

R-Separated Spaces

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Abstract

In this paper we have generalized the axioms of the separated spaces T_i , ($i = \overline{0,4}$), by replacing the equality relation on a topological space X , Δ_X , by an arbitrary binary relation, R . Many theorems in general topology may be generalized in this way. It will be interesting to study spaces separated by functions, equivalence relations or order relations. In section 1 are presented axioms and characterizing theorems of R -separation, in section 2 are presented some properties of spaces separated by equivalence relations and in section 3 we will obtain some results concerning spaces separated by functions.

Mathematics Subject Classification: 54A05

Key words: topological space, normal, regular space, quotient space, equivalence relation.

1 Axioms and theorems of R separation

First let us make the following notations: (X, \mathcal{T}) is a topological space; $R \subset X \times X$ is a binary relation on X ; \overline{R} is the dual of R (i.e. $x\overline{R}y \Leftrightarrow (x, y) \notin R$); R^{-1} is the inverse of R (i.e. $xR^{-1}y \Leftrightarrow (y, x) \in R$); $xRA \Leftrightarrow xRy$, $(\forall) y \in A$; $x\overline{R}A \Leftrightarrow x\overline{R}y$ $(\forall) y \in A$; $ARB \Leftrightarrow xRy$, $(\forall) x \in A$, $(\forall) y \in B$; $A\overline{R}B \Leftrightarrow x\overline{R}y$, $(\forall) x \in A$, $(\forall) y \in B$; $R(x) = \{y \mid xRy\}$ and $R^{-1}(x) = \{y \mid yRx\}$; $R(A) = \{y \mid (\exists) x \in A \text{ so that } xRy\}$; $R^{-1}(A) = \{y \mid (\exists) x \in A \text{ so that } yRx\}$; \mathcal{V}_x is the neighborhood filter of $x \in X$; For any $A \subset X$ we note by $\mathcal{V}_A = \{B \mid (\exists) D \in \mathcal{T} \text{ so that } A \subset D \subset B\}$ and we note $CA = X \setminus A$.

We will replace in the classical definitions of the separated spaces $x \neq y$ by $x\overline{R}y$; $x \notin A$ by $x\overline{R}A$ or $A\overline{R}x$ and $A \cap B$ by $A\overline{R}B$. Replacing in the following considerations the relation R by Δ_X (i.e. $xRy \Leftrightarrow x = y$), we shall find the classical case of T_i spaces ($i = \overline{0,4}$).

Definition 1. X is T_0^R -space iff $(\forall) x, y \in X$ with $x\overline{R}y$, $(\exists) V_x \in \mathcal{V}_x$ so that $V_x\overline{R}y$ or $(\exists) V_y \in \mathcal{V}_y$ so that $x\overline{R}V_y$.

Definition 2. X is T_1^R -space iff $(\forall) x, y \in X$ with $x\overline{R}y$, $(\exists) V_x \in \mathcal{V}_x$ and $(\exists) V_y \in \mathcal{V}_y$ so that $x\overline{R}V_y$ and $V_x\overline{R}y$.

Definition 3. X is T_2^R -space iff $(\forall) x, y \in X$ with $x\overline{R}y$, $(\exists) V_x \in \mathcal{V}_x$, $(\exists) V_y \in \mathcal{V}_y$ so that $V_x\overline{R}V_y$.

Remark 1. $T_2^R \subset T_1^R \subset T_0^R$.

Remark 2. If in the definitions 1,2,3 we take xRy iff $x = y$, then they become those in the classical case.

Theorem 4. X is T_0^R - space iff for each $x, y \in X$ we have: $y \in \overline{R(x)}$ and $x \in \overline{R^{-1}(y)} \Rightarrow xRy$, where by \overline{A} we note the closure of the subset A of X .

Proof. $V_x \overline{Ry}$ or $x \overline{R} V_y \Leftrightarrow V_x \cap R^{-1}(y) = \emptyset$ or $R(x) \cap V_y = \emptyset \Leftrightarrow x \notin \overline{R^{-1}(y)}$ or $y \notin \overline{R(x)}$ for $x \overline{R} y \Leftrightarrow$ if $x \in \overline{R^{-1}(y)}$ and $y \in \overline{R(x)}$, then xRy . \square

Theorem 5. X is T_1^R - space iff $R(x)$ and $R^{-1}(x)$ are closed subsets, for every $x \in X$.

Proof. " \Rightarrow ". Suppose that X is T_1^R - space and $x \overline{R} y$. Then exists $V_x \in \mathcal{V}_x$ and $V_y \in \mathcal{V}_y$ so that $x \overline{R} V_y$ and $V_x \overline{R} y$. From here: $R(x) \cap V_y = \emptyset$ and $V_x \cap R^{-1}(y) = \emptyset \Rightarrow y \notin \overline{R(x)}$ and $x \in \overline{R^{-1}(y)}$. So:

(a) X is T_1^R - space \Rightarrow for each $y \notin \overline{R(x)}$ we have $y \notin \overline{R(x)} \Rightarrow \overline{R(x)} \subset R(x) \Rightarrow R(x) = \overline{R(x)}$;

(b) $\overline{R^{-1}(y)} = R^{-1}(y)$, just like in (a).

" \Leftarrow ". (a) If $R^{-1}(y)$ is a closed subset of X , then for each $x \overline{R} y$, we have $x \notin \overline{R^{-1}(y)}$. From here: $(\exists) V_x \in \mathcal{V}_x$ with $V_x \cap R^{-1}(y) = \emptyset \Rightarrow V_x \overline{R} y$. (b) $(\exists) V_y \in \mathcal{V}_y$ with $x \overline{R} V_y$ just like in (a). \square

Theorem 6. X is T_2^R - space iff R is a closed subset of $X \times X$.

Proof. X is T_2^R - space iff $(\forall) x \overline{R} y$, $(\exists) V_x \in \mathcal{V}_x$ and $(\exists) V_y \in \mathcal{V}_y$ with $V_x \overline{R} V_y \Leftrightarrow (\forall) (x, y) \notin R$, $(\exists) V_x \in \mathcal{V}_x$, $(\exists) V_y \in \mathcal{V}_y$, so that $V_x \times V_y \subset \overline{R} \Leftrightarrow (\forall) (x, y) \in \overline{R}$, then it is interior point of $\overline{R} \Leftrightarrow \overline{R}$ is an open subset of $X \times X$. \square

Theorem 7. X is T_2^R - space iff for each $x \in X$ we have $\bigcap_{V \in \mathcal{V}_x} \overline{R(V)} = R(x)$ and for every $y \in X$ we have $\bigcap_{U \in \mathcal{V}_y} \overline{R^{-1}(U)} = R^{-1}(y)$.

Proof. " \Rightarrow ". Suppose X is T_2^R - space and $y \in \bigcap_{V \in \mathcal{V}_x} \overline{R(V)} \Rightarrow y \in \overline{R(V)}$, $(\forall) V \in \mathcal{V}_x$.

If $y \notin R(x) \Rightarrow x \overline{R} y \Rightarrow (\exists) V_x \in \mathcal{V}_x$ and $(\exists) V_y \in \mathcal{V}_y$, so that $V_x \overline{R} V_y \Rightarrow R(V_x) \cap V_y = \emptyset \Rightarrow y \notin \overline{R(V_x)}$, contradiction, so $y \in R(x)$ and from here $\bigcap_{V \in \mathcal{V}_x} \overline{R(V)} \subset R(x) \Rightarrow$

$$\bigcap_{V \in \mathcal{V}_x} \overline{R(V)} = R(x).$$

In the same way we infer $\bigcap_{U \in \mathcal{V}_y} \overline{R^{-1}(U)} = R^{-1}(y)$.

" \Leftarrow ". Suppose $x \overline{R} y \Rightarrow y \notin R(x) \Rightarrow y \notin \bigcap_{V \in \mathcal{V}_x} \overline{R(V)} \Rightarrow (\exists) V_x \in \mathcal{V}_x$ so that $y \notin \overline{R(V_x)} \Rightarrow (\exists) V_y \in \mathcal{V}_y$ so that $V_y \cap R(V_x) = \emptyset \Rightarrow V_x \overline{R} V_y \Rightarrow X$ is T_2^R . \square

Remark 3. It is enough to replace in Theorem 7, \mathcal{V}_x with a neighborhood basis of x .

Definition 8. (a) X is a R_l - regular space iff for each F , closed subset so that $F \overline{R} y$, there exists $V_F \in \mathcal{V}_F$ (neighborhood of F) and there exists $V_y \in \mathcal{V}_y$ so that $V_F \overline{R} V_y$.

(b) X is a R_r - regular space iff for each F , closed subset with $x \overline{R} F$, there exists $V_F \in \mathcal{V}_F$ and there exists $V_x \in \mathcal{V}_x$ so that $V_x \overline{R} V_F$.

(c) X is a R - regular space iff X is a R_l and R_r space.

Remark 4. If R is a simetric relation (i.e. $R \subset R^{-1}$) then (a) \Leftrightarrow (b) \Leftrightarrow (c).

Remark 5. If in this definition we take xRy iff $x = y$ then they become those in the classical case.

Theorem 9. (a) X is a R_l - regular space iff for each $y \in X$ and $U \in \mathcal{V}_{R^{-1}(y)}$ there exists $V \in \mathcal{V}_y$ so that $\overline{R^{-1}(V)} \subset U$.

(b) X is a R_r - regular space iff for each $x \in X$ and $U \in \mathcal{V}_{R(x)}$ there exists $V \in \mathcal{V}_x$ so that $\overline{R(V)} \subset U$.

(c) X is a R - regular space iff (a) and (b) are both trues.

Proof. (a) " \Rightarrow ". Suppose X is R_l regular space. Let $y \in X$ be and $U \in \mathcal{V}_{R^{-1}(y)}$. Suppose that U is an open set. Then $F = CU$ is a closed set $\Rightarrow F \cap R^{-1}(y) = \emptyset \Rightarrow F\overline{R}y \Rightarrow$ there exists $V_F \in \mathcal{V}_F$ so that $V_F\overline{R}V_y \Rightarrow V_F \cap R^{-1}(V_y) = \emptyset$. Without loss the generality we can suppose V_F open set $\Rightarrow CF$ is a closed set $\Rightarrow \overline{R^{-1}(V_y)} \subset \overline{CV_F} = CV_F$. But $F \subset V_F \Rightarrow CV_F \subset CF = U \Rightarrow \overline{R^{-1}(V_y)} \subset U$.

" \Leftarrow ". Let F be a closed set and $y \in X$ so that $F\overline{R}y \Rightarrow F \cap R^{-1}(y) = \emptyset \Rightarrow R^{-1}(V_y) \subset CF; CF = U$ is an open set $\Rightarrow U \in \mathcal{V}_{R^{-1}(y)} \Rightarrow$ there exists $V \in \mathcal{V}_y$ so that $\overline{R^{-1}(V_y)} \subset U \Rightarrow \overline{CR^{-1}(V_y)} \supset CU = F$. But $\overline{CR^{-1}(V_y)} = V_F$ is an open set $\Rightarrow V_F \in \mathcal{V}_F$. Because $R^{-1}(V_y) \subset \overline{R^{-1}(V_y)}$ we have:

$$CR^{-1}(V_y) \supset \overline{CR^{-1}(V_y)} \Rightarrow CR^{-1}(V_y) \supset V_F \Rightarrow V_F \cap R^{-1}(y) = \emptyset \Rightarrow V_F\overline{R}V_y.$$

(b) In the same way as (a).

(c) Is the consequence of (a) and (b). \square

Definition 10. X is a R - normal space iff for each F_1, F_2 closed sets so that $F_1\overline{R}F_2$. there exists:

$$V_1 \in \mathcal{V}_{F_1}, V_2 \in \mathcal{V}_{F_2} \text{ so that } V_1\overline{R}V_2.$$

Theorem 11. X is a R - normal space iff for each F closed set and $U \in \mathcal{V}_{R(F)}$, there exists $V \in \mathcal{V}_F$ so that (a) $\overline{R(V)} \subset U$ and (b) for each $U \in \mathcal{V}_{R^{-1}(y)}$, there exists $V \in \mathcal{V}_F$ so that $\overline{R^{-1}(V)} \subset U$.

Proof. " \Rightarrow ". Suppose X is a R - normal space. Let F be a closed set and $U \in \mathcal{V}_{R(F)} \Rightarrow R(F) \subset U$. Suppose U is a open set $\Rightarrow F_1 = CU$ is a closed set. As $CR(F) \supset F_1 \Rightarrow R(F) \cap F_1 = \emptyset \Rightarrow F\overline{R}F_1 \Rightarrow$ exists $V_F \in \mathcal{V}_F$ and $V_1 \in \mathcal{V}_{F_1}$ so that $V_F\overline{R}V_1$. But $V_1 \supset F_1 \Rightarrow CV_1 \subset CF_1$. Suppose V_1 is an open set $\Rightarrow CV_1 = \overline{CV_1}$ is a closed set. But $V_F\overline{R}V_1 \Rightarrow R(V_F) \subset CV_1 = \overline{CV_1} \Rightarrow \overline{R(V_F)} \subset CV_1 \subset CF_1 = U$. So $\overline{R(V_F)} \subset U$. In the same way for each $U \in \mathcal{V}_{R^{-1}(y)}$, there exists $V \in \mathcal{V}_F$ so that $\overline{R^{-1}(V)} \subset U$.

" \Leftarrow ". Let F_1, F_2 be closed sets so that $F_1\overline{R}F_2 \Rightarrow R(F_1) \cap F_2 = \emptyset \Rightarrow \overline{R(F_1)} \subset CF_2; CF_2 = U$ is a open set $\Rightarrow U \in \mathcal{V}_{R(F_1)} \Rightarrow$ there exists $V_1 \in \mathcal{V}_{F_1}$ so that $\overline{R(V_1)} \subset U \Rightarrow \overline{CR(V_1)} = V_2$, is an open set and $V_2 \supset F_2 \Rightarrow V_2 \in \mathcal{V}_{F_2}$. Observe that $CV_2 = \overline{R(V_1)} \supset R(V_1) \Rightarrow R(V_1) \cap V_2 = \emptyset \Rightarrow V_1\overline{R}V_2$. \square

Remark 6. Definition 5 \Leftrightarrow condition (a) \Leftrightarrow condition (b), as we can see from the proof. The R - separated spaces can be characterized by using sequences. First we define the T_3^R and T_4^R spaces.

Dfinition 12. (a) X is T_3^R space iff X is a T_1 space and R - regular space.

(b) X is T_4^R space iff X is a T_1 space and an R - normal space.

Theorem 13. $T_4^R \subset T_3^R \subset T_2^R \subset T_1^R \subset T_0^R$.

Proof. Observe that if X is T_1 space then $\{x\} = \overline{\{x\}}$ for each $x \in X$; using this condition results the first and the second inclusion of theorem 7. \square

Remark 7. A naturally condition for T_3^R and T_4^R spaces would to be T_1^R space, but it is not good enough to Theorem 7.

Theorem 14. X is T_0^R space iff for the generalized sequences $(x_\alpha)_{\alpha \in I}$ and $(y_\beta)_{\beta \in J}$ we have

$$\left. \begin{array}{l} x_\alpha \rightarrow x \\ y_\beta \rightarrow y \\ x_\alpha R y \\ x R y_\beta \end{array} \right\} \Rightarrow x R y.$$

Proof. We will use Theorem 1.

" \Rightarrow ". Suppose $y \in \overline{R(x)}$ and $x \in \overline{R^{-1}(y)} \Rightarrow x R y$. See that:

$$\left. \begin{array}{l} x_\alpha \rightarrow x \\ x_\alpha R y \Rightarrow x \in R^{-1}(y) \end{array} \right\} \Rightarrow x \in \overline{R^{-1}(y)}.$$

Also:

$$\left. \begin{array}{l} y_\beta \rightarrow y \\ x R y_\beta \Rightarrow y \in R(x) \end{array} \right\} \Rightarrow y \in \overline{R(x)}.$$

By using Theorem 4 we have $x R y$.

" \Leftarrow ". $y \in \overline{R(x)} \Rightarrow$ there exists $(y_\beta)_{\beta \in J}$ so that $y_\beta \rightarrow y$ and $y_\beta \in R(x) \Rightarrow x R y_\beta$, $x \in \overline{R^{-1}(y)} \Rightarrow$ there exists $(x_\alpha)_{\alpha \in I}$ so that $x_\alpha \rightarrow x$ and $x \in R^{-1}(y) \Rightarrow x_\alpha R y$. From here it follows $x R y$. \square

Theorem 15. X is T_1^R space iff for the generalized sequences $(x_\alpha)_{\alpha \in I}$ and $(y_\beta)_{\beta \in J}$ we have

$$\left. \begin{array}{l} x_\alpha \rightarrow x \\ x_\alpha R y \end{array} \right\} \Rightarrow x R y \quad \text{and} \quad \left. \begin{array}{l} y_\beta \rightarrow y \\ x R y_\beta \end{array} \right\} \Rightarrow x R y.$$

Proof. We use Theorem 5.

" \Rightarrow ". $x_\alpha \rightarrow x; x_\alpha R y \Rightarrow x_\alpha \in R^{-1}(y) = \overline{R^{-1}(y)} \Rightarrow x \in R^{-1}(y) \Rightarrow x R y$, $y_\beta \rightarrow y; x R y_\beta \Rightarrow y_\beta \in \overline{R(x)} = \overline{R(x)} \Rightarrow y \in \overline{R(x)} \Rightarrow x R y$

" \Leftarrow ". $y \in \overline{R(x)} \Rightarrow$ there exists $(y_\beta)_{\beta \in J}$ a generalized sequence so that $y_\beta \in R(x)$ and $y_\beta \rightarrow y$. Observe that $x R y_\beta \Rightarrow x R y \Rightarrow y \in R(x) \Rightarrow R(x) = \overline{R(x)}$, $x \in \overline{R^{-1}(y)} \Rightarrow$ there exists $(x_\alpha)_{\alpha \in I}$ a generalized sequence so that $x_\alpha \in R^{-1}(y)$ and $x_\alpha \rightarrow x$. Observe that $x_\alpha R y \Rightarrow x R y \Rightarrow x \in R^{-1}(y) \Rightarrow x R y$. \square

Theorem 16. X is T_2^R space iff for each generalized sequences $(x_\alpha)_{\alpha \in I}$ and $(y_\alpha)_{\alpha \in I}$ so that $x_\alpha R y_\alpha, x_\alpha \rightarrow x$ and $y_\alpha \rightarrow y$, we have $x R y$.

Proof. We use Theorem 6.

X is T_2^R space iff $R \subset X \times X$ is a closed set \Leftrightarrow for each $(x_\alpha, y_\alpha) \in R$ so that $(x_\alpha, y_\alpha) \rightarrow (x, y)$ we have $(x, y) \in R \Leftrightarrow$ for each $x_\alpha \rightarrow x; y_\alpha \rightarrow y$, $x_\alpha R y_\alpha$ it follows $x R y$. \square

Example 17. If " \leq " is an order relation on X and if X is a T_1^R space then $x_\alpha \leq y; x_\alpha \rightarrow x \Rightarrow x \leq y; x \leq y_\beta; y_\beta \rightarrow y \Rightarrow x \leq y$.

If X is a T_2^R space, then from $x_\alpha \leq y_\alpha; x_\alpha \rightarrow x; y_\alpha \rightarrow y$ results $x \leq y$.

Example 18. Let " \prec " be an order relation on \mathbf{R} : $x \prec y$ iff $y - x \in \mathbf{N}$. \mathbf{R} is T_i^{\prec} space for $i \in \{0, 1, 2\}$ but it is not T_i^{\prec} space for $i \in \{3, 4\}$.

Proof. (a) It is obvious that " \prec " is an order relation on \mathbf{R} and more: $\prec(x) = x + \mathbf{N}$, $\prec^{-1}(y) = y - \mathbf{N}$, for each $x, y \in \mathbf{R}$. $x_n \prec y_n \Rightarrow y_n - x_n \in \mathbf{N}$, where $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ are real sequences. $x_n \rightarrow x, y_n \rightarrow y \Rightarrow y_n - x_n \rightarrow y - x$. From here exists $n_0 \in \mathbf{N}$ so that $y_n - x_n \in (y - x - 1/2, y - x + 1/2)$ for each $n \geq n_0$. Because $y_n - x_n \in \mathbf{N}$ it follows $y_n - x_n = m \in \mathbf{N}$, so $y_n - x_n$ is constant for

$n \geq n_0 \Rightarrow y_n - x_n = y - x \in \mathbf{N} \Rightarrow x \prec y$. Using Theorem 16 results \mathbf{R} is T_2^\prec space, so T_i^\prec space for every $i \in \{0, 1, 2\}$.

(b) \mathbf{R} is not \prec_r regular space. From Theorem 9 we have that for $x = 0$, if $V = \bigcup_{n=0}^{\infty} \left(n - \frac{1}{2^n}; n + \frac{1}{2^n} \right) \in \mathcal{V}_{\prec(0)}$, because $\prec(0) = \mathbf{N}$. \mathbf{R} is \prec_r regular space \Rightarrow exists $U \in \mathcal{V}_0$ so that $\overline{\prec(U)} \subset V$. Let $\varepsilon > 0$ be so that $U_0 = (-\varepsilon, \varepsilon) \subset U$. We have $\prec(U_0) \subset \overline{\prec(U_0)} \subset \overline{\prec(U)} \subset V$ and then $U_0 + \mathbf{N} \subset V$. So $\bigcup_{n=0}^{\infty} (n - \varepsilon; n + \varepsilon) \subset \bigcup_{n=0}^{\infty} \left(n - \frac{1}{2^n}; n + \frac{1}{2^n} \right)$. It follows $0 < \varepsilon \leq \frac{1}{2^n}$ for each $n \in \mathbf{N} \Rightarrow \varepsilon = 0$, contradiction. Therefore \mathbf{R} is not T_i^\prec space for $i \in \{3, 4\}$. \square

2 Spaces separated by equivalence relations

In this section we will consider the case of an equivalence relation $R = \rho$. First we define a notion of continuity of a binary relation.

Definition 19. (a) A binary relation R on a topological space (X, \mathcal{T}) is continuous iff for every $D \in \mathcal{T}$ an open set, $R^{-1}(D) \in \mathcal{T}$ is an open set.

(b) R is an open relation iff for each $D \in \mathcal{T}$, $R(D) \in \mathcal{T}$ is an open set.

Remark 8. If $R = \rho$ is an equivalence relation, then $\rho = \rho^{-1}$ so (a) \Leftrightarrow (b).

Theorem 20. Let ρ be an equivalence relation on X and $(\hat{X}, \hat{\mathcal{T}})$ be the quotient space. If \hat{X} is T_i space then X is T_i^R space for each $i = \overline{0, 4}$.

Proof. Let $\hat{x} = \rho(x)$ be the equivalence class of each $x \in X$, and $p : X \rightarrow \hat{X}$, $p(x) = \hat{x}$ be the canonical projection. Suppose \hat{X} is T_2 space and let $x, y \in X$ be so that $x\bar{\rho}y \Leftrightarrow \hat{x} \neq \hat{y}$, more, $\rho(x) \cap \rho(y) = \emptyset$. There exists $V_x \in \mathcal{V}_x$ and $V_y \in \mathcal{V}_y$ neighborhoods of \hat{x}, \hat{y} so that $V_x \cap V_y = \emptyset$.

If $V_x = p^{-1}(V_x)$ and $V_y = p^{-1}(V_y)$, then, as p is continuous we have $V_x \in \mathcal{V}_x$ and $V_y \in \mathcal{V}_y$ and more because $V_x \cap V_y = \emptyset \Rightarrow V_x \bar{\rho} V_y \Rightarrow X$ is T_2^ρ space. The cases of $i \in \{0, 1, 2, 3, 4\}$ can be proved analogously. \square

This theorem has a converse given by:

Theorem 21. If ρ is an equivalence relation on X and if ρ is continous then X is T_i^ρ space $\Rightarrow \hat{X}$ is T_i space, for each $i = \overline{0, 4}$.

Proof. We will prove this result only in the case of $i = 2$. Suppose X is T_2^ρ space. Fisrt, observe that p is an open map $\Leftrightarrow \rho$ is an open relation. Let be $\hat{x}, \hat{y} \in \hat{X}$ so that $\hat{x} \neq \hat{y} \Rightarrow x\bar{\rho}y \Rightarrow$ there exists $V_x \in \mathcal{V}_x$ and $V_y \in \mathcal{V}_y$ so that $V_x \bar{\rho} V_y$. Because p is open map we have that $V_x = p(V_x)$ and $V_y = p(V_y)$ are neighborhoods of \hat{x} and \hat{y} and more $V_x \bar{\rho} V_y \Rightarrow V_x \cap V_y = \emptyset$ so \hat{X} is T_2 space. \square

Example 22. 1) Let $X = \mathbf{R}$ be with usualy topology and ρ the equivalence on \mathbf{R} defined by: $x\rho y \Leftrightarrow x - y \in \mathbf{Q}$. Observe that for each D open set, $\rho(D) = D + \mathbf{Q} = \mathbf{R}$, so ρ is a continuous relation. From here we can see that R is not T_0^ρ space because for each $x\bar{\rho}y$, $V_x \in \mathcal{V}_x$, $V_y \in \mathcal{V}_y$ results $\rho(V_x) = \rho(V_y) = \mathbf{R}$ so $y \in \rho(V_x)$ and $x \in \rho(V_y)$.

If $\hat{\mathbf{R}} = \mathbf{R}/\rho$ would be T_0 space, then \mathbf{R} would be T_0^ρ space, so \mathbf{R}/ρ is not T_i space, $i = \overline{0, 4}$.

2) Some surfaces can be obtained as quotient spaces by identifying points of the border of a plane quadrate $P \subset \mathbf{R}^2$. For example the 2-sphere, S^2 . Construction of

S^2 : We note by $\text{Int } P$ the interior of P and by bP the border of P . We consider P as a topological subspace of \mathbf{R}^2 . See that if D is an open set of P , then $D \cup bP$ is also an open set of P . We define on P the relation $\rho : x\rho y \Leftrightarrow x = y$ or $x, y \in bP$. We see that ρ is a continuous equivalence relation, since if D is an open subset of P :

- a) If $D \cap bP = \emptyset$ then $\rho(D) = D$ is an open subset of P .
- b) If $D \cap bP \neq \emptyset$ then $\rho(D) = D \cup bP$ is also an open subset of P .

We define $S^2 = P/\rho$ the quotient space of P . It is not difficult to see that S^2 is homeomorphe with any sphere of \mathbf{R}^3 . P is a T_2^ρ space, using Theorem 16 of Section 1.

3 Function relations

Let $f : X \rightarrow X$ be a function and (X, \mathcal{T}) be a topological space. We will establish some properties of T_i^f spaces, $i = \overline{0, 4}$.

Theorem 23. *If X is T_0 space and f is continuous, then X is T_0^f space.*

Proof. Observe that iff f is continuous then $f(\overline{A}) \subset \overline{f(A)}$ and $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for each A, B subsets of X . X is T_0 space iff:

$$(1) \quad x \in \overline{\{y\}} \text{ and } y \in \overline{\{x\}} \Rightarrow x = y.$$

We shall prove:

$$(2) \quad y \in \overline{\{f(x)\}} \text{ and } x \in \overline{\{f^{-1}(y)\}} \Rightarrow f(x) = y.$$

$x \in \overline{\{f^{-1}(y)\}}$ and $\overline{\{f^{-1}(y)\}} \subset f^{-1}(\overline{\{y\}}) \Rightarrow x \in f^{-1}(\overline{\{y\}})$, because f is continuous.

From here $f(x) \in \overline{\{y\}}$ and $\left. \begin{array}{l} f(x) \in \overline{\{y\}} \\ y \in \overline{\{f(x)\}} \end{array} \right\} \stackrel{(1)}{\Rightarrow} y = f(x)$, so X is T_0^f space (2). \square

Theorem 24. *If X is T_0^f space and if f is bijection, having the inverse f^{-1} continuous, then X is T_0 space.*

Proof. Let $x, y \in X$ be. We shall prove that: $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}} \Rightarrow x = y$, i.e. X is T_0^f space. Since f is bijection \Rightarrow exists $z \in X$ so that $y = f(z)$. Suppose that $x \in \overline{\{f(z)\}}$ and $f(z) \in \overline{\{x\}} \Leftrightarrow x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. Because f^{-1} is continuous we have: $z \in f^{-1}(\overline{\{x\}}) \subset \overline{\{f^{-1}(x)\}}$ and so $z \in \overline{\{f^{-1}(x)\}}$. From here $x \in \overline{\{f(z)\}}$ and $z \in \overline{\{f^{-1}(x)\}} \Rightarrow x = f(z)$, because X is T_0^f space. So we have $x = f(z) = y$, and then X is T_0 space. \square

Theorem 25. *If X is T_1 space and f is continuous, then X is T_1^f space.*

Proof. Observe that X is T_1^f space iff $\{f(x)\}$ and $f^{-1}(y)$ are closed sets, for each $x, y \in X$. X is T_1 space $\Rightarrow \{f(x)\}$ is closed set (1). Because $\{y\}$ is a closed set and f is continuous, we have that $f^{-1}(y)$ is a closed set (2). From (1) and (2) we have that X is T_1^f space. \square

Theorem 26. *If X is T_1^f space and f is onto, then X is T_1 space.*

Proof. X is T_1^f space $\Rightarrow \{f(x)\}$ is a closed set for each $x \in X$. Because for each $y \in X$, exists $x \in X$ so that $y = f(x)$, we have that $\{y\} = \{f(x)\}$ is a closed set so X is T_1 space. \square

Theorem 27. *If X is T_2 space and f is continuous, then X is T_2^f space.*

Proof. Note G_f the graph of f . We shall prove that CG_f is an open subset of $X \times X$. Let be $(x, y) \in CG_f$. Then $f(x) \neq y$. Because X is T_2 space, there exists $V \in \mathcal{V}_y$

and $V_1 \in \mathcal{V}_{f(x)}$ so that $V \cap V_1 = \emptyset$. But f is continuous and from here there exists $U \in \mathcal{V}_x$ so that $f(U) \subset V_1 \Rightarrow f(U) \cap V = \emptyset \Rightarrow U \times V \subset CG_f \Rightarrow CG_f$ is an open set $\Rightarrow G_f$ is a closed set $\Rightarrow X$ is T_2^f (see Theorem 4 Section 1). \square

Remark 9. Theorem 27 states that each continuous function on a T_2 space has a closed graphical.

Theorem 28. *If X is T_2^f space and f is bijection having the inverse f^{-1} continuous, then X is T_2 space.*

Proof. Let $y \in X$ be \Rightarrow there exists $x \in X$ so that $y = f(x) \Rightarrow x = f^{-1}(y)$. Because f^{-1} is continuous we have: For each $U \in \mathcal{V}_x$ there exists $V \in \mathcal{V}_y$ so that $f^{-1}(V) \subset U \Rightarrow V \subset f(U) \Rightarrow \overline{V} \subset \overline{f(U)}$. From here we infer

$$y \in \bigcap_{V \in \mathcal{V}_y} \overline{V} = \bigcap_{V \in \mathcal{V}_{f(x)}} \overline{V} \subset \bigcap_{V \in \mathcal{V}_y} \overline{f(U)} = \{f(x)\} = \{y\},$$

because X is T_2^f (Theorem 4 Section 1). So we have: $\bigcap_{V \in \mathcal{V}_y} \overline{V} = \{y\}$ for each $y \in X$,

and then X is T_2^f space. \square

Theorem 29. *If X is T_2^f space and compact, then f is continuous.*

Proof. X is T_2^f space $\Rightarrow \bigcap_{U \in \mathcal{V}_x} \overline{f(U)} = \{f(x)\}$ for each $x \in X$. Hence $\bigcup_{U \in \mathcal{V}_x} C\overline{f(U)} = X \setminus \{f(x)\}$, for an arbitrary point x of X . Let $V \in \mathcal{V}_{f(x)}$; then $X = \bigcup_{U \in \mathcal{V}_x} C\overline{f(U)} \cup V$.

Because X is a compact space there exist $U_1, U_2, \dots, U_n \in \mathcal{V}_x$ such that

$$X = C\overline{f(U_1)} \cup C\overline{f(U_2)} \cup \dots \cup C\overline{f(U_n)} \cup V = C \left[\overline{f(U_1)} \cup \overline{f(U_2)} \cup \dots \cup \overline{f(U_n)} \right] \cup V.$$

Let $U \in \mathcal{V}_x$ be so that $U \subset \bigcap_{i=1}^n U_i$. We have $\overline{f(U)} \subset \bigcap_{i=1}^n \overline{f(U_i)}$. Hence

$$C\overline{f(U)} \supset C \left[\bigcap_{i=1}^n \overline{f(U_i)} \right] \Rightarrow C\overline{f(U)} \cup V = X \Rightarrow C\overline{f(U)} \cap CV = \emptyset \Rightarrow \overline{f(U)} \subset V.$$

We have proved that f is continuous in x . So f is continuous on X . \square

Remark 10. This theorem is in fact an alternative of the principle of the closed graphical.

Theorem 30. *If X is f_r -regular space, then f is continuous.*

Proof. It is a consequence of Theorem 9 Section 1. \square

Theorem 31. *If X is f -regular space and f is bijection, then f is homeomorphism.*

Proof. It is a consequence of Theorem 9 Section 1. \square

Theorem 32. *If X is a regular space and f is continuous, then X is f_r -regular space.*

Proof. Let $x \in X$ be and $V \in \mathcal{V}_{f(x)}$ be \Rightarrow there exists $V_1 = \overline{V_1} \in \mathcal{V}_{f(x)}$ so that $V_1 \subset V$, because X is regular space. As f is continuous there exists $U \in \mathcal{V}_x$ so that

$$f(U) \subset V_1 \Rightarrow \overline{f(U)} \subset V_1 = \overline{V_1} \Rightarrow \overline{f(U)} \subset V.$$

Using now Theorem 9 Section 1, it follows that X is f_r -regular space. \square

Theorem 33. *If X is regular space and f is homeomorphism, then X is f regular space.*

Proof. It is a consequence of Theorem 9 Section 1. \square

Theorem 34. *If X is T_4^f space (i.e. f - normal and T_1) then f is continuous. If f is bijection, then it is a homeomorphism.*

Proof. It is a consequence of Theorem 11 Section 1, if we remark that X is T_1 space $\Rightarrow \{x\}$ is a closed set for each $x \in X$. \square

Theorem 35. (a) *If f is continuous and X is regular space, then X is f_r -regular.*

(b) *If f is homeomorphism and X is regular space, then X is f - regular.*

Proof. (a) Let $x \in X$ be; for each $V \in \mathcal{V}_{f(x)}$, there exists $U \in \mathcal{V}_x$ so that $f(U) \subset V$. Suppose $V = \overline{V}$, because X is regular space. From here $\overline{f(U)} \subset \overline{V}$; because $V = \overline{V} \Rightarrow X$ is f_r -regular space.

(b) It is a consequence of (a), observing that f^{-1} is continuous. \square

Theorem 36. *If f is homeomorphism and X is normal space, then X is f -normal.*

Proof. Let F be a closed subset of $X \Rightarrow f(F)$ is a closed set, because f is homeomorphism. If $V \in \mathcal{V}_{f(F)}$ then $V \in \mathcal{V}_{f(x)}$ for each $x \in F$. But f is continuous \Rightarrow there exists $U_x \in \mathcal{V}_x$ so that $f(U_x) \subset V$. Let $U = \bigcup_{x \in F} U_x$ be; notice U is neighborhood of F it follows $f(U) = \bigcup_{x \in F} f(U_x) \subset V$. Because X is normal space, suppose

$V = \overline{V} \Rightarrow \overline{f(U)} \subset V$ (1).

In the same way replacing f with f^{-1} that is continuous too, we will have : for each F a closed subset of X , and for each $U \in \mathcal{V}_{f^{-1}(x)}$ there exists $V \in \mathcal{V}_F$ so that $f^{-1}(V) \subset U$, supposing $U = \overline{U}$ (2) From (1), and (2), using Theorem 11 Section 1 we have that X is f -normal space. \square

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