Riemann-Lagrange Geometrical Background for Multi-Time Physical Fields

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Abstract

The aim of this paper is to create a large geometrical background which to allow the including of famous equations of mathematical physics (Maxwell or Einstein equations) as particular cases. The geometrical construction is realized on the 1-jet fibre bundle $J^1(T,M) \to T \times M$, using only a Kronecker h-regular quadratic Lagrangian function

$$L = h_{\alpha\beta}(t)g^{ij}(t,x)x^i_\alpha x^j_\beta + U^{(i)}_{(\alpha)}(t,x)x^i_\alpha + F(t,x).$$

Section 1 exposes the main reasons that determined our study. Also, it contains certain physical and geometrical aspects of the already classical Lagrangian geometry of physical fields from [12], whose ideas represent the start point in our generalized metrical multi-time Lagrangian approach of the theory of physical fields. Section 2 introduces the ML^n_p spaces that represent the natural houses for our generalized field theory. A characterization theorem for these spaces is given. At the same time, the main local features of geometrical objects produced on a ML^n_p space are described. We reffer to the canonical nonlinear connection Γ, the generalized Cartan Γ-linear connection $C\Gamma$, together with its torsion and curvature d-tensors. Section 3 presents the metrical multi-time Lagrange theory of electromagnetism and describes its generalized Maxwell equations. Section 4 presents the generalized Einstein equations which govern the metrical multi-time Lagrange theory of gravitational field. The generalized conservation laws of the gravitational field are derived.

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1 Geometrical and physical aspects

On the one hand, it is well known that the 1-jet fibre bundle $J^1(T,M) \to T \times M$ is a basic object in the study of classical and quantum field theories [21]. On the other hand, the construction of a new field theory, described in multi-time terms on $J^1(T,M)$, was imposed of certain relativistic invariant equations involving many time

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variables (chiral fields, sine-Gordon etc.), and of KP-hierarchy of integrable systems, in which the arbitrary variables t^α and t^β are quite equal in rights and there is no reason to prefer one to another by choosing it as time [3].

Using the covariant Hamiltonian multisymplectic or polisymplectic formalism, many researchers were studied in this direction [5], [6]. Therefore, we consider that a contravariant Lagrangian development of a field theory on $J^1(T,M)$, created by Riemannian geometrical methodes, promises unpublished new points of view in this topic. Moreover, we believe that a contravariant generalized multi-time Riemann-Lagrange formalism may have interesting conections with covariant Hamiltonian multisymplectic or polysymplectic formalism of great interest in relativity, numerical theory and geometrical integrators theory [2], [23], [28].

We should like to point out that the construction of a generalized multi-time Riemann-Lagrange field theory was begun in [13]. In order to have a good distinction between the theory from paper [13] and the Riemann-Lagrange theory developed in this paper, let us try to emphasize the main differences and similitudes of these multi-time field theories.

Firstly, the geometrical construction exposed in [13] is realized on the configuration bundle $\oplus_{\alpha=1}^{p} \boldsymbol{TM} \to M$, where the coordinates of α-th copy of $\boldsymbol{TM}$ are denoted (x^i, x^i_α). The geometrical invariance group of this vector bundle stands out by the ignoring of multi-time reparametrizations:

$$(1.1) \qquad \begin{cases} \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{x}^i_\alpha = \dfrac{\partial \tilde{x}^i}{\partial x^j} x^j_\alpha. \end{cases}$$

This fact emphasizes the absolute character of the multi-time coordinates involved in theory, which are regarded as fixed coordinates on $I\!R^p$. Comparatively, our geometrical construction is realized yet on the more suitable physical configuration bundle $J^1(T,M) \to T \times M$, whose local coordinates are $(t^\alpha, x^i, x^i_\alpha)$. The geometrical invariance group of the jet fibre bundle of order one $J^1(T,M)$, induced by the transformation group of $T \times M$, stands out by the relativistic character of the multi-time coordinates t^α:

$$(1.2) \qquad \begin{cases} \tilde{t}^\alpha = \tilde{t}^\alpha(t^\beta) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{x}^i_\alpha = \dfrac{\partial \tilde{x}^i}{\partial x^j} \dfrac{\partial t^\beta}{\partial \tilde{t}^\alpha} x^j_\beta, \end{cases}$$

where the meaning of x^i_α is that of *partial velocities* or, alternatively, of *partial directions*.

Secondly, the geometrical development from [13] is realized by a given semi-Riemannian metric $h_{\alpha\beta}$ on $I\!R^p$, together with an *"a priori"* fixed nonlinear connection $N = (N^{(i)}_{(\alpha)j})$ and a given "Lagrangian" $L : \oplus_{\alpha=1}^{p} \boldsymbol{TM} \to I\!R$.

Remark 1.1 In order to have a clearer understanding of geometrical concepts used in [13], comparatively with the concepts from this paper, we point out that in the paper [13] a Lagrangian is viewed as a real smooth function on $E = \oplus_{\alpha=1}^{p} \boldsymbol{TM}$. In contrast, we use the following distinct notions on $E = J^1(T,M)$:

i) *multi-time dependent Lagrangian function* – A smooth map $L : J^1(T,M) \to I\!R$.

ii) *multi-time Lagrangian* (Olver's terminology) – A local function $\mathcal{L}$ on $J^1(T, M)$ which transform by the rule $\tilde{\mathcal{L}} = \mathcal{L}|\det J|$, where J is the Jacobian matrix of coordinate transformations $t^\alpha = t^\alpha(\tilde{t}^\beta)$. If L is a Lagrangian function on 1-jet fibre bundle, then $\mathcal{L} = L\sqrt{|h|}$ represent a Lagrangian on $J^1(T, M)$.

For that reason, we used the quotation marks for the above notion of Lagrangian.

The geometrical objects taken in study in the the paper [13] allow the construction of the Sasakian-like metric on $\oplus_{\alpha=1}^{p} \boldsymbol{T}M$, whose physical meaning is that of multi-time gravitational potential, namely

$$G = g_{ij}dx^i \otimes dx^j + G^{(\alpha)(\beta)}_{(i)(j)}\delta x^i_\alpha \otimes \delta x^j_\beta, \tag{1.3}$$

where

$$\delta x^i_\alpha = dx^i_\alpha + N^{(i)}_{(\alpha)j}dx^j,$$

$$G^{(\alpha)(\beta)}_{(i)(j)}(x^k, x^k_\gamma) = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta},$$

$$g_{ij}(x^k, x^k_\gamma) = h_{\alpha\beta}G^{(\alpha)(\beta)}_{(i)(j)}(x^k, x^k_\gamma).$$

Now, it is important to note that, after the local description of the generalized Cartan connection induced by G, together with its local d-torsions and d-curvatures, the geometrical theory developed in [13] is stoped. Consequently, the generalized multi-time field theory, in the sense of generalized Maxwell and Einstein equations, is not described in [13]. We believe that the theory from [13] was stoped because of the very complicated computations that were involved in the local description of the Bianchi identities attached to the generalized Cartan connection. In our opinion the description of these local Bianchi identities should be decisive to create the subsequent generalized field theory because the multi-time geometrical approach from [13] try to extend the geometrical ideas from [12].

In this paper, using as a pattern the geometrical ideas from [12], we try to construct a generalized Riemann-Lagrange geometry of physical fields, but on the 1-jet fibre bundle $J^1(T, M)$. On this bundle of configurations, our generalized field theory is geometrical created by a given *vertical fundamental metrical d-tensor* of particular form

$$G^{(\alpha)(\beta)}_{(i)(j)} = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k),$$

where $h = (h_{\alpha\beta})$ is temporal semi-Riemannian metric, which is produced by a Kronecker h-regular quadratic Lagrangian function $L : J^1(T, M) \to I\!R$, that is,

$$G^{(\alpha)(\beta)}_{(i)(j)}(x^k, x^k_\gamma) = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k).$$

Using this geometrical object, we derive a nonlinear connection $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ and then we construct the Sasakian-like metric G on $J^1(T, M)$,

$$G = h_{\alpha\beta}dt^\alpha \otimes dt^\beta + g_{ij}dx^i \otimes dx^j + G^{(\alpha)(\beta)}_{(i)(j)}\delta x^i_\alpha \otimes \delta x^j_\beta, \tag{1.4}$$

where $\delta x^i_\alpha = dx^i_\alpha + M^{(i)}_{(\alpha)\beta}dt^\beta + N^{(i)}_{(\alpha)j}dx^j$. The physical meaning of G is that of a *multi-time gravitational h-potential.* In this geometrical context, the canonical generalized

Cartan connection $C\Gamma$ produced by G, together with its thirty Bianchi identities, extremely used in the development of the generalized multi-time field theory, can be locally described [15]. Therefore, it naturally follows the geometrical construction of our generalized multi-time theory for electromagnetic and gravitational fields.

Remark 1.2 In our opinion, from a physical point of view, our generalized multi-time theory of physical fields appears as a generalized unified field theory because the *multi-time electromagnetic field* F, that we will take in our study, will be directly derived from G. As a consequence, our generalized multi-time field theory can be included in the set of *"metrical"* field theories. At the same time, taking into account the form of the geometrical invariance group (1.2) of this theory, we appreciate that the metrical multi-time Riemann-Lagrange theory developed by us, can be also included in the set of *"parametrized"* field theories. For more details upon the clasification of field theories, see [6].

Remark 1.3 We consider that the key of succes in our generalized multi-time field theory, comparatively with the theory developed in [13], is provided by the *Kronecker h-regularity* of $G^{(\alpha)(\beta)}_{(i)(j)}$, which allows the description of the geometrical objects or identities on $J^1(T,M)$ by a reduced number of local adapted components. Altough the Kronecker h-regularity condition imposed to $G^{(\alpha)(\beta)}_{(i)(j)}$ may be interpreted as a too restrictive one, we appreciate that this fact is not true. In this direction, let us analyse the form of vertical metrical d-tensors $G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta}$ produced by several Lagrangian functions which govern various famous physical domains (bosonic string theory, electrodynamics, elasticity, Kaluza-Klein dynamics of ideal fluids without viscosity, hydrodinamics etc.) studied in [6], [7], [16] and [22]. The Kronecker decomposition of the vertical metrical d-tensors derived from these important physical Lagrangian functions show the naturalness of the Kronecker h-regularity condition.

In the sequel, we try to expose the main geometrical and physical aspects of *the Lagrangian theory of physical fields* from [11], [12]. From our point of view, all these aspects may be easily extended to our multi-time geometrical and physical backgrounds. In this sense, we recall that a Lagrange space $L^n = (M, L(x,y))$ (this concept represents the geometrical background in [11] and [12]) is defined as a pair which consists of a real, smooth, n-dimensional manifold M, whose coordinates are $(x^i)_{i=\overline{1,n}}$, and a regular "Lagrangian" $L : \boldsymbol{TM} \to I\!\!R$, not necessarily homogenous with respect to the direction $(y^i)_{i=\overline{1,n}}$. Let us consider

$$g_{ij}(x^k, y^k) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}, \tag{1.5}$$

the *fundamental metrical d-tensor* attached to the "Lagrangian" L. From physical point of view, this d-tensor has the physical meaning of an *"unified"* gravitational field on $\boldsymbol{TM}$, which consists of one *"external"* (x)-gravitational field spanned by points $\{x\}$, and the other *"internal"* (y)-gravitational field spanned by directions $\{y\}$. It should be emphasized that y is endowed with some microscopic character of the space-time structure. Moreover, since y is a vertical vector field on $\boldsymbol{TM}$, the y-dependence has combined with the concept of *anisotropy.*

The field theory developed on a Lagrange space L^n relies on a nonlinear connection $\Gamma = (N^i_j(x,y))$ attached naturally to the given "Lagrangian" L. This plays the role of mapping operator of internal (y)-field on the external (x)-field, and prescribes the *"interaction"* between (x)- and (y)- fields. From geometrical point of view, the nonlinear connection allows the construction of the *adapted bases*

$$\left\{\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N_i^j\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^i}\right\}\subset\mathcal{X}(\boldsymbol{T}M),$$
(1.6)
$$\{dx^i,\delta y^i=dy^i+N_j^i dx^j\}\subset\mathcal{X}^*(\boldsymbol{T}M).$$

Concerning the *"unified"* field $g_{ij}(x,y)$ of L^n, the authors construct a Sasakian-like metric on $\boldsymbol{T}M$,

$$G=g_{ij}dx^i\otimes dx^j+g_{ij}\delta y^i\otimes\delta y^j. \tag{1.7}$$

As to the spatial structure, the most important thing is to determine the *Cartan canonical connection* $C\Gamma=(L^i_{jk},C^i_{jk})$ with respect to g_{ij}, which comes from the metrical conditions

$$\begin{cases} g_{ij|k}=\dfrac{\delta g_{ij}}{\delta x^k}-L^m_{ik}g_{mj}-L^m_{jk}g_{mi}=0\\ g_{ij}|_k=\dfrac{\partial g_{ij}}{\partial y^k}-C^m_{ik}g_{mj}-C^m_{jk}g_{mi}=0,\end{cases} \tag{1.8}$$

where "$_{|k}$" and "$|_k$" are the local h- and v- covariant derivatives of $C\Gamma$. The importance to the Cartan canonical connection comes from its main role played in the Lagrangian theory of physical fields.

In this context, the *Einstein-Miron-Anastasiei equations* of the gravitational potentials $g_{ij}(x,y)$ of a Lagrange space L^n, $n>2$, are postulated as being the abstract geometrical Einstein equations attached to $C\Gamma$ and G, namely [12]

$$\begin{cases} R_{ij}-\dfrac{1}{2}Rg_{ij}=\mathcal{K}\mathcal{T}^H_{ij}, & {}'P_{ij}=\mathcal{K}\mathcal{T}^1_{ij},\\ S_{ij}-\dfrac{1}{2}Sg_{ij}=\mathcal{K}\mathcal{T}^V_{ij}, & {}''P_{ij}=-\mathcal{K}\mathcal{T}^2_{ij},\end{cases} \tag{1.9}$$

where $R_{ij}=R^m_{ijm}$, $S_{ij}=S^m_{ijm}$, ${}'P_{ij}=P^m_{ijm}$, ${}''P_{ij}=P^m_{imj}$ are the Ricci d-tensors of $C\Gamma$, $R=g^{ij}R_{ij}$, $S=g^{ij}S_{ij}$ are the scalar curvatures, $\mathcal{T}^H_{ij}$, $\mathcal{T}^V_{ij}$, $\mathcal{T}^1_{ij}$, $\mathcal{T}^2_{ij}$ are the components of the stress-energy tensor $\mathcal{T}$ (equal to 0 for vacuum), and $\mathcal{K}$ is the Einstein constant. Moreover, the stress-energy d-tensors $\mathcal{T}^H_{ij}$ and $\mathcal{T}^V_{ij}$ satisfy the *conservation laws*

$$\mathcal{K}\mathcal{T}^H{}^m_{j|m}=-\frac{1}{2}(P^{hm}_{js}R^s_{hm}+2R^s_{mj}P^m_s),\quad \mathcal{K}\mathcal{T}^V{}^m_{j|m}=0, \tag{1.10}$$

where all notations are described in [12].

The Lagrangian theory of electromagnetism relies on the *canonical Liouville vector field* $\mathbf{C}=y^i(\partial/\partial y^i)$ and the Cartan canonical connection $C\Gamma$ of the Lagrange space L^n. In this context, the authors introduce the *electromagnetic 2-form* on $\boldsymbol{T}M$,

$$F=F_{ij}\delta y^i\wedge dx^j+f_{ij}\delta y^i\wedge\delta y^j, \tag{1.11}$$

where

$$(1.12)\qquad \begin{aligned} F_{ij} &= \frac{1}{2}[(g_{im}y^m)_{|j} - (g_{jm}y^m)_{|i}],\\ f_{ij} &= \frac{1}{2}[(g_{im}y^m)|_j - (g_{jm}y^m)|_i]. \end{aligned}$$

Using certain geometrical identities, they deduce that the vertical electromagnetic components f_{ij} vanish always. At the same time, using the Bianchi identities attached to the Cartan canonical connection $C\Gamma$, they conclude that the horizontal electromagnetic components F_{ij} are governed by the following *Maxwell-Miron-Anastasiei equations*: ,

$$(1.13)\qquad \begin{cases} F_{ij|k} + F_{jk|i} + F_{ki|j} = -\sum_{\{i,j,k\}} C_{imr}R^r_{jk}y^m \\ F_{ij}|_k + F_{jk}|_i + F_{ki}|_j = 0. \end{cases}$$

Finally, we point out that physical aspects of the Lagrangian electromagnetism are studied by Ikeda in [8].

2 The geometry of ML^n_p spaces

Let us consider T (resp. M) a *"temporal"* (resp. *"spatial"*) manifold of dimension p (resp. n), whose coordinates are $(t^\alpha)_{\alpha=\overline{1,p}}$ (resp. $(x^i)_{i=\overline{1,n}}$). Let

$$E = J^1(T, M) \to T \times M$$

be the jet fibre bundle of order one associated to these manifolds. We recall that the *bundle of configuration* $J^1(T, M)$ has the local coordinates $(t^\alpha, x^i, x^i_\alpha)$, where $\alpha = \overline{1,p}$ and $i = \overline{1,n}$. We underline that, throughout this paper, the indices $\alpha, \beta, \gamma, \ldots$ run from 1 to p, and the indices $i, j, k, \ldots$ run from 1 to n.

Remark 2.1 In the particular case $T = I\!R$ (i. e., the temporal manifold T is the usual time axis represented by the set of real numbers), the coordinates (t^1, x^i, x^i_1) of the 1-jet space $J^1(I\!R, M) \equiv I\!R \times \boldsymbol{T}M$ are denoted (t, x^i, y^i). Note that the spaces $J^1(I\!R, M)$ and $I\!R\times \boldsymbol{T}M$ identify only punctually, the geometrical invariance groups of these spaces being different. Thus, the geometrical transformation group of $J^1(I\!R, M)$ is given by

$$(2.1)\qquad \begin{cases} \tilde{t} = \tilde{t}(t) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j}\dfrac{dt}{d\tilde{t}}y^j. \end{cases}$$

Comparatively, the transformation coordinates on $I\!R \times \boldsymbol{T}M$ are

$$(2.2)\qquad \begin{cases} \tilde{t} = t \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j}y^j. \end{cases}$$

We start our study considering a smooth multi-time dependent Lagrangian function $L : E \to I\!R$, which is locally expressed by

$$E \ni (t^\alpha, x^i, x^i_\alpha) \to L(t^\alpha, x^i, x^i_\alpha) \in I\!R.$$

Definition 2.1 The distinguished tensor field on $J^1(T, M)$,

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta}, \tag{2.3}$$

is called the *vertical fundamental metrical d-tensor* attached to L.

Now, let $h = (h_{\alpha\beta}(t^\gamma))$ be a fixed semi-Riemannian metric on the temporal manifold T and $g_{ij}(t^\gamma, x^k, x^k_\gamma)$ be a *"spatial"* metrical d-tensor on E, symmetric, of rank n, and having a constant signature.

Definition 2.2 A multi-time dependent Lagrangian function $L : E \to I\!R$, whose vertical fundamental metrical d-tensor (2.3) is of the form

$$G^{(\alpha)(\beta)}_{(i)(j)}(t^\gamma, x^k, x^k_\gamma) = h^{\alpha\beta}(t^\gamma) g_{ij}(t^\gamma, x^k, x^k_\gamma), \tag{2.4}$$

is called a *Kronecker h-regular multi-time dependent Lagrangian function with respect to the temporal semi-Riemannian metric* $h = (h_{\alpha\beta})$.

In this context, we can introduce the following

Definition 2.3 A pair $ML^n_p = (J^1(T, M), L)$, where $p = \dim T$ and $n = \dim M$, which consists of the 1-jet fibre bundle and a Kronecker h-regular multi-time dependent Lagrangian function $L : J^1(T, M) \to I\!R$ is called a *metrical multi-time Lagrange space* or a ML^n_p *space.*

Remarks 2.2 i) In the particular case $(T, h) = (I\!R, \delta)$, a metrical multi-time Lagrange space is called a *relativistic rheonomic Lagrange space* and is denoted $RRL^n = (J^1(I\!R, M), L)$. The Riemann-Lagrange geometry of RRL^n spaces is now completely developed in [17]. From our point of view, the geometrical framework exposed in [17] establishes a generalized geometric foundation for *relativistic* time-dependent Lagrangian mechanics of first order variational problems because the geometrical invariance group is given by (2.1). This transformation group emphasizes the relativistic character of the time t. We invite the reader to compare the geometrization from [17] with that realized in [12]. The geometrical background constructed in [12] is created in order to geometrize the *absolute* time-dependent Lagrangian mechanics. The absolute character of the time t involved in study comes from the form (2.2) of the geometrical invariance group that governs the theory from [12].

ii) If the temporal manifold T is 1-dimensional, then, via a temporal reparametrization, we have $J^1(T, M) \equiv J^1(I\!R, M)$. In other words, a metrical multi-time Lagrangian space having $\dim T = 1$ can be regarded as a *reparametrized relativistic rheonomic Lagrange space.*

Example 2.1 Suppose that the spatial manifold M is also endowed with a semi-Riemannian metric $g = (g_{ij}(x))$. Then, the multi-time dependent Lagrangian function representing a basic object in the physical theory of bosonic strings,

$$L_1 : J^1(T, M) \to I\!R, \quad L_1 = h^{\alpha\beta}(t) g_{ij}(x) x^i_\alpha x^j_\beta \tag{2.5}$$

is a Kronecker h-regular multi-time dependent Lagrangian function. Consequently, the pair

$$BSML^n_p = (J^1(T, M), L_1)$$

is a ML^n_p space that is called the *metrical multi-time Lagrange space of bosonic strings.* We underline that the multi-time Lagrangian $\mathcal{L}_1 = L_1\sqrt{|h|}$ is exactly the energy multi-time Lagrangian whose extremals are the harmonic maps between the semi-Riemannian manifolds (T, h) and (M, g). For more details, see [4].

Example 2.2 In above notations, taking $U^{(\alpha)}_{(i)}(t, x)$ as a d-tensor field on E and $F : T \times M \to I\!R$ a smooth map, the more general multi-time dependent Lagrangian function

$$L_2 : E \to I\!R, \quad L_2 = h^{\alpha\beta}(t) g_{ij}(x) x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t, x) x^i_\alpha + F(t, x) \tag{2.6}$$

is also a Kronecker h-regular multi-time dependent Lagrangian function. The metrical multi-time Lagrange space

$$EDML^n_p = (J^1(T, M), L_2)$$

is called the *autonomous metrical multi-time Lagrange space of electrodynamics* because, in the particular case $(T, h) = (I\!R, \delta)$, we discover a natural relativistic generalization of the classical absolute rheonomic Lagrangian space of electrodynamics [12] which governs the movement law of a particle placed concomitantly into a gravitational field and an electromagnetic one. For that reason, from physical point of view, the semi-Riemannian metric $h_{\alpha\beta}(t)$ (resp. $g_{ij}(x)$) represents the *gravitational potentials* of the space T (resp. M), the d-tensor $U^{(\alpha)}_{(i)}(t, x)$ stands for the *electromagnetic potentials* and F is a function which is called *potential function.* The non-dynamical character of spatial gravitational potentials $g_{ij}(x)$ motivates us to use the terminology *"autonomous"*. We point out that the main geometrical and physical aspects of $EDML^n_p$ spaces are deeply studied [16]. The Riemann-Lagrange geometry of these spaces stands out by the multi-time and partial directions dependence of geometrical or physical objects that involves.

Example 2.3 More general, if we consider $g_{ij}(t, x)$ a d-tensor field on E, symmetric, of rank n and having constant signature on E, we can define the Kronecker h-regular multi-time dependent Lagrangian function

$$L_3 : E \to I\!R, \quad L_3 = h^{\alpha\beta}(t) g_{ij}(t, x) x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t, x) x^i_\alpha + F(t, x). \tag{2.7}$$

In this context, the pair

$$NEDML^n_p = (J^1(T, M), L_3)$$

is a metrical multi-time Lagrange space which is called the *non-autonomous metrical multi-time Lagrange space of electrodynamics.* Physically, we remark that the gravitational potentials $g_{ij}(t, x)$ of the spatial manifold M are dependent of the temporal coordinates t^γ, emphasizing their dynamic character.

An important role and, at the same time, an obstruction in the subsequent development of the metrical multi-time Lagrangian geometry, is played by the following theorem proved in [19]:

Theorem 2.1 *(characterization of metrical multi-time Lagrange spaces)*
If we have $p = \dim T \geq 2$, then the following statements are equivalent:

i) L is a Kronecker h-regular multi-time dependent Lagrangian function on $J^1(T, M)$.

ii) The multi-time dependent Lagrangian function L reduces to a non-autonomous electrodynamics multi-time dependent Lagrangian function, that is,

$$L = h^{\alpha\beta}(t)g_{ij}(t,x)x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t,x)x^i_\alpha + F(t,x).$$

In other words, in the case $p \geq 2$, any ML^n_p space is equivalent with a $NEDML^n_p$ space.

A direct consequence of the previous characterization theorem is

Corollary 2.2 *The fundamental vertical metrical d-tensor of an ML^n_p space has the Kronecker form*

$$G^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = \begin{cases} h^{11}(t)g_{ij}(t, x^k, y^k), & p = 1 \\ h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k), & p \geq 2. \end{cases} \tag{2.8}$$

Remarks 2.3 i) It is obvious that the Theorem 2.1 is an obstruction in the development of a fertile geometrical theory for ML^n_p spaces. This obstruction will be removed in a subsequent paper by the introduction of a more general notion, that of *generalized metrical multi-time Lagrange space* [14]. The generalized metrical multi-time Lagrange geometry and its derived theory of physical fields are constructed in [14] using a given h-regular fundamental vertical metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}$ on the 1-jet space $J^1(T, M)$, which can be not provided by a multi-time dependent Lagrangian function, together with an *"a priori"* fixed nonlinear connection $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$.

ii) In the case $p = \dim T \geq 2$, the Theorem 2.1 obliges us to continue the study of the metrical multi-time Lagrangian space theory, channeling our attention upon the non-autonomous metrical multi-time Lagrange spaces of electrodynamics.

Following the Riemann-Lagrange geometrical development from the paper [19], the fundamental vertical metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}$ of the metrical multi-time Lagrange space $ML^n_p = (J^1(T, M), L)$ naturally induces a *canonical nonlinear connection*, defined by the local components $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ on $J^1(T, M)$.

Theorem 2.3 *The canonical nonlinear connection Γ of the metrical multi-time Lagrange space $ML^n_p = (J^1(T, M), L)$ is defined by the temporal adapted components*

$$M^{(i)}_{(\alpha)\beta} = \begin{cases} -H^1_{11}y^i, & p = 1 \\ -H^\gamma_{\alpha\beta}x^i_\gamma, & p \geq 2 \end{cases} \tag{2.9}$$

and the spatial adapted components

$$N^{(i)}_{(\alpha)j} = \begin{cases} h_{11}\dfrac{\partial \mathcal{G}^i}{\partial y^j}, & p=1 \\ \Gamma^i_{jk}x^k_\alpha + \dfrac{g^{ik}}{2}\dfrac{\partial g_{jk}}{\partial t^\alpha} + \dfrac{g^{ik}}{4}h_{\alpha\beta}U^{(\beta)}_{(k)j}, & p\geq 2, \end{cases} \tag{2.10}$$

where

$$\begin{aligned} \mathcal{G}^i &= \frac{g^{ik}}{4}\left(\frac{\partial^2 L}{\partial x^j \partial y^k}y^j - \frac{\partial L}{\partial x^k} + \frac{\partial^2 L}{\partial t \partial y^k} + \frac{\partial L}{\partial x^k}H^1_{11} + 2h^{11}H^1_{11}g_{kl}y^l\right), \\ H^\gamma_{\alpha\beta} &= \frac{h^{\gamma\eta}}{2}\left(\frac{\partial h_{\eta\alpha}}{\partial t^\beta} + \frac{\partial h_{\eta\beta}}{\partial t^\alpha} - \frac{\partial h_{\alpha\beta}}{\partial t^\eta}\right), \\ \Gamma^i_{jk} &= \frac{g^{im}}{2}\left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right), \\ U^{(\beta)}_{(k)j} &= \frac{\partial U^{(\beta)}_{(k)}}{\partial x^j} - \frac{\partial U^{(\beta)}_{(j)}}{\partial x^k}. \end{aligned} \tag{2.11}$$

Remarks 2.4 i) Considering the particular case $(T,h) = (I\!R, \delta)$, we emphasize that the canonical nonlinear connection $\Gamma = (0, N^{(i)}_{(1)j})$ of the relativistic rheonomic Lagrange space $RRL^n = (J^1(I\!R, M), L)$ represents a natural generalization of the canonical nonlinear connection used in the classical Lagrangian geometry from [12].

ii) The canonical nonlinear connection $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ of a ML^n_p space allows the construction of *adapted bases*

$$\begin{aligned} &\left\{\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^i_\alpha}\right\} \subset \mathcal{X}(E), \\ &\{dt^\alpha, dx^i, \delta x^i_\alpha\} \subset \mathcal{X}^*(E), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \frac{\delta}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha} - M^{(j)}_{(\beta)\alpha}\frac{\partial}{\partial x^j_\beta} \\ \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N^{(j)}_{(\beta)i}\frac{\partial}{\partial x^j_\beta} \\ \delta x^i_\alpha &= dx^i_\alpha + M^{(i)}_{(\alpha)\beta}dt^\beta + N^{(i)}_{(\alpha)j}dx^j. \end{aligned} \tag{2.13}$$

The simple tensorial transformation rules of the elements of above adapted bases determine us to study the geometrical and physical objects on ML^n_p spaces, at level of local adapted components.

The naturalness of the construction of the nonlinear connection of a metrical multi-time Lagrange space $ML^n_p = (J^1(T,M), L)$ comes from the following result, proved also in [19]:

Theorem 2.4 *The Euler-Lagrange equations of the energy action functional*

$$\mathcal{E}(f) = \int_T L(t^\gamma, x^k, x^k_\gamma)\sqrt{|h|}dt,$$

where $f = (x^i(t^\gamma))$, *are equivalent with the harmonic maps equations attached to the nonlinear connection* $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$. *In other words, the extremals of* $\mathcal{E}$ *are equivalent with the solutions of the harmonic map equations [18]*

$$h^{\alpha\beta}\{x^i_{\alpha\beta} + M^{(i)}_{(\alpha)\beta} + N^{(i)}_{(\alpha)j}x^j_\beta\} = 0,$$

where $x^i_{\alpha\beta}$ *represent the second derivatives of* $x^i(t^\gamma)$.

The main result of the metrical multi-time Lagrange geometry is the theorem of existence of the *Cartan canonical h-normal* Γ*-linear connection* $C\Gamma$ which allows the subsequent development of the *metrical multi-time Lagrangian theory of physical fields.*

Theorem 2.5 *(of existence and uniqueness of Cartan canonical connection)*

On the metrical multi-time Lagrange space $ML^n_p = (J^1(T, M), L)$, *endowed with its canonical nonlinear connection* Γ, *there is a unique h-normal* Γ*-linear connection [15], defined by adapted components*

$$C\Gamma = (H^\gamma_{\alpha\beta}, G^k_{j\gamma}, L^i_{jk}, C^{i(\gamma)}_{j(k)}),$$

having the metrical properties:

i) $g_{ij|k} = 0, \quad g_{ij}|^{(\gamma)}_{(k)} = 0,$

ii) $G^k_{j\gamma} = \dfrac{g^{ki}}{2}\dfrac{\delta g_{ij}}{\delta t^\gamma}, \quad L^k_{ij} = L^k_{ji}, \quad C^{i(\gamma)}_{j(k)} = C^{i(\gamma)}_{k(j)}.$

Moreover, the components L^i_{jk} *and* $C^{i(\gamma)}_{j(k)}$ *of the Cartan canonical connection have the expressions*

$$\begin{aligned} L^i_{jk} &= \frac{g^{im}}{2}\left(\frac{\delta g_{mj}}{\delta x^k} + \frac{\delta g_{mk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m}\right), \\ C^{i(\gamma)}_{j(k)} &= \frac{g^{im}}{2}\left(\frac{\partial g_{mj}}{\partial x^k_\gamma} + \frac{\partial g_{mk}}{\partial x^j_\gamma} - \frac{\partial g_{jk}}{\partial x^m_\gamma}\right). \end{aligned} \tag{2.14}$$

Remarks 2.5 i) A proof of the Theorem 2.5 can be found in the paper [14], in the more general context of generalized metrical multi-time Lagrange spaces.

ii) In the particular case $(T, h) = (I\!R, \delta)$, the Cartan canonical δ-normal Γ-linear connection of the relativistic rheonomic Lagrange space $RRL^n = (J^1(I\!R, M), L)$ naturally generalizes the canonical Cartan connection used in the classical rheonomic Lagrange geometry from [12].

iii) As a rule, the Cartan canonical connection of a metrical multi-time Lagrange space ML^n_p verifies also the metrical properties

$$h_{\alpha\beta/\gamma} = h_{\alpha\beta|k} = h_{\alpha\beta}|^{(\gamma)}_{(k)} = 0, \quad g_{ij/\gamma} = 0.$$

iv) In the case $p = \dim T \geq 2$, the components of the Cartan connection of a metrical multi-time Lagrange space reduce to those of a $NEDML^n_p$ space, namely

$$\bar{G}^{\gamma}_{\alpha\beta} = H^{\gamma}_{\alpha\beta},\ G^k_{j\gamma} = \frac{g^{ki}}{2}\frac{\partial g_{ij}}{\partial t^{\gamma}},\ L^i_{jk} = \Gamma^i_{jk},\ C^{i(\gamma)}_{j(k)} = 0.$$

The general theorems from [15], which characterize the d-torsion and d-curvature of a h-normal Γ-linear connection, applied to the Cartan connection of a ML^n_p space, imply the following important results:

Theorem 2.6 *The torsion d-tensor* **T** *of the Cartan canonical connection of a metrical multi-time Lagrange space ML^n_p is determined by the local adapted components:*

(2.15)

	h_T		h_M		v	
	$p=1$	$p\geq 2$	$p=1$	$p\geq 2$	$p=1$	$p\geq 2$
h_Th_T	0	0	0	0	0	$R^{(m)}_{(\mu)\alpha\beta}$
h_Mh_T	0	0	T^m_{1j}	$T^m_{\alpha j}$	$R^{(m)}_{(1)1j}$	$R^{(m)}_{(\mu)\alpha j}$
h_Mh_M	0	0	0	0	$R^{(m)}_{(1)ij}$	$R^{(m)}_{(\mu)ij}$
vh_T	0	0	0	0	$P^{(m)\ (1)}_{(1)1(j)}$	$P^{(m)\ (\beta)}_{(\mu)\alpha(j)}$
vh_M	0	0	$P^{m(1)}_{i(j)}$	0	$P^{(m)\ (1)}_{(1)i(j)}$	0
vv	0	0	0	0	0	0

where,

i) for $p = \dim T = 1$, we have

$$T^m_{1j} = -G^m_{j1},\ P^{m(1)}_{i(j)} = C^{m(1)}_{i(j)},\ P^{(m)\ (1)}_{(1)1(j)} = -G^m_{j1},$$

$$P^{(m)\ (1)}_{(1)i(j)} = \frac{\partial N^{(m)}_{(1)i}}{\partial y^j} - L^m_{ji},\quad \frac{\delta N^{(m)}_{(1)i}}{\delta x^j} - \frac{\delta N^{(m)}_{(1)j}}{\delta x^i},$$

$$R^{(m)}_{(1)1j} = -\frac{\partial N^{(m)}_{(1)j}}{\partial t} + H^1_{11}\left[N^{(m)}_{(1)j} - \frac{\partial N^{(m)}_{(1)j}}{\partial y^k}y^k\right];$$

ii) for $p = \dim T \geq 2$, denoting

$$F^m_{i(\mu)} = \frac{g^{mp}}{2}\left[\frac{\partial g_{pi}}{\partial t^{\mu}} + \frac{1}{2}h_{\mu\beta}U^{(\beta)}_{(p)i}\right],$$

$$H^{\gamma}_{\mu\alpha\beta} = \frac{\partial H^{\gamma}_{\mu\alpha}}{\partial t^{\beta}} - \frac{\partial H^{\gamma}_{\mu\beta}}{\partial t^{\alpha}} + H^{\eta}_{\mu\alpha}H^{\gamma}_{\eta\beta} - H^{\eta}_{\mu\beta}H^{\gamma}_{\eta\alpha},$$

$$r^m_{pij} = \frac{\partial \Gamma^m_{pi}}{\partial x^j} - \frac{\partial \Gamma^m_{pj}}{\partial x^i} + \Gamma^k_{pi}\Gamma^m_{kj} - \Gamma^k_{pj}\Gamma^m_{ki},$$

we have

$$T^m_{\alpha j} = -G^m_{j\alpha},\ P^{m\ (\beta)}_{(\mu)\alpha(j)} = -\delta^{\beta}_{\gamma}G^m_{j\alpha},\ R^{(m)}_{(\mu)\alpha(j)} = -H^{\gamma}_{\mu\alpha\beta}x^m_{\gamma},$$

$$R^{(m)}_{(\mu)\alpha j} = -\frac{\partial N^{(m)}_{(\mu)j}}{\partial t^{\alpha}} + \frac{g^{mk}}{2}H^{\beta}_{\mu\alpha}\left[\frac{\partial g_{jk}}{\partial t^{\beta}} + \frac{h_{\beta\gamma}}{2}U^{(\gamma)}_{(k)j}\right],$$

$$R^{(m)}_{(\mu)ij} = r^m_{ijk}x^k_{\mu} + \left[F^m_{i(\mu)|j} - F^m_{j(\mu)|i}\right];$$

Theorem 2.7 *The curvature d-tensor* $\mathbf{R}$ *of the Cartan canonical connection of a* ML_p^n *space is determined by the local adapted components:*

	h_T		h_M		v	
	$p=1$	$p\geq 2$	$p=1$	$p\geq 2$	$p=1$	$p\geq 2$
h_Th_T	0	$H^{\alpha}_{\eta\beta\gamma}$	0	$R^l_{i\beta\gamma}$	0	$R^{(l)(\alpha)}_{(\eta)(i)\beta\gamma}$
h_Mh_T	0	0	R^l_{i1k}	$R^l_{i\beta k}$	$R^{(l)(1)}_{(1)(i)1k}=R^l_{i1k}$	$R^{(l)(\alpha)}_{(\eta)(i)\beta k}$
h_Mh_M	0	0	R^l_{ijk}	R^l_{ijk}	$R^{(l)(1)}_{(1)(i)jk}=R^l_{ijk}$	$R^{(l)(\alpha)}_{(\eta)(i)jk}$
vh_T	0	0	$P^{(l)\ (1)}_{i1(k)}$	0	$P^{(l)(1)\ (1)}_{(1)(i)1(k)}=P^{(l)\ (1)}_{i1(k)}$	0
vh_M	0	0	$P^{l\ (1)}_{ij(k)}$	0	$P^{(l)(1)\ (1)}_{(1)(i)j(k)}=P^{l\ (1)}_{ij(k)}$	0
vv	0	0	$S^{l(1)(1)}_{i(j)(k)}$	0	$S^{(l)(1)(1)(1)}_{(1)(i)(j)(k)}=S^{l(1)(1)}_{i(j)(k)}$	0

where $R^{(l)(\alpha)}_{(\eta)(i)\beta\gamma}=\delta^{\alpha}_{\eta}R^l_{i\beta\gamma}+\delta^l_iH^{\alpha}_{\eta\beta\gamma}$, $R^{(l)(\alpha)}_{(\eta)(i)\beta k}=\delta^{\alpha}_{\eta}R^l_{i\beta k}$, $R^{(l)(\alpha)}_{(\eta)(i)jk}=\delta^{\alpha}_{\eta}R^l_{ijk}$ *and*

i) for $p=\dim T=1$, *we have*

$$R^l_{i1k}=\frac{\delta G^l_{i1}}{\delta x^k}-\frac{\delta L^l_{ik}}{\delta t}+G^m_{i1}L^l_{mk}-L^m_{ik}G^l_{m1}+C^{l(1)}_{i(m)}R^{(m)}_{(1)1k},$$

$$R^l_{ijk}=\frac{\delta L^l_{ij}}{\delta x^k}-\frac{\delta L^l_{ik}}{\delta x^j}+L^m_{ij}L^l_{mk}-L^m_{ik}L^l_{mj}+C^{l(1)}_{i(m)}R^{(m)}_{(1)jk},$$

$$P^{l\ (1)}_{i1(k)}=\frac{\partial G^l_{i1}}{\partial y^k}-C^{l(1)}_{i(k)/1}+C^{l(1)}_{i(m)}P^{(m)\ (1)}_{(1)1(k)},$$

$$P^{l\ (1)}_{ij(k)}=\frac{\partial L^l_{ij}}{\partial y^k}-C^{l(1)}_{i(k)|j}+C^{l(1)}_{i(m)}P^{(m)\ (1)}_{(1)j(k)},$$

$$S^{l(1)(1)}_{i(j)(k)}=\frac{\partial C^{l(1)}_{i(j)}}{\partial y^k}-\frac{\partial C^{l(1)}_{i(k)}}{\partial y^j}+C^{m(1)}_{i(j)}C^{l(1)}_{m(k)}-C^{m(1)}_{i(k)}C^{l(1)}_{m(j)};$$

ii) for $p=\dim T\geq 2$, *we have*

$$H^{\alpha}_{\eta\beta\gamma}=\frac{\partial H^{\alpha}_{\eta\beta}}{\partial t^{\gamma}}-\frac{\partial H^{\alpha}_{\eta\gamma}}{\partial t^{\beta}}+H^{\mu}_{\eta\beta}H^{\alpha}_{\mu\gamma}-H^{\mu}_{\eta\gamma}H^{\alpha}_{\mu\beta},$$

$$R^l_{i\beta\gamma}=\frac{\delta G^l_{i\beta}}{\delta t^{\gamma}}-\frac{\delta G^l_{i\gamma}}{\delta t^{\beta}}+G^m_{i\beta}G^l_{m\gamma}-G^m_{i\gamma}G^l_{m\beta},$$

$$R^l_{i\beta k}=\frac{\delta G^l_{i\beta}}{\delta x^k}-\frac{\delta\Gamma^l_{ik}}{\delta t^{\beta}}+G^m_{i\beta}\Gamma^l_{mk}-\Gamma^m_{ik}G^l_{m\beta},$$

$$R^l_{ijk}=r^l_{ijk}=\frac{\partial\Gamma^l_{ij}}{\partial x^k}-\frac{\partial\Gamma^l_{ik}}{\partial x^j}+\Gamma^m_{ij}\Gamma^l_{mk}-\Gamma^m_{ik}\Gamma^l_{mj}.$$

3 Generalized Maxwell equations for a multi-time electromagnetic field

Let $ML_p^n=(J^1(T,M),L)$ be a metrical multi-time Lagrange space, together with $\Gamma=(M^{(i)}_{(\alpha)\beta},N^{(i)}_{(\alpha)j})$ its canonical nonlinear connection. Let us consider the Cartan canonical connection of ML_p^n, locally expresed by $C\Gamma=(H^{\gamma}_{\alpha\beta},G^k_{i\gamma},L^k_{ij},C^{k(\gamma)}_{i(j)})$.

Using the *canonical Liouville d-tensor* $\mathbf{C}= x^i_\alpha(\partial/\partial x^i_\alpha)$ and the fundamental vertical metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(k)}$ of the metrical multi-time Lagrange space ML^n_p, we can construct the *metrical deflection d-tensors*

$$\begin{aligned} \bar{D}^{(\alpha)}_{(i)\beta} &= G^{(\alpha)(\gamma)}_{(i)(k)}\bar{D}^{(k)}_{(\gamma)\beta} = x^{(\alpha)}_{(i)/\beta}, \\ D^{(\alpha)}_{(i)j} &= G^{(\alpha)(\gamma)}_{(i)(k)}D^{(k)}_{(\gamma)j} = x^{(\alpha)}_{(i)|j}, \\ d^{(\alpha)(\beta)}_{(i)(j)} &= G^{(\alpha)(\gamma)}_{(i)(k)}d^{(k)(\beta)}_{(\gamma)(j)} = x^{(\alpha)}_{(i)}|^{(\beta)}_{(j)}, \end{aligned} \tag{3.16}$$

where $x^{(\alpha)}_{(i)} = G^{(\alpha)(\gamma)}_{(i)(k)}x^k_\gamma$ and "$_{/\beta}$", "$_{|j}$" and "$|^{(\beta)}_{(j)}$" are the local covariant derivatives induced by $C\Gamma$.

Taking into account the expressions of the local covariant derivatives of $C\Gamma$ (see the papers [15], [20]), by a direct calculation, we obtain

Proposition 3.1 *The metrical deflection d-tensors of a metrical multi-time Lagrange space ML^n_p have the expressions:*

i) for $p = 1$,

$$\begin{aligned} \bar{D}^{(1)}_{(i)1} &= \frac{h^{11}}{2}\frac{\delta g_{im}}{\delta t}y^m, \\ D^{(1)}_{(i)j} &= h^{11}g_{ik}\left[-N^{(k)}_{(1)j} + L^k_{jm}y^m\right], \\ d^{(1)(1)}_{(i)(j)} &= h^{11}\left[g_{ij} + g_{ik}C^{k(1)}_{m(j)}y^m\right]; \end{aligned} \tag{3.17}$$

ii) for $p \geq 2$,

$$\begin{aligned} \bar{D}^{(\alpha)}_{(i)\beta} &= \frac{h^{\alpha\gamma}}{2}\frac{\partial g_{km}}{\partial t^\beta}x^m_\gamma, \\ D^{(\alpha)}_{(i)j} &= -\frac{h^{\alpha\gamma}}{2}\frac{\partial g_{ij}}{\partial t^\gamma} - \frac{1}{4}U^{(\alpha)}_{(i)j}, \\ d^{(i)(\beta)}_{(\alpha)(j)} &= h^{\alpha\beta}g_{ij}. \end{aligned} \tag{3.18}$$

In order to construct the metrical multi-time Lagrangian theory of electromagnetism, we introduce

Definition 3.1 The distinguished 2-form on $J^1(T, M)$, locally defined by

$$F = F^{(\alpha)}_{(i)j}\delta x^i_\alpha \wedge dx^i + f^{(\alpha)(\beta)}_{(i)(j)}\delta x^i_\alpha \wedge \delta x^j_\beta, \tag{3.19}$$

where

$$\begin{aligned} F^{(\alpha)}_{(i)j} &= \frac{1}{2}\left[D^{(\alpha)}_{(i)j} - D^{(\alpha)}_{(j)i}\right], \\ f^{(\alpha)(\beta)}_{(i)(j)} &= \frac{1}{2}\left[d^{(\alpha)(\beta)}_{(i)(j)} - d^{(\alpha)(\beta)}_{(j)(i)}\right], \end{aligned}$$

is called the *multi-time electromagnetic field* of the space ML^n_p.

Remark 3.1 The naturalness of the previous definition comes considering the particular case of a relativistic rheonomic Lagrange space (i. e., $(T, h) = (\mathbb{R}, \delta)$). In this particular case, we find a natural relativistic generalization of the electromagnetic d-tensors used in [12].

Proposition 3.2 *The components $F^{(\alpha)}_{(i)j}$ and $f^{(\alpha)(\beta)}_{(i)(j)}$ of the electromagnetic d-form F of the metrical multi-time Lagrange space ML^n_p are described by the formulas:*

i) in the case $p = 1$,

$$F^{(1)}_{(i)j} = \frac{h^{11}}{2}\left[g_{jm}N^{(m)}_{(1)i} - g_{im}N^{(m)}_{(1)j} + (g_{ik}L^k_{jm} - g_{jk}L^k_{im})y^m\right], \quad f^{(1)(1)}_{(i)(j)} = 0;$$

ii) in the case $p \geq 2$,

$$F^{(\alpha)}_{(i)j} = \frac{1}{8}\left[U^{(\alpha)}_{(j)i} - U^{(\alpha)}_{(i)j}\right], \quad f^{(\alpha)(\beta)}_{(i)(j)} = 0.$$

Because the vertical electromagnetic components $f^{(\alpha)(\beta)}_{(i)(j)}$ vanish, it follows that the main laws that govern the electromagnetic metrical multi-time Lagrangian theory must be intimately conected by the horizontal electromagnetic components $F^{(\alpha)}_{(i)j}$.

Theorem 3.3 *The electromagnetic components $F^{(\alpha)}_{(i)j}$ of the metrical multi-time Lagrange space ML^n_p are governed by the following generalized Maxwell equations:*

i) for $p = 1$,

$$\begin{cases} F^{(1)}_{(i)k/1} = \frac{1}{2}\mathcal{A}_{\{i,k\}}\left\{\bar{D}^{(1)}_{(i)1|k} + D^{(1)}_{(i)m}T^m_{1k} + d^{(1)(1)}_{(i)(m)}R^{(m)}_{(1)1k} - \left[T^p_{1i|k} + C^{p(1)}_{k(m)}R^{(m)}_{(1)1i}\right]y_{(p)}\right\} \\ \sum_{\{i,j,k\}} F^{(1)}_{(i)j|k} = -\frac{1}{2}\sum_{\{i,j,k\}} C^{(1)(1)(1)}_{(i)(l)(m)}R^{(m)}_{(1)jk}y^l \\ \sum_{\{i,j,k\}} F^{(1)}_{(i)j}|^{(1)}_{(k)} = 0, \end{cases}$$

ii) for $p \geq 2$,

$$\begin{cases} F^{(\alpha)}_{(i)k/\beta} = \frac{1}{2}\mathcal{A}_{\{i,k\}}\left\{\bar{D}^{(\alpha)}_{(i)\beta|k} + D^{(\alpha)}_{(i)m}T^m_{\beta k} + d^{(\alpha)(\mu)}_{(i)(m)}R^{(m)}_{(\mu)\beta k} - \left[T^p_{\beta i|k} + C^{p(\mu)}_{k(m)}R^{(m)}_{(\mu)\beta i}\right]x^{(\alpha)}_{(p)}\right\} \\ \sum_{\{i,j,k\}} F^{(\alpha)}_{(i)j|k} = 0 \\ \sum_{\{i,j,k\}} F^{(\alpha)}_{(i)j}|^{(\gamma)}_{(k)} = 0, \end{cases}$$

where $y_{(p)} = G^{(1)(1)}_{(p)(q)}y^q$, $C^{(1)(1)(1)}_{(i)(l)(m)} = G^{(1)(1)}_{(l)(q)}C^{q(1)}_{i(m)} = \dfrac{h^{11}}{2}\dfrac{\partial^3 L}{\partial y^i \partial y^l \partial y^m}$, $x^{(\alpha)}_{(p)} = G^{(\alpha)(\beta)}_{(p)(q)}x^q_\beta$.

Proof. Firstly, we point out that the Ricci identities [20] applied to the spatial metrical d-tensor g_{ij} imply that the following curvature d-tensor identities:

$$R_{mi\beta k} + R_{im\beta k} = 0, \quad R_{mijk} + R_{imjk} = 0, \quad P_{mij(k)}^{\ \ \ \ \ (\gamma)} + P_{imj(k)}^{\ \ \ \ \ (\gamma)} = 0,$$

where $R_{mi\beta k} = g_{ip}R^p_{m\beta k}$, $R_{mijk} = g_{ip}R^p_{mjk}$ and $P_{mij(k)}^{\ \ \ \ \ (\gamma)} = g_{ip}P^{p\ \ \ (\gamma)}_{mj(k)}$, are true.

Now, let us consider the following general deflection d-tensor identities [20]:

$d_1)$ $\bar{D}^{(p)}_{(\nu)\beta|k} - D^{(p)}_{(\nu)k/\beta} = x^m_\nu R^p_{m\beta k} - D^{(p)}_{(\nu)m} T^m_{\beta k} - d^{(p)(\mu)}_{(\nu)(m)} R^{(m)}_{(\mu)\beta k}$,

$d_2)$ $D^{(p)}_{(\nu)j|k} - D^{(p)}_{(\nu)k|j} = x^m_\nu R^p_{mjk} - d^{(p)(\mu)}_{(\nu)(m)} R^{(m)}_{(\mu)jk}$,

$d_3)$ $D^{(p)}_{(\nu)j}|^{(\gamma)}_{(k)} - d^{(p)(\gamma)}_{(\nu)(k)|j} = x^m_\nu P^{p\ (\gamma)}_{mj(k)} - D^{(p)}_{(\nu)m} C^{m(\gamma)}_{j(k)} - d^{(p)(\mu)}_{(\nu)(m)} P^{(m)\ (\gamma)}_{(\mu)j(k)}$,

where $\bar{D}^{(i)}_{(\alpha)\beta} = x^i_{\alpha/\beta}$, $D^{(i)}_{(\alpha)j} = x^i_{\alpha|j}$, $d^{(i)(\beta)}_{(\alpha)(j)} = x^i_\alpha|^{(\beta)}_{(j)}$. Contracting these deflection d-tensor identities by $G^{(\alpha)(\nu)}_{(i)(p)}$ and using the above curvature d-tensor equalities, we obtain the metrical deflection d-tensors identities:

$d'_1)$ $\bar{D}^{(\alpha)}_{(i)\beta|k} - D^{(\alpha)}_{(i)k/\beta} = -x^{(\alpha)}_{(m)} R^m_{i\beta k} - D^{(\alpha)}_{(i)m} T^m_{\beta k} - d^{(\alpha)(\mu)}_{(i)(m)} R^{(m)}_{(\mu)\beta k}$,

$d'_2)$ $D^{(\alpha)}_{(i)j|k} - D^{(\alpha)}_{(i)k|j} = -x^{(\alpha)}_{(m)} R^m_{ijk} - d^{(\alpha)(\mu)}_{(i)(m)} R^{(m)}_{(\mu)jk}$,

$d'_3)$ $D^{(\alpha)}_{(i)j}|^{(\gamma)}_{(k)} - d^{(\alpha)(\gamma)}_{(i)(k)|j} = -x^{(\alpha)}_{(m)} P^{m\ (\gamma)}_{ij(k)} - D^{(\alpha)}_{(i)m} C^{m(\gamma)}_{j(k)} - d^{(\alpha)(\mu)}_{(i)(m)} P^{(m)\ (\gamma)}_{(\mu)j(k)}$.

At the same time, we recall that the following Bianchi identities hold good [15]:

$b_1)$ $\mathcal{A}_{\{j,k\}} \left\{ R^l_{j\alpha k} + T^l_{\alpha j|k} + C^{l(\mu)}_{k(m)} R^{(m)}_{(\mu)\alpha j} \right\} = 0$,

$b_2)$ $\sum_{\{i,j,k\}} \left\{ R^l_{ijk} - C^{l(\mu)}_{k(m)} R^{(m)}_{(\mu)ij} \right\} = 0$,

$b_3)$ $\mathcal{A}_{\{j,k\}} \left\{ P^{l\ (\varepsilon)}_{jk(p)} + C^{l(\varepsilon)}_{j(p)|k} + C^{l(\mu)}_{k(m)} P^{(m)\ (\varepsilon)}_{(\mu)j(p)} \right\} = 0$,

where $\mathcal{A}_{\{j,k\}}$ means alternate sum and $\sum_{\{i,j,k\}}$ means cyclic sum.

Now, in order to obtain the first Maxwell identity, we permute i and k in d'_1 and we subtract the new identity from the initial one. Finally, using the Bianchi identity b_1, we obtain what we were looking for.

Doing a cyclic sum by the indices $\{i, j, k\}$ in d'_2 and using the Bianchi identity b_2, it follows the second Maxwell equation.

Applying a Christoffel process to the indices $\{i, j, k\}$ in d'_3 and combining with the Bianchi identity b_3 and the relation $P^{(m)\ (\varepsilon)}_{(\mu)j(p)} = P^{(m)\ (\varepsilon)}_{(\mu)p(j)}$, we get a new identity. The cyclic sum by the indices $\{i, j, k\}$ applied to this last identity implies the third Maxwell equation. ∎

Remark 3.2 In the particular case $(T, h) = (I\!R, \delta)$, we discover a natural generalization of the Maxwell equations appeared in the absolute rheonomic Lagrangian electromagnetism from [12].

4 Multi-time gravitational h-potential on ML^n_p. Generalized Einstein equations and conservation laws

Let $h = (h_{\alpha\beta})$ be a fixed semi-Riemannian metric on the temporal manifold T and $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ a fixed nonlinear connection on the 1-jet space $E = J^1(T, M)$. In order to develope the metrical multi-time Lagrange theory of gravitational field, we introduce the following abstract physical concept:

Definition 4.1 From physical point of view, an adapted metrical d-tensor G on E, locally expressed by

$$G = h_{\alpha\beta}dt^\alpha \otimes dt^\beta + g_{ij}dx^i \otimes dx^j + h^{\alpha\beta}g_{ij}\delta x^i_\alpha \otimes \delta x^j_\beta,$$

where $g_{ij} = g_{ij}(t^\gamma, x^k, x^k_\gamma)$ is a d-tensor field on E, symmetric, of rank $n = \dim M$ and having a constant signature on E, is called a *multi-time gravitational h-potential.*

Remark 4.1 In the particular case $(T, h) = (I\!R, \delta)$, the gravitational δ-potentials naturally generalize the gravitational potentials used in [12].

Now, taking $ML^n_p = (J^1(T, M), L)$ a metrical multi-time Lagrange space, via its fundamental vertical metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}$ (see (2.8)) and its canonical nonlinear connection $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ (see (2.9), (2.10)), we can construct a multi-time gravitational h-potential, naturally attached to the space ML^n_p, setting

$$G = h_{\alpha\beta}dt^\alpha \otimes dt^\beta + g_{ij}dx^i \otimes dx^j + h^{\alpha\beta}g_{ij}\delta x^i_\alpha \otimes \delta x^j_\beta.$$

Let us consider $C\Gamma = (H^\gamma_{\alpha\beta}, G^k_{j\gamma}, L^i_{jk}, C^{i(\gamma)}_{j(k)})$ the Cartan canonical connection of ML^n_p.

We *postulate* that the generalized Einstein equations, which govern the multi-time gravitational h-potential G of the metrical multi-time Lagrange space ML^n_p, are the abstract geometrical Einstein equations attached to the Cartan canonical connection $C\Gamma$ of ML^n_p and the adapted metric G on E, that is,

$$Ric(C\Gamma) - \frac{Sc(C\Gamma)}{2}G = \mathcal{K}\mathcal{T}, \tag{4.1}$$

where $Ric(C\Gamma)$ represents the Ricci d-tensor of the Cartan connection, $Sc(C\Gamma)$ is its scalar curvature, $\mathcal{K}$ is the Einstein constant and $\mathcal{T}$ is an intrinsec tensor of matter (equal to 0 for vacuum), which is called the *stress-energy* d-tensor.

In the adapted basis $(X_A) = \left(\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^i_\alpha}\right)$ attached to the canonical nonlinear connection Γ of ML^n_p, the curvature d-tensor $\mathbf{R}$ of the canonical Cartan connection is locally expressed by $\mathbf{R}(X_C, X_B)X_A = R^D_{ABC}X_D$. It follows that we have $R_{AB} = Ric(X_A, X_B) = R^D_{ABD}$ and $Sc(C\Gamma) = G^{AB}R_{AB}$, where

$$G^{AB} = \begin{cases} h_{\alpha\beta}, & \text{for } A = \alpha,\ B = \beta \\ g^{ij}, & \text{for } A = i,\ B = j \\ h_{\alpha\beta}g^{ij}, & \text{for } A = {}^{(i)}_{(\alpha)},\ B = {}^{(j)}_{(\beta)} \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Taking into account, on the one hand, the form of the fundamental vertical metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}$ of the metrical multi-time Lagrange space ML^n_p, and, on the other hand, the expressions of local curvature d-tensors attached to the Cartan canonical connection $C\Gamma$, by direct computations, we deduce

Theorem 4.1 *The Ricci d-tensor $Ric(C\Gamma)$ of the Cartan canonical connection $C\Gamma$ of a metrical multi-time Lagrange space ML^n_p is determined by the following adapted components:*

i) for $p = \dim T = 1$,

$$R_{11} \overset{not}{=} H_{11} = 0, \quad R_{i1} = R^m_{i1m}, \quad R_{ij} = R^m_{ijm}, \quad R^{\ (1)}_{i(j)} \overset{not}{=} P^{\ (1)}_{i(j)} = -P^{m\ (1)}_{im(j)},$$

$$R^{(1)}_{(i)j} \overset{not}{=} P^{(1)}_{(i)j} = P^{m\ (1)}_{ij(m)}, \quad R^{(1)}_{(i)1} \overset{not}{=} P^{(1)}_{(i)1} = P^{m\ (1)}_{i1(m)}, \quad R^{(1)(1)}_{(i)(j)} \overset{not}{=} S^{(1)(1)}_{(i)(j)} = S^{m(1)(1)}_{i(j)(m)};$$

ii) for $p = \dim T \geq 2$,

$$R_{(\alpha)(\beta)} \overset{not}{=} H_{\alpha\beta} = H^{\mu}_{\alpha\beta\mu}, \quad R_{i\alpha} = R^m_{i\alpha m}, \quad R_{ij} = R^m_{ijm}, \quad R^{\ (\alpha)}_{i(j)} \overset{not}{=} P^{\ (\alpha)}_{i(j)} = 0,$$

$$R^{(\alpha)}_{(i)j} \overset{not}{=} P^{(\alpha)}_{(i)j} = 0, \quad R^{(\alpha)}_{(i)\beta} \overset{not}{=} P^{(\alpha)}_{(i)\beta} = 0, \quad R^{(\alpha)(\beta)}_{(i)(j)} \overset{not}{=} S^{(\alpha)(\beta)}_{(i)(j)} = 0.$$

Moreover, denoting $H = h^{\alpha\beta}H_{\alpha\beta}$, $R = g^{ij}R_{ij}$ and $S = h_{\alpha\beta}g^{ij}S^{(\alpha)\beta)}_{(i)(j)}$, it follows

Corollary 4.2 *The scalar curvature* $Sc(C\Gamma)$ *of the Cartan canonical connection* $C\Gamma$ *of a metrical multi-time Lagrange space* ML^n_p *is given by the formulas*

i) for $p = \dim T = 1$, $\qquad Sc(C\Gamma) = R + S$;

ii) for $p = \dim T \geq 2$, $\qquad Sc(C\Gamma) = H + R$.

The main result of the metrical multi-time Lagrange theory of gravitational field is given by the following theorem:

Theorem 4.3 *The generalized Einstein equations, which govern the multi-time gravitational h-potential* G *induced by the Kronecker h-regular Lagrangian of a metrical multi-time Lagrange space* ML^n_p*, have the following adapted local form:*

i) for $p = \dim T = 1$,

$$(E_1) \qquad \begin{cases} -\dfrac{R+S}{2}h_{11} = \mathcal{K}\mathcal{T}_{11} \\ R_{ij} - \dfrac{R+S}{2}g_{ij} = \mathcal{K}\mathcal{T}_{ij} \\ S^{(1)(1)}_{(i)(j)} - \dfrac{R+S}{2}h^{11}g_{ij} = \mathcal{K}\mathcal{T}^{(1)(1)}_{(i)(j)}, \end{cases}$$

$$(E_2) \qquad \begin{cases} 0 = \mathcal{T}_{1i}, \quad R_{i1} = \mathcal{K}\mathcal{T}_{i1}, \quad P^{(1)}_{(i)1} = \mathcal{K}\mathcal{T}^{(1)}_{(i)1} \\ 0 = \mathcal{T}^{\ (1)}_{1(i)}, \quad P^{\ (1)}_{i(j)} = \mathcal{K}\mathcal{T}^{\ (1)}_{i(j)}, \quad P^{(1)}_{(i)j} = \mathcal{K}\mathcal{T}^{(1)}_{(i)j}, \end{cases}$$

ii) for $p = \dim T \geq 2$,

$$(E_1) \qquad \begin{cases} H_{\alpha\beta} - \dfrac{H+R}{2}h_{\alpha\beta} = \mathcal{K}\mathcal{T}_{\alpha\beta} \\ R_{ij} - \dfrac{H+R}{2}g_{ij} = \mathcal{K}\mathcal{T}_{ij} \\ -\dfrac{H+R}{2}h^{\alpha\beta}g_{ij} = \mathcal{K}\mathcal{T}^{(\alpha)(\beta)}_{(i)(j)}, \end{cases}$$

$$(E_2)\qquad \begin{cases} 0=\mathcal{T}_{\alpha i}, & R_{i\alpha}=\mathcal{K}\mathcal{T}_{i\alpha}, \quad 0=\mathcal{T}^{(\alpha)}_{(i)\beta} \\ 0=\mathcal{T}^{\ (\beta)}_{\alpha(i)}, & 0=\mathcal{T}^{\ (\alpha)}_{i(j)}, \qquad 0=\mathcal{T}^{(\alpha)}_{(i)j}, \end{cases}$$

where $\mathcal{T}_{AB},\ A,B\in\{\alpha,i,\binom{(\alpha)}{(i)}\}$ *are the adapted local components of the stress-energy d-tensor* $\mathcal{T}$.

Remark 4.2 Asumming that $p=\dim T>2$ and $n=\dim M>2$, the set (E_1) of the generalized Einstein equations can be rewritten in the more natural form

$$(E_1')\qquad \begin{cases} H_{\alpha\beta}-\dfrac{H}{2}h_{\alpha\beta}=\mathcal{K}\tilde{\mathcal{T}}_{\alpha\beta} \\ R_{ij}-\dfrac{R}{2}g_{ij}=\mathcal{K}\tilde{\mathcal{T}}_{ij}, \end{cases}$$

where $\tilde{\mathcal{T}}_{AB},\ A,B\in\{\alpha,i\}$ are the adapted local components of a new stress-energy d-tensor $\tilde{\mathcal{T}}$. This new form of the generalized Einstein equations is derived in the more general case of a *generalized metrical multi-time Lagrange space* [14].

It is obvious that, in order to have the compatibility of the generalized Einstein equations, it is necessary that certain adapted local components of the stress-energy d-tensor $\mathcal{T}$ to vanish *"a priori"*. Moreover, from physical point of view, it is well known that a good stress-energy d-tensor $\mathcal{T}$ must verify the local *conservation laws*

$$\mathcal{T}^B_{A|B}=0,\ \forall\, A\in\{\alpha,i,\ {(\alpha)\atop (i)}\},$$

where $\mathcal{T}^B_A=G^{BD}\mathcal{T}_{DA}$. Consequently, by a direct calculation, we find

Theorem 4.4 *The conservation laws of the generalized Einstein equations of a metrical multi-time Lagrange space* ML^n_p *are given by the formulas:*

i) for $p=1$,

$$(4.3)\qquad \begin{cases} \left[\dfrac{R+S}{2}\right]_{/1}=R^m_{1|m}-P^{(m)}_{(1)1}|^{(1)}_{(m)} \\ \left[R^m_j-\dfrac{R+S}{2}\delta^m_j\right]_{|m}=-P^{(m)}_{(1)j}|^{(1)}_{(m)} \\ \left[S^{(m)(1)}_{(1)(j)}-\dfrac{R+S}{2}\delta^m_j\right]\Big|^{(1)}_{(m)}=-P^{m(1)}_{\ \ (j)|m}, \end{cases}$$

where $R^i_1=g^{im}R_{m1}$, $P^{(i)}_{(1)1}=h_{11}g^{im}P^{(1)}_{(m)1}$, $R^i_j=g^{im}R_{mj}$, $P^{(i)}_{(1)j}=h_{11}g^{im}P^{(1)}_{(m)j}$, $P^{i(1)}_{\ \ (j)}=g^{im}P^{\ \ (1)}_{m(j)}$ *and* $S^{(i)(1)}_{(1)(j)}=h_{11}g^{im}S^{(1)(1)}_{(m)(j)}$;

i) for $p\geq 2$,

$$(4.4)\qquad \begin{cases} \left[H^\mu_\beta-\dfrac{H+R}{2}\delta^\mu_\beta\right]_{/\mu}=-R^m_{\beta|m} \\ \left[R^m_j-\dfrac{H+R}{2}\delta^m_j\right]_{|m}=0, \end{cases}$$

where $H^{\mu}_{\beta} = h^{\mu\gamma}H_{\gamma\mu}$, $R^i_j = g^{im}R_{mj}$ *and* $R^i_{\beta} = g^{im}R_{m\beta}$.

Remarks 4.3 i) In the case $p > 2$, $n > 2$, taking into account the components $\tilde{\mathcal{T}}_{\alpha\beta}$ and $\tilde{\mathcal{T}}_{ij}$ of the new stress-energy d-tensor $\tilde{\mathcal{T}}$ from (E_1'), we point out that the conservation laws modify in the following simple and natural new form (see the paper [14]):

$$\tilde{\mathcal{T}}^{\mu}_{\beta/\mu} = 0, \quad \tilde{\mathcal{T}}^{m}_{j|m} = 0. \tag{4.5}$$

ii) In the particular case $(T, h) = (\mathbb{R}, \delta)$, the generalized Einstein equations, together with their generalized conservation laws, naturally generalize the similar equations described in [12].

Open problems.

i) What is the real physical interpretation of our abstract geometrical theory?

ii) The development of an analogous metrical multi-time Lagrangian geometry of physical fields on the jet space of order two $J^2(T, M)$ is in our attention.

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