzian inner product as dual Lorentzian space and denote this by D_1^3. It is clear that any dual vector $\hat{x}$ in D_1^3, consists of any two real vectors x and x^* in L^3, which are expressed in the natural right handed orthonormal frame in the $3-$dimensional Lorentzian space L^3. We call the elements of D_1^3 the dual vectors. If $x \neq 0$ the norm $\|\hat{x}\|$ of $\hat{x}$ is defined by $\|\hat{x}\| = \sqrt{|\langle \hat{x}, \hat{x} \rangle|}$.

Let $\hat{x}$ be dual vector. $\hat{x}$ is said to be spacelike, timelike, lightlike (null) if the vector x is spacelike, timelike, lightlike (null), respectively. Then

$$S_1^2 = \{\, \hat{x} = x + \xi x^* | \; \|\hat{x}\| = (1, \, 0); \, x, \, x^* \in R_1^3, \, x \text{ spacelike } \}$$

is called *the dual Lorentzian unit sphere* in D_1^3

$$H_0^2 = \{\, \hat{x} = x + \xi x^* | \; \|\hat{x}\| = (1, \, 0); \, x, \, x^* \in \mathbf{R}_1^3, \, x \text{ timelike } \}$$

is called the dual hyperbolic unit sphere in D_1^3. Oriented timelike and spacelike lines in L^3 may be represented by timelike and spacelike unit vectors with three-components in the Dual Lorentzian space D_1^3, respectively. A differentiable curve on the dual hyperbolic unit sphere H_0^2 corresponds to a timelike ruled surface while a differentiable curve on the dual Lorentzian unit sphere S_1^2 corresponds to any ruled surface [11].

The drall of ruled surface determining of generator line $X(t) = x(t) + \xi x^*(t)$ is defined by

$$\delta = \frac{< x(t), x^*(t) >}{< x(t), x^*(t) >} = \frac{p^*}{p}.$$

If $\delta = 0$, $p^* = 0$ then the ruled surface is developable [2, 4].

2 Preliminaries

Definition 2.1. A symmetric bilinear b on vector space V is

i) positive [negative] definite provided $v \neq 0$ implies $b(v, v) > 0$ [< 0],

ii) positive [negative] semidefinite provided $b(v, v) \geq 0$ [≤ 0] for all $v \in V$,

iii) nondegenerate provided $b(v, w) = 0$ for all $w \in V$ implies $v = 0$ [8].

Definition 2.2. A scalar product g on a vector space V is a nondegenerate symmetric bilinear form on V [8].

Definition 2.3. The index ν of symmetric bilinear form b on $V$is the largest integer that is the dimension of a subspace $W \subset V$ on which $g|_W$ is negative definite [8].

Lemma 2.1 *A scalar product space $V \neq 0$ has an orthonormal basis* [8].

Lemma 2.2 *Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis for V, with $\varepsilon_i =< e_i, e_i >$. Then each $v \in V$ has a uniqe expression [8],*

$$v = \sum_{i=1}^{n} \varepsilon_i < v, e_i > e_i.$$

Lemma 2.3 *For any orthonormal basis $\{e_1, e_2, ..., e_n\}$ for V, the number of negative signs in the signature $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ is the index ν of V* [8].

Definition 2.4. A metric tensor g on a smooth manifold M is a symmetric nondegenerate $(0, 2)$ tensor field on M of constant index [8].

Definition 2.5. A semi-Riemannian manifold is a smooth manifold furnished with a metric tensor g [8].

Definition 2.6. A semi-Riemannian submanifold $\overline{M}$ with $(n-1)$-dimensional of a semi-Riemannian manifold M with n-dimenisonal is called semi-Riemannian hypersurface of M [8].

Definition 2.7. A geodesic in a semi-Riemannian manifold M is a curve $\alpha : I \to M$ whose vector field α' is parallel. Equivalently, geodesics are the curves of acceleration zero, $\alpha' = 0$ [8].

3 Differential-Geometrical Conditions Between Geodesic Curves and Timelike-Ruled Surfaces in Lorentz Space

Given a curve $\alpha(s)$ on a surface M in Lorentz space as arc-lenght parameter. Let $\{V_1(s), V_2(s), V_3(s)\}$ be Frenet trihedron at the point $\alpha(s)$ of given curve $\alpha(s)$.

In this situation, equalities relation with covariant derivations curve throughout of Frenet vectors $V_i(s), 1 \leq i \leq 3$, are written in form

$$(1) \qquad \begin{bmatrix} V_1'(s) \\ V_2'(s) \\ V_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix}.$$

Let $\{\eta_1(s), \eta_2(s), \eta_3(s)\}$ be Darboux trihedron at the point $\alpha(s)$ of surface. Then the derivatives of Darboux vectors are

$$\begin{bmatrix} \eta_1^{'}(s) \\ \eta_2^{'}(s) \\ \eta_3^{'}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ \kappa_g & 0 & \tau_g \\ \kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \eta_1(s) \\ \eta_2(s) \\ \eta_3(s) \end{bmatrix}. \tag{2}$$

A timelike-ruled surface in lines space such that dual unit vectorel function $\vec{X} = \vec{X(t)}$ depending on a parameter t is written as

$$\vec{X} = \vec{X(t)} = \vec{x(t)} + \xi \vec{x^\star(t)},$$

where $\xi = (0,1)$ is a dual unit.

Now, let

$$\begin{aligned} \vec{X_1(t)} &= \vec{x_1(t)} + \xi \vec{x_1^\star(t)}, \\ \vec{X_2(t)} &= \frac{\vec{X_1^{'}(t)}}{\|\vec{X_1^{'}(t)}\|} = \frac{\vec{X_1^{'}(t)}}{P}, \\ \vec{X_3(t)} &= \vec{X_1(t)} \times \vec{X_2(t)} \end{aligned}$$

be a trihedron depending on timelike-ruled surface. In that case, it is written

$$\begin{bmatrix} \vec{X_1^{'}(t)} \\ \vec{X_2^{'}(t)} \\ \vec{X_3^{'}(t)} \end{bmatrix} = \begin{bmatrix} 0 & P & 0 \\ P & 0 & Q \\ 0 & -Q & 0 \end{bmatrix} \begin{bmatrix} \vec{X_1(t)} \\ \vec{X_2(t)} \\ \vec{X_3(t)} \end{bmatrix}. \tag{3}$$

In this equation, $P = p + \xi p^\star$ and $Q = q + \xi q^\star$ are the dual invariants of the timelike-ruled surface, which are defined as

$$P = \left\|\vec{X_1^{'}(t)}\right\|, \quad Q = < \vec{X_2^{'}(t)}, \vec{X_3(t)} > .$$

If the moments of the Darboux vectors at the point $\alpha(s)$ of the curve $\alpha(s)$ are taken with respect to origin in the $(O; x, y, z)$ coordinate system, the dual unit vectors which are defined as

$$\begin{aligned} X_i(s) &= \eta_i(s) + \xi \eta_i^\star(s) \\ &= \eta_i(s) + \xi(\vec{\alpha(s)} \Lambda \eta_i(s)), 1 \le i \le 3 \end{aligned}$$

form the base $\{X_1(s), X_2(s), X_3(s)\}$ in the lines space. When the point $\alpha(s)$ traces the curve $\alpha(s)$ in Lorentz space, the dual unit vector $X_1(s)$ generates surface in the lines space. The $\{X_1(s), X_2(s), X_3(s)\}$ belonging to the generator $X_1(s)$ is Blaschke trihedron of this surface. The vector $\alpha(s)$ is written according to Darboux vectors of the curve $\alpha(s)$ as

$$\alpha(s) = m(s)\eta_1(s) + n(s)\eta_2(s) + k(s)\eta_3(s). \tag{4}$$

Hence, we obtain the relations

$$\begin{aligned} X_1(s) &= \eta_1(s) + \xi[-k(s)\eta_2(s) + n(s)\eta_3(s)] \\ X_2(s) &= \eta_2(s) + \xi[-k(s)\eta_1(s) - m(s)\eta_3(s)] \\ X_3(s) &= \eta_3(s) + \xi[n(s)\eta_1(s) + m(s)\eta_2(s)]. \end{aligned}$$

If $m(s)$, $n(s)$ and $k(s)$ coefficients are found, timelike-ruled surfaces $X_1(s)$ can be determined with respect to the invariants of the given curve $\alpha(s)$. If the equation (4) is differentiated with respect to s and making use of the derivative formulas (2), we obtain

$$\left\{\begin{array}{lcl} m^{'}(s) + \ n(s)\kappa_g + \ k(s)\kappa_n & = & 1 \\ m^{'}(s)\kappa_g + \ n^{'}(s) - \ k(s)\tau_g & = & 0 \\ m(s)\kappa_n + \ n(s)\tau_g + \ k^{'}(s) & = & 0. \end{array}\right. \tag{5}$$

For geodesic curves ($\kappa_g = 0$). This system becomes

$$\left\{\begin{array}{lcl} m^{'}(s) + \ k(s)\kappa_n & = & 1 \\ n^{'}(s) - \ k(s)\tau_g & = & 0 \\ k^{'}(s) + \ m(s)\kappa_n + \ n(s)\tau_g & = & 0. \end{array}\right. \tag{6}$$

Now we solve the system of differential equation (6) for the certain special cases.

3.1. If $m(s) = 0$, the curve $\alpha(s)$ is located in affine subspace, combined with the vector space $S_p\{\eta_2(s), \eta_3(s)\}$ at the point $\alpha(s)$. In this situation, from the system (6)

$$\left\{\begin{array}{lcl} k(s)\kappa_n & = & 1 \\ n^{'}(s) - \ k(s)\tau_g & = & 0 \\ k^{'}(s) + \ n(s)\tau_g & = & 0 \end{array}\right. \tag{7}$$

are found. From the last two equations we get

$$n^{''}(s) - \frac{\tau_g^{'}}{\tau_g} n^{'}(s) + \tau_g^2 n(s) = 0.$$

If we make the parameter change as $t = \displaystyle\int_0^s \tau_g ds$ in this equation we obtain $\dfrac{d^2n}{dt^2} + n = 0$, and get the solution of this equation

$$n(s) = c_1 \cos\left(\int_0^s \tau_g ds\right) + c_2 \sin\left(\int_0^s \tau_g ds\right),$$

where c_1 and c_2 are real constants. From the first equation of (7), it is clear that

$$k(s) = \frac{1}{\kappa_n}.$$

In this case with the aid of equation (4), $\alpha(s)$ can be written as

$$\alpha(s) = \left[c_1 \cos\left(\int_0^s \tau_g ds\right) + c_2 \sin\left(\int_0^s \tau_g ds\right)\right] \eta_2(s) + \frac{1}{\kappa_n}\eta_3(s).$$

Therby, Blaschke vectors of timelike-ruled surface $X_1(s)$ are determined by dual unit vectors such as

$$\begin{array}{lcl} X_1(s) & = & \eta_1(s) + \xi \left[-\dfrac{1}{\kappa_n}\eta_2(s) + c_1 \cos\left(\displaystyle\int_0^s \tau_g ds\right) + c_2 \sin\left(\displaystyle\int_0^s \tau_g ds\right) \eta_3(s)\right] \\ X_2(s) & = & \eta_2(s) + \xi \left[-\dfrac{1}{\kappa_n}\eta_1(s)\right] \\ X_3(s) & = & \eta_3(s) + \xi \left[c_1 \cos\left(\displaystyle\int_0^s \tau_g ds\right) + c_2 \sin\left(\displaystyle\int_0^s \tau_g ds\right) \eta_1(s)\right]. \end{array}$$

If $c_1 = c_2 = 0$, then $n(s) = 0$ and $k(s) = \dfrac{1}{\kappa_n}$ are found. Therefore,

$$\begin{cases} X_1(s) &= \eta_1(s) + \xi\left[-\frac{1}{\kappa_n}\eta_2(s)\right] \\ X_2(s) &= \eta_2(s) + \xi\left[-\frac{1}{\kappa_n}\eta_1(s)\right] \\ X_3(s) &= \eta_3(s). \end{cases}$$

The first and third equalities of this system are differentiated with respect to s and if the values found are used in (2) and (3), we obtain

$$p = \kappa_g, \quad p^\star = 0, \qquad q = \tau_g, \quad q^\star = 0.$$

Corollary 3.1 *The ruled surface determining of generator line $X_3 = X_3(s)$ is a developable surface.*

3.2. If $n(s) = 0$, then the curve $\alpha(s)$ is located in affine subspace, combined with vector space $S_p\{\eta_1(s), \eta_3(s)\}$ at the point $\alpha(s)$. In this case,

$$\begin{aligned} m'(s) + \ k(s)\kappa_n &= 1 \\ -k(s)\tau_g &= 0 \\ k'(s) + \ m(s)\kappa_n &= 0 \end{aligned}$$

is obtained from the system (6). The solutions of the first and third equations of this system are in the form

$$m(s) = c_1 e^{-\kappa_n s} - c_2 e^{\kappa_n s}, \quad k(s) = c_1 e^{-\kappa_n s} + c_2 e^{\kappa_n s} + \frac{1}{\kappa_n},$$

where c_1 and c_2 are real constants. Thus, the vector $\alpha(s)$ is in the form

$$\alpha(s) = \ (c_1 e^{-\kappa_n s} - \ c_2 e^{\kappa_n s})\eta_1(s) + (c_1 e^{-\kappa_n s} + \ c_2 e^{\kappa_n s} + \frac{1}{\kappa_n})\eta_3(s).$$

However, Blaschke vectors of timelike-ruled surface $X_1(s)$ are derived

$$\begin{aligned} X_1(s) &= \eta_1(s) + \xi\left[-\left(c_1 e^{-\kappa_n s} + c_2 e^{\kappa_n s} + \frac{1}{\kappa_n}\right)\eta_2(s)\right] \\ X_2(s) &= \eta_2(s) + \xi\left[-\left(c_1 e^{-\kappa_n s} + c_2 e^{\kappa_n s} + \frac{1}{\kappa_n}\right)\eta_1(s) - \left(c_1 e^{-\kappa_n s} - c_2 e^{\kappa_n s}\right)\eta_3(s)\right] \\ X_3(s) &= \eta_3(s) + \xi[(c_1 e^{-\kappa_n s} - c_2 e^{\kappa_n s})\eta_2(s)]. \end{aligned}$$

In this system, the first and third equalities are differentiated in terms of s and if the values found are used in the equalities (2) and (3), we obtain

$$p = \kappa_g, \quad p^\star = c_1\kappa_n e^{-\kappa_n s} - c_2\kappa_n e^{\kappa_n s} \quad q = \tau_g, \quad q^\star = c_1\kappa_n e^{-\kappa_n s} + c_2\kappa_n e^{\kappa_n s}.$$

Corollary 3.2 *If $c_1 = c_2 = 0$ then the ruled surface determining of generator line $X_3 = X_3(s)$ is a developable surface.*

3.3. If $k(s) = 0$, the curve $\alpha(s)$ is located in an affine subspace, combined with vector space $S_p\{\eta_1(s), \eta_2(s)\}$ at the point $\alpha(s)$. In this case, the curve $\alpha(s)$ becomes a plane curve. Because the vector $\alpha(s)$ lies in the osculating plane of the curve. Thus, from the system (6), we obtain

$$\begin{cases} m^{'}(s) &= 1 \\ n^{'}(s) &= 0 \\ m(s)\kappa_n + n(s)\tau_g &= 0. \end{cases}$$

and further,

$$m(s) = s + c_1, \qquad n(s) = c_2.$$

In this case the vector $\alpha(s)$ by the aid of equation (4) can be written as

$$\alpha(s) = (s + c_1)\eta_1(s) + (c_2)\eta_2(s).$$

So, Blaschke vectors of timelike-ruled surface $X_1(s)$ are found

$$\begin{cases} X_1(s) &= \eta_1(s) + \xi[c_2\eta_3(s)] \\ X_2(s) &= \eta_2(s) + \xi[-(s + c_1)\eta_3(s)] \\ X_3(s) &= \eta_3(s) + \xi[c_2\eta_1(s) + (s + c_1)\eta_2(s)]. \end{cases}$$

The first and third equalities of this system are differentiated in terms of s and if the values found are used in the equalities (2) and (3), we obtain

$$p = \kappa_g, \quad p^\star = -c_2\tau_g, \quad q = \tau_g, \quad q^\star = 1 + c_2\kappa_g,$$

and hence we state

Corollary 3.3 *The ruled surface determining of generator line $X_3 = X_3(s)$ is not a developable surface.*

4 Differential-Geometrical Conditions Between Geodesic Curves and Spacelike-Ruled Surfaces

Given a curve $\alpha(s)$ on a surface M in Lorentz space as arc-length parameter.Let $\{V_1(s), V_2(s), V_3(s)\}$ be Frenet trihedron at the point $\alpha(s)$ of a curve defined as

$$\alpha : I \subset IR \to L^3, \quad s \to \alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s)).$$

In this situation, the Frenet equations satisfied by the Frenet vectors $V_i(s), 1 \leq i \leq 3$, formally given by

$$\begin{pmatrix} V_1^{'}(s) \\ V_2^{'}(s) \\ V_3^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{pmatrix}. \tag{8}$$

Let $\{\eta_1(s), \eta_2(s), \eta_3(s)\}$ be Darboux trihedron at the point $\alpha(s)$ of surface. Then the Darboux equations are

$$\begin{pmatrix} \eta_1'(s) \\ \eta_2'(s) \\ \eta_3'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ \kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{pmatrix} \begin{pmatrix} \eta_1(s) \\ \eta_2(s) \\ \eta_3(s) \end{pmatrix}, \tag{9}$$

where $\kappa_g, \kappa_n, \tau_g$ are the geodesic curvature, normal curvature and geodesic torsion respectively.

A spacelike-ruled surface is given by the dual unit vectorial function $\vec{X} = \vec{X(t)}$ depending on a parameter t as

$$\vec{X} = \vec{X(t)} = \vec{x(t)} + \xi \vec{x^\star(t)},$$

where $\xi = (0, 1)$ is a dual unit.

Now, let us study to conduct a trihedron depending on spacelike-ruled surfaces. Respectively, let the first, the second and the third axes be

$$\begin{aligned} \vec{X_1(t)} &= \vec{x_1(t)} + \xi \vec{x_1^\star(t)}, \\ \vec{X_2(t)} &= \frac{\vec{X_1'(t)}}{\|\vec{X_1'(t)}\|} = \frac{\vec{X_1'(t)}}{P}, \\ \vec{X_3(t)} &= \vec{X_1(t)} \times \vec{X_2(t)}. \end{aligned}$$

In that case, it is written

$$\begin{pmatrix} \vec{X_1'(t)} \\ \vec{X_2'(t)} \\ \vec{X_3'(t)} \end{pmatrix} = \begin{pmatrix} 0 & P & 0 \\ -P & 0 & Q \\ 0 & Q & 0 \end{pmatrix} \begin{pmatrix} \vec{X_1(t)} \\ \vec{X_2(t)} \\ \vec{X_3(t)} \end{pmatrix}. \tag{10}$$

In this equation, $P = p + \xi p^\star$ and $Q = q + \xi q^\star$ are the dual invariants of the spacelike-ruled surface, which are defined as

$$P = \left\| \vec{X_1'(t)} \right\|, \quad Q = \langle \vec{X_2'(t)}, \vec{X_3(t)} \rangle.$$

If the moments of the Darboux vectors at the point $\alpha(s)$ of the curve $\alpha(s)$ are taken with respect to origin in the coordinate system $\{O; x, y, z\}$, the dual unit vectors which are defined as

$$\begin{aligned} X_i(s) &= \eta_i(s) + \xi \eta_i^\star(s) \\ &= \eta_i(s) + \xi(\vec{\alpha(s)} \Lambda \eta_i(s)), \ \ 1 \leq i \leq 3 \end{aligned}$$

form the base $\{X_1(s), X_2(s), X_3(s)\}$ in the affine space and these vectors have the following property

$$\langle X_i(s), X_j(s) \rangle = \begin{cases} \varepsilon(X_i), & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

When the point $\alpha(s)$ traces the curve α in Lorentz space, the dual unit vector $X_1(s)$ generates surface in the affine space. The $\{X_1(s), X_2(s), X_3(s)\}$ belonging to the generator $X_1(s)$ is Blaschke trihedron of this surface. The vector $\alpha(s)$ is written according to Darboux vectors of the curve $\alpha(s)$ as

$$\alpha(s) = \ m(s)\eta_1(s) + \ n(s)\eta_2(s) + \ k(s)\eta_3(s). \tag{11}$$

Hence,

$$\begin{cases} X_1(s) &= \eta_1(s) + \xi[-\ k(s)\eta_2(s) -\ n(s)\eta_3(s)] \\ X_2(s) &= \eta_2(s) + \xi[\ k(s)\eta_1(s) +\ m(s)\eta_3(s)] \\ X_3(s) &= \eta_3(s) + \xi[-\ n(s)\eta_1(s) +\ m(s)\eta_2(s)]. \end{cases}$$

If are found $m(s)$, $n(s)$ and $k(s)$, then the spacelike-ruled surface $X_1(s)$ can be determined with respect to the invariants of the given curve $\alpha(s)$. If the equation (11) is differentiated with respect to s, and making use of the derivative formulas (9), we obtain

$$\begin{cases} m^{'}(s) +\ n(s)\kappa_g +\ k(s)\kappa_n &= 1 \\ m(s)\kappa_g +\ n^{'}(s) +\ k(s)\tau_g &= 0 \\ m(s)\kappa_n +\ n(s)\tau_g +\ k^{'}(s) &= 0. \end{cases} \tag{12}$$

If the curve be a geodesic curve ($\kappa_g = 0$), the system (12) takes the following form

$$\begin{cases} m^{'}(s) +\ k(s)\kappa_n &= 1 \\ n^{'}(s) +\ k(s)\tau_g &= 0 \\ k^{'}(s) +\ m(s)\kappa_n +\ n(s)\tau_g &= 0. \end{cases} \tag{13}$$

Now we solve the system of differential equation (13) for certain special cases.

4.1. If $m(s) = 0$, then the curve $\alpha(s)$ is located in an affine subspace, combined with vector space $S_p\{\eta_2(s), \eta_3(s)\}$ at the point $\alpha(s)$. In this situation, from the system (13) we infer

$$\begin{cases} k(s)\kappa_n &= 1 \\ n^{'}(s) +\ k(s)\tau_g &= 0 \\ k^{'}(s) +\ n(s)\tau_g &= 0. \end{cases} \tag{14}$$

Hence, from the last two equations, we get

$$n^{''}(s) - \frac{\tau_g^{'}}{\tau_g} n^{'}(s) - \tau_g^2 n(s) = 0.$$

If we make the parameter change as $t = \int_0^s \tau_g ds$ in this equation, we obtain

$$\frac{d^2 n}{dt^2} - n = 0.$$

The solution of this equation is

$$n(s) = c_1 e^{-(\int_0^s \tau_g ds)} + c_2 e^{(\int_0^s \tau_g ds)},$$

where c_1 and c_2 are real constants. From (14) it is clear that

$$k(s) = \frac{1}{\kappa_n}.$$

In this case with the aid of equation (11) $\alpha(s)$ can be written as

$$\alpha(s) = [c_1 e^{-(\int_0^s \tau_g ds)} + c_2 e^{(\int_0^s \tau_g ds)}]\eta_2(s) + \frac{1}{\kappa_n}\eta_3(s).$$

Therby, Blaschke vectors of the spacelike-ruled surface $X_1(s)$ are determined by the dual unit vectors

$$\begin{cases} X_1(s) &=& \eta_1(s)+\xi\left[-\frac{1}{\kappa_n}\eta_2(s)-\left(c_1e^{-(\int_0^s\tau_g ds)}+c_2e^{\int_0^s\tau_g ds}\right)\eta_3(s)\right] \\ X_2(s) &=& \eta_2(s)+\xi\left[\frac{1}{\kappa_n}\eta_1(s)\right] \\ X_3(s) &=& \eta_3(s)+\xi\left[-\left(c_1e^{-(\int_0^s\tau_g ds)}+c_2e^{\int_0^s\tau_g ds}\right)\eta_1(s)\right] \end{cases}$$

If $c_1=c_2=0$, then $n(s)=0$ and $k(s)=\frac{1}{\kappa_n}$ are found. Therefore,

$$\begin{cases} X_1(s) &=& \eta_1(s)+\xi\left[-\frac{1}{\kappa_n}\eta_2(s)\right] \\ X_2(s) &=& \eta_2(s)+\xi\left[\frac{1}{\kappa_n}\eta_1(s)\right] \\ X_3(s) &=& \eta_3(s). \end{cases}$$

The first and third equalities of this system are differentiated with respect to s and if the values found are used in the (9) and (10), we obtain

$$p=\kappa_g,\quad p^\star=0\quad q=\tau_g,\quad q^\star=0.$$

Corollary 4.1 *The spacelike-ruled surfaces $X_3=X_3(s)$ is a developable surface.*

4.2. If $n(s)=0$, the curve $\alpha(s)$ is located in an affine subspace, combined with the vector space $S_p\{\eta_1(s),\eta_3(s)\}$ at the point $\alpha(s)$. In this case,

$$\begin{cases} m'(s)+\ k(s)\kappa_n &=& 1 \\ k(s)\tau_g &=& 0 \\ k'(s)+\ m(s)\kappa_n &=& 0 \end{cases}$$

are obtained from the system (13). The solutions of the first and third equations of this system are in the form

$$m(s)=\ c_1e^{-\kappa_n s}-\ c_2e^{\kappa_n s},\qquad k(s)=\ c_1e^{-\kappa_n s}+\ c_2e^{\kappa_n s}+\tfrac{1}{\kappa_n}$$

where c_1 and c_2 are real constants. Thus, the vector $\alpha(s)$ has the shape

$$\alpha(s)=\ (c_1e^{-\kappa_n s}-\ c_2e^{\kappa_n s})\eta_1(s)+(c_1e^{-\kappa_n s}+\ c_2e^{\kappa_n s}+\tfrac{1}{\kappa_n})\eta_3(s).$$

However, Blaschke vectors of the spacelike-ruled surface $X_1(s)$ hence

$$\begin{cases} X_1(s) &=& \eta_1(s)+\xi\left[-\left(c_1e^{-\kappa_n s}+c_2e^{\kappa_n s}+\frac{1}{\kappa_n}\right)\eta_2(s)\right] \\ X_2(s) &=& \eta_2(s)+\xi\left[\left(c_1e^{-\kappa_n s}+c_2e^{\kappa_n s}+\frac{1}{\kappa_n}\right)\eta_1(s)+(c_1e^{-\kappa_n s}-c_2e^{\kappa_n s})\eta_3(s)\right] \\ X_3(s) &=& \eta_3(s)+\xi[(c_1e^{-\kappa_n s}-c_2e^{\kappa_n s})\eta_2(s)]. \end{cases}$$

In this system, the first and third equalities are differentiated in terms of s and if the values found are used in the equalities (9) and (10), we obtain

$$p=\kappa_g,\quad p^\star=c_1\kappa_ne^{-\kappa_n s}-c_2\kappa_ne^{\kappa_n s}\quad q=\tau_g,\quad q^\star=-c_1\kappa_ne^{-\kappa_n s}-c_2\kappa_ne^{\kappa_n s}.$$

Corollary 4.2 *If $c_1 = c_2 = 0$ then the spacelike-ruled surface $X_3 = X_3(s)$ is a developable surface.*

4.3. If $k(s) = 0$, the curve $\alpha(s)$ is located in affine subspace, combined with the vector space $S_p\{\eta_1(s), \eta_2(s)\}$ at the point $\alpha(s)$. In this case, the curve $\alpha(s)$ becomes a plane curve, since the vector $\alpha(s)$ lies in the osculating plane of the curve. Thus, from the (13), we obtain

$$\begin{cases} m^{'}(s) & = & 1 \\ n^{'}(s) & = & 0 \\ m(s)\kappa_n + n(s)\tau_g & = & 0. \end{cases}$$

whence

$$m(s) = \; s + c_1, \qquad n(s) = \; c_2.$$

In this case, using (11), the vector $\alpha(s)$ can be written as

$$\alpha(s) = \; (s + c_1)\eta_1(s) + (c_2)\eta_2(s).$$

Then, Blaschke vectors of the spacelike-ruled surface $X_1(s)$ are

$$\begin{cases} X_1(s) & = & \eta_1(s) + \xi[-c_2\eta_3(s)] \\ X_2(s) & = & \eta_2(s) + \xi[(s + c_1)\eta_3(s)] \\ X_3(s) & = & \eta_3(s) + \xi[-c_2\eta_1(s) + (s + c_1)\eta_2(s)]. \end{cases}$$

The first and third equations in this system differentiated in terms of s and if the values found and used in the equalities (9) and (10), lead to

$$p = \kappa_g, \quad p^\star = -c_2\tau_g, \quad q = \tau_g, \quad q^\star = 1 - c_2\kappa_g.$$

Hence we infer

Corollary 4.3 *The ruled surface determined by the generator line $X_3 = X_3(s)$ is not a developable surface.*

References

[1] K. Akutagawa and S. Nishikawa, *The Gauss Map and Spacelike Surfaces with Prescribed Mean Curvature in Minkowski* 3*-Space*, Tohoku Math. J., 42, 67-82, 1990.

[2] W. Blaschke, *Vorlesungen Über Differential Geometrie I, Ban I*, Verlag Von Julius Springer-Verlag in Berlin, 1930.

[3] Çöken, A. Ceylan, *Joachimsthal Theorem for Hypersurfaces in the Semi-Euclidean Spaces*,Doctoral Dissertation, Osman Gazi University Graduate School of Natural and Applied Science Department of Mathematics, Eskişehir, 1995.

[4] H.W. Guggenheimer, *Differential Geometry*, Mc. Graw-Hill Book Company, New York, 1963.

[5] S. Izumiya, D. Pei and T. Sano, *The lightcone Gauss Map and The lightcone Developable of a Spacelike Curve in Minkowski* 3-*Space*, Glasgow Math. J., 42 (200), 75-89.

[6] B. O 'Neill, *Elemantary Differential Geometry*, Academic Press, New York, 1966.

[7] B. O 'Neill, *Semi-Riemannian Geometry with Applications to relativity*,Academic press Inc, London, 1983.

[8] Ü, Pekmen, *Differential-Geometrical Conditions Between Geodesic Curves and Ruled Surfaces*, Journal of Faculty of Science Ege University, Vol.16 (1995), No.1.

Department of Mathematics,
Süleyman Demirel University
32260 Isparta, TURKEY.