Pseudo-Umbilical Spacelike Submanifolds in the Indefinite Space Form

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Abstract

We prove an integral inequality for compact pseudo-umbilical spacelike submanifolds in the indefinite space form. As an application of the inequality, we give a necessary and sufficient condition for such submanifolds to be totally geodesic.

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1 Introduction

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p, which is called as *indefinite space form of index* p. Let M^n be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. As the semi-Riemannian metric of $M_p^{n+p}(c)$ induces the Riemannian metric of M^n, M^n is called a spacelike submanifold. Let h be the second fundamental form of the immersion, and ξ the mean curvature vector. Denote by $\langle\cdot,\cdot\rangle$ the scalar product of $M_p^{n+p}(c)$. If there exists a function λ on M^n such that

$$\langle h(X,Y),\xi\rangle = -\lambda\langle X,Y\rangle \tag{1.1}$$

for any tangent vector X, Y on M^n, then M^n is called a ***pseudo-umbilical spacelike submanifold*** of $M_p^{n+p}(c)$. It is clear that $\lambda \geq 0$. If the mean curvature vector ξ vanishes identically, then M^n is called a maximal spacelike submanifold of $M_p^{n+p}(c)$. Every maximal spacelike submanifold of $M_p^{n+p}(c)$ is itself a pseudo-umbilical spacelike submanifold of $M_p^{n+p}(c)$.

Maximal and pseudo-umbilical spacelike submanifolds have been studied by many researches. For example in 1988 Ishihara [1] proved that if M^n is a complete and maximal spacelike submanifold of $M_p^{n+p}(c)$, then either M^n is totally geodesic (when $c \geq 0$) or $0 \leq S \leq -npc$(when $c < 0$), where S is the square length of the second fundamental form of M^n. In 1995 Sun [2] first proved that the mean curvature H of

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the pseudo-umbilical submanifolds M^n in $M_p^{n+p}(c)$ is constant, then he generalized Ishihara's result to pseudo-umbilical submanifolds, obtaining the inequality

$$nH^2 \le S \le \frac{1}{2}np\Big[H^2 - c - \sqrt{(H^2-c)^2} + 4H^2c/p\Big].$$

In this paper, we prove an integral inequality for compact pseudo-umbilical spacelike submanifolds in the indefinite space form and as an application of the inequality, we give a necessary and sufficient condition for such submanifolds to be totally geodesic. We will prove the following

Theorem 1. *Let M^n be an n-dimensional compact pseudo-umbilical spacelike submanifold in $M_p^{n+p}(c)$, then*

$$\int_{M^n}\left\{\frac{1}{2}\sum R^2_{mijk} + \sum R^2_{mj} - ncR + nH^2R\right\} * 1 \le 0. \tag{1.2}$$

Theorem 2. *Let M^n be an n-dimensional compact pseudo-umbilical spacelike submanifold in $M_p^{n+p}(c)$, then*

$$\int_{M^n}\left\{\frac{1}{2}\sum R^2_{mijk} + (n-2)cS - n(n-1)c^2 + n^2(n-1)cH^2 + nH^2S\right\} * 1 \le 0, \tag{1.3}$$

and equality holds if and only if M^n is totally geodesic.

In the above Theorem, $\sum R^2_{mijk}$ is the square length of the Riemannian curvature tensor of M^n, $\sum R^2_{mj}$ the square length of the Ricci curvature tensor and R the scalar curvature. We shall make use of the following convention on the ranges of indices:

$$1 \le A, B, C, \cdots \le n+p; \quad 1 \le i, j, k, \cdots \le n, \quad n+1 \le \alpha, \beta, \gamma, \cdots \le n+p,$$

and we shall agree that repeated indices are summed over the respective ranges.

2 Local formulas

We choose a local field of semi-Riemannian orthonormal frames $e_1, \cdots, e_{n+p}$ in $M_p^{n+p}(c)$ such that, restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n. Let $\omega_1, \cdots, \omega_{n+p}$ be its dual frame field such that the semi-Riemannian metric of $M_p^{n+p}(c)$ is given by

$$ds^2 = \sum(\omega_i)^2 - \sum(\omega_\alpha)^2 = \sum \varepsilon_A(\omega_A)^2,$$

where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$. Then the structure equations of $M_p^{n+p}(c)$ are given by

$$d\omega_A = -\sum \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum K_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = c\varepsilon_A\varepsilon_B(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}). \tag{2.3}$$

We restrict these forms to M^n, then

$$\omega_\alpha = 0, \quad \omega_{\alpha i} = \sum h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji} \tag{2.4}$$

$$d\omega_{ij} = -\sum \omega_{ik}\wedge\omega_{kj} - \frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l, \tag{2.5}$$

$$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) - \sum(h^\alpha_{il}h^\alpha_{jk} - h^\alpha_{ik}h^\alpha_{jl}), \tag{2.6}$$

$$R_{jk} = \sum R_{ljkl} = c(n-1)\delta_{jk} + \sum h^\alpha_{lk}h^\alpha_{jl}, \tag{2.7}$$

$$R = \sum R_{jj} = n(n-1)c + S, \tag{2.8}$$

$$d\omega_{\alpha\beta} = -\sum \omega_{\alpha\gamma}\wedge\omega_{\gamma\alpha} - \frac{1}{2}\sum R_{\alpha\beta ij}\omega_i\wedge\omega_j, \tag{2.9}$$

$$R_{\alpha\beta ij} = -\sum(h^\alpha_{il}h^\beta_{jl} - h^\alpha_{jl}h^\beta_{il}). \tag{2.10}$$

For indefinite Riemannian manifolds, refer to O'Neill [3].

We call $h = \sum h^\alpha_{ij}\omega_i\omega_j e_\alpha$ the second fundamental form of the immersed manifold M^n. Denote by $S = \sum(h^\alpha_{ij})^2$ the square length of h , $\xi = \frac{1}{n}\sum \mathrm{tr}H_\alpha e_\alpha$ the mean curvature vector and

$$H = \frac{1}{n}\sqrt{\sum(\mathrm{tr}H_\alpha)^2}$$

the mean curvature of M^n respectively. Here tr is the trace of the matrix $H_\alpha = (h^\alpha_{ij})$. Now let e_{n+1} be parallel to ξ. Then we have

$$\mathrm{tr}H_{n+1} = nH, \quad \mathrm{tr}H_\alpha = 0, \quad \alpha \neq n+1. \tag{2.11}$$

Let h^α_{ijk} and h^α_{ijkl} denote the covariant derivative and the second covariant derivative of h^α_{ij} respectively, defined by

$$\sum h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum h^\alpha_{ik}\omega_{kj} - \sum h^\alpha_{jk}\omega_{ki} - \sum h^\beta_{ij}\omega_{\beta\alpha}, \tag{2.12}$$

$$\sum h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum h^\alpha_{ijl}\omega_{lk} - \sum h^\alpha_{ilk}\omega_{lj} - \sum h^\alpha_{ljk}\omega_{li} - \sum h^\beta_{ijk}\omega_{\beta\alpha}, \tag{2.13}$$

then we have

$$h^\alpha_{ijk} - h^\alpha_{ikj} = 0, \tag{2.14}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum h^\alpha_{im}R_{mjkl} - \sum h^\alpha_{jm}R_{mikl} - \sum h^\beta_{ij}R_{\alpha\beta kl}. \tag{2.15}$$

The Laplacian $\triangle h^\alpha_{ij}$ of h^α_{ij} is defined by $\triangle h^\alpha_{ij} = \sum h^\alpha_{ijkk}$. By a direct calculation we have (cf. [4])

$$\begin{aligned} \sum h^\alpha_{ij}\triangle h^\alpha_{ij} &= \sum h^\alpha_{ij}h^\alpha_{kkij} - \sum h^\alpha_{ij}h^\alpha_{km}R_{mijk} - \\ &- \sum h^\alpha_{ij}h^\alpha_{mi}R_{mkjk} - \sum h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}. \end{aligned} \tag{2.16}$$

3 Proof of Theorem

Proof of Theorem 1. From (1.1) and (2.11)

$$\langle h(e_i, e_j), He_{n+1}\rangle = -H^2\delta_{ij} \quad \text{i.e.} \quad h_{ij}^{n+1} = -H\delta_{ij}, \tag{3.1}$$

therefore

$$\sum h_{ij}^{\alpha} h_{kkij}^{\alpha} = -nH \triangle H. \tag{3.2}$$

Since H is constant (see [2]),

$$\sum h_{ij}^{\alpha} h_{kkij}^{\alpha} = 0. \tag{3.3}$$

On the other hand, from (2.6)

$$\begin{aligned} &-\sum h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk} = -\frac{1}{2}\sum (h_{ij}^{\alpha} h_{km}^{\alpha} - h_{mj}^{\alpha} h_{ik}^{\alpha}) R_{mijk} \\ &= \frac{1}{2}\sum \{R_{imkj} - c(\delta_{ij}\delta_{mk} - \delta_{mj}\delta_{ik})\} R_{mijk} = \frac{1}{2}\sum R_{mijk}^2 - cR, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &-\sum h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mkjk} = \sum (h_{ij}^{\alpha} h_{mi}^{\alpha} - h_{ii}^{\alpha} h_{mj}^{\alpha} + nH^2\delta_{mj}) R_{mj} = \\ &= \sum \{c(\delta_{ij}\delta_{mi} - \delta_{ii}\delta_{mj}) - R_{imij} + nH^2\delta_{mj}\} R_{mj} = \\ &= \sum R_{mj}^2 - (n-1)cR + nH^2 R. \end{aligned} \tag{3.5}$$

From (2.10)

$$-\sum h_{ij}^{\alpha} h_{ki}^{\beta} R_{\alpha\beta jk} = \sum h_{ij}^{\alpha} h_{ki}^{\beta} h_{jl}^{\alpha} h_{kl}^{\beta} - \sum h_{ij}^{\alpha} h_{ki}^{\beta} h_{kl}^{\alpha} h_{jl}^{\beta}. \tag{3.6}$$

Since

$$\begin{aligned} \sum h_{ij}^{\alpha} h_{jl}^{\beta} h_{lk}^{\alpha} h_{ki}^{\beta} &= \sum \operatorname{tr}(H^{\alpha}H^{\beta})^2 \leq \\ &\leq \sum \operatorname{tr}((H^{\alpha})^2 (H^{\beta})^2) = \sum h_{ij}^{\alpha} h_{jl}^{\alpha} h_{lk}^{\beta} h_{ki}^{\beta}, \end{aligned} \tag{3.7}$$

therefore

$$-\sum h_{ij}^{\alpha} h_{ki}^{\beta} R_{\alpha\beta jk} \geq 0. \tag{3.8}$$

From (2.14), (3.3-3.8)

$$\sum h_{ij}^{\alpha} \triangle h_{ij}^{\alpha} \geq \frac{1}{2}\sum R_{mijk}^2 + \sum R_{mj}^2 - ncR + nH^2 R. \tag{3.9}$$

Since $\int_{M^n} \left\{\sum h_{ij}^{\alpha} \triangle h_{ij}^{\alpha}\right\} * 1 \leq 0$ (see [4]), we have

$$\int_{M^n} \left\{\frac{1}{2}\sum R_{mijk}^2 + \sum R_{mj}^2 - ncR + nH^2 R\right\} * 1 \leq 0. \tag{3.10}$$

Theorem 1 is proved.
Proof of Theorem 2. From (2.7)

$$\sum R_{mj}^2 = n(n-1)^2c^2 + 2(n-1)cS + \sum_{m,j}\left(\sum_{i,\alpha} h_{im}^\alpha h_{ij}^\alpha\right)^2, \tag{3.11}$$

$$\sum_{m,j}\left(\sum_{i,\alpha} h_{im}^\alpha h_{ij}^\alpha\right)^2 \geq \sum_j\left(\sum_{i,\alpha}(h_{ij}^\alpha)^2\right)^2 \geq \frac{1}{n}\left(\sum(h_{ij}^\alpha)^2\right)^2 = \frac{1}{n}S^2, \tag{3.12}$$

therefore from (3.10),
(3.13)

$$\int_{M^n}\left\{\frac{1}{2}\sum R_{mijk}^2 + (n-2)cS - n(n-1)c^2 + n^2(n-1)cH^2 + nH^2S\right\} * 1 \leq 0.$$

If M^n is totally geodesic, i.e., $S=0$, $h_{ij}^\alpha = 0$, then from (2.6)

$$\frac{1}{2}\sum R_{mijk}^2 = n(n-1)c^2, \tag{3.14}$$

in this case, (3.13) becomes an equality; Inversely, if (3.13) becomes an equality, then from (3.12) $S=0$, i.e. M^n is totally geodesic. Theorem 2 is proved.

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