

Variational Solutions of Stationary Hamilton-Jacobi Equations

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Abstract

This work deals with the stationary Hamilton-Jacobi equation $F(B^* \psi_y) - (Ay, \psi_y) = g$ in the class of convex continuous functions ψ on a real Hilbert space H . After obtaining an asymptotic result, the existence, the uniqueness and a Galerkin approximation of the solution to the above equation are established.

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Let H and U be two real Hilbert spaces with the scalar products (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$, respectively. The norms of H and U will be denoted by $|\cdot|$ and $|\cdot|_U$, respectively. We define the function $\psi^\infty : H \rightarrow R$,

$$(1) \quad \psi^\infty(y) = \inf \left\{ \int_0^\infty (g(x(t)) + h(u(t))) dt; x' = Ax + Bu, x(0) = y; u \in L^1(R^+; U) \right\}$$

in the following hypotheses:

- (i) A is the infinitesimal generator of a C_0 -semigroup of contractions on H i.e., A is m -dissipative on H ;
 B is a linear continuous operator from U to H .
- (ii) The function $g : H \rightarrow R$ belongs to $K \cap C_{Lip}^1(H)$ and satisfies the following two conditions:

$$(2) \quad \text{if } g(y_n) \rightarrow 0 \text{ for some sequences } \{y_n\} \text{ then } y_n \rightarrow 0;$$

$$(3) \quad g(e^{-(\lambda I - A)t} y) \in L^1(R^+) \text{ for } \lambda > 0 \text{ and all } y \in H.$$

Here $e^{-(\lambda I - A)t}$ is the semigroup generated by the operator $-\lambda I + A$, K is the closed convex cone of $C(H)$ consisting of all convex functions $\varphi : H \rightarrow R$ which satisfy the condition $\varphi(y) \geq \varphi(0) = 0$ for all $y \in H$ and $C_{Lip}^1(H)$ is the space of all $\varphi \in C^1(H)$ such that the Fréchet derivative $\varphi^{(1)}$ is Lipschitz on H .

- (iii) $F : U \rightarrow R$ is a convex function which is bounded on bounded subsets and belongs to $C^1(U)$. We associate the function $h : U \rightarrow \bar{R}$, defined by

$$h(u) = \sup\{-\langle p, u \rangle - F(p); p \in U\} \text{ i.e., } h(u) = F^*(-u),$$

to the function F . Then the assumption (iii) implies $\lim_{|u|_U \rightarrow \infty} \frac{h(u)}{|u|_U} = +\infty$. Further assume that F and F^* are strictly convex.

(iv) $\psi_0 : H \rightarrow R$ is a function which belongs to $K \cap C_{Lip}^1(H)$.

Under the above hypotheses, we see that $\Psi^\infty < +\infty$ on H and for every $y \in H$ the infimum defining $\Psi^\infty(y)$ is attained in a unique pair (x^*, u^*) . Moreover, the function Ψ^∞ is convex and lower semicontinuous on H . Since it is everywhere finite we conclude that it is continuous on H .

Now we shall prove the following asymptotic result.

Theorem 1. *Let assumptions (i), (ii), (iii) and (iv) be satisfied. Then the solution Ψ to the problem*

$$(4) \quad \begin{cases} \psi_t + F(B^* \psi_y) - (Ay, \psi_y) = g; & t \geq 0, y \in D(A) \\ \psi(0, y) = \psi_0(y), & y \in H \end{cases}$$

satisfies

$$(5) \quad \lim_{t \rightarrow \infty} \psi(t, y) = \psi^\infty(y) \text{ for all } y \in H.$$

We have denoted by Ψ_t and Ψ_y the partial derivatives (in a generalized sense) of the function Ψ with respect to t and y , respectively. Here B^* is the adjoint of B .

Proof. The solution Ψ to the Cauchy problem (4) is given by (see [1], p.54)

$$(6) \quad \psi(t, y) = \inf \left\{ \int_0^t (g(x(s)) + h(u(s))) ds + \psi_0(x(t)); x' = Ax + Bu, \right. \\ \left. x(0) = y; u \in L^1(0, t; U) \right\}$$

for $t \geq 0, y \in H$. This yields

$$(7) \quad \int_0^t (g(x^t(s)) + h(u^t(s))) ds + \psi_0(x^t(t)) = \psi(t, y) \leq \int_0^t (g(x^*(s)) + \\ + h(u^*(s))) ds + \psi_0(x^*(t))$$

where (x^*, u^*) is the optimal pair in Problem (1) and (x^t, u^t) in (6), i.e.,

$$\begin{cases} (x^t)' = Ax^t + Bu^t \text{ on } [0, t] \\ x^t(0) = y. \end{cases}$$

Letting $t \rightarrow +\infty$, we see by (7) that

$$(8) \quad \limsup_{t \rightarrow \infty} \psi(t, y) \leq \psi^\infty(y).$$

We set

$$\tilde{u}^t(s) = \begin{cases} u^t(s) & \text{for } 0 \leq s \leq t \\ 0 & \text{for } s > t \end{cases}$$

and

$$\tilde{x}^t(s) = e^{As}y + \int_0^s e^{A(s-\tau)}B\tilde{u}^t(\tau)d\tau.$$

Let some sequence $t_n \rightarrow \infty$ and let $\{\tilde{x}^{t_n}\}$ and $\{\tilde{u}^{t_n}\}$ be such that

$$(9) \quad \psi^\infty(y) \leq \int_0^\infty (g(\tilde{x}^{t_n}(s)) + h(\tilde{u}^{t_n}(s))) ds \leq \psi^\infty(y) + \frac{1}{n},$$

$$(\tilde{x}^{t_n})' = A\tilde{x}^{t_n} + B\tilde{u}^{t_n} \quad \text{on } R^+$$

$$\tilde{x}^{t_n}(0) = y.$$

Since $|\tilde{x}^{t_n}(s)| \leq C \left(|y| + \int_0^s \|\tilde{u}^{t_n}(\tau)\| d\tau \right)$, $\forall s \geq 0$ and $g(\tilde{x}^{t_n}(s)) \geq g(0) - |\partial g(0)| |\tilde{x}^{t_n}(s)|$, $\forall s \geq 0$ by (9) it follows that $\int_0^\infty h(\tilde{u}^{t_n}(s))ds \leq C \left(\int_0^\infty \|\tilde{u}^{t_n}(s)\| ds + 1 \right)$ where C is independent of n . The latter implies that $\{\tilde{u}^{t_n}\}$ is weakly compact in $L^1(R^+; U)$ (by virtue of the Dunford-Pettis theorem). Hence on some sequence $t_n \rightarrow \infty$, $\tilde{u}^{t_n} \rightarrow \tilde{u}$ weakly in $L^1(R^+; U)$ and $\tilde{x}^{t_n}(s) \rightarrow \tilde{x}(s)$ weakly in H for every $s \geq 0$, where \tilde{x} is the mild solution to

$$\begin{cases} x' = Ax + B\tilde{u} & \text{on } R^+ \\ x(0) = y. \end{cases}$$

Since the functional $(x, u) \rightarrow \int_0^T (g(x) + h(u))dt$ is weakly lower semicontinuous on every $C([0, T]; H) \times L^1(0, T; U)$ (because it is convex and lower semicontinuous), it follows by (7) and (8) that

$$\int_0^\infty (g(\tilde{x}^t(s)) + h(\tilde{u}^t(s)))ds \leq \limsup_{t \rightarrow \infty} \psi(t, y) \leq \psi^\infty(y)$$

which obviously implies (5) as claimed. \square

Now let us consider the stationary Hamilton-Jacobi equation

$$(10) \quad F(B^*\psi_y) - (Ay, \psi_y) = g, \quad \forall y \in D(A)$$

in the real Hilbert space H .

By a *solution* to (10) we shall mean a function $\psi : H \rightarrow R$ which is Gâteaux differentiable and satisfies Eq. (10) for all $y \in D(A)$.

Equation (10) is relevant in the calculus of variations with infinite horizon and in nonlinear analysis. The main result is

Theorem 2. *Let assumptions (i), (ii) part (3) and (iii) be satisfied. Then Eq. (10) has at least one solution $\psi^\infty \in K$. If also condition (2) holds then the solution ψ^∞ to Eq. (10) is unique.*

Proof. For every $t \geq 0$, the unique optimal pair (x^*, u^*) in Problem (1) is also an optimal pair for the finite horizon control problem

$$(11) \quad \psi^\infty(y) = \inf \left\{ \int_0^t (g(x(s)) + h(u(s)))ds + \psi^\infty(x(t)); x' = Ax + Bu, x(0) = y \right\}$$

$$= \int_0^t (g(x^*(s)) + h(u^*(s)))ds + \psi^\infty(x^*(t)).$$

By the maximum principle ([1], p. 13), for every $t \geq 0$ there exists $p^t \in C([0, t]; H)$ which satisfies the system

$$\begin{cases} (p^t)' + A^*p^t \in \partial g(x^*), & 0 \leq s \leq t \\ B^*p^t \in \partial h(u^*), & 0 \leq s \leq t \\ p^t(t) \in -\partial\psi^\infty(x^*(t)) & . \end{cases}$$

(Here ∂ is the subdifferentiation symbol). Next by (11) we see that

$$(12) \quad \psi^\infty(x^*(t)) = \int_t^\infty (g(x^*(s)) + h(u^*(s)))ds, \quad \forall t \geq 0.$$

Assume that $y \in D(A)$. Then x^* is right differentiable at $t = 0$ and $\frac{d^+x^*}{dt}(0) = Ay + Bu^*(0)$. Since

$$\frac{\psi^\infty(x^*(t)) - \psi^\infty(y)}{t} \xrightarrow{t \rightarrow 0} \left(\eta, \frac{d^+x^*}{dt}(0) \right)$$

where η is an element of $\partial\Psi^\infty(y)$, it follows by (12) that

$$(13) \quad (\eta, Ay + Bu^*(0)) + g(y) + h(u^*(0)) = 0$$

where $u^*(0) = \nabla h^*(-B^*\eta)$ and $\eta \in \partial\psi^\infty(y)$ (we may take $\eta = -p^t(0)$). By using (13) and the conjugacy formula, we get

$$F(B^*\eta) - (Ay, \eta) = g(y).$$

To conclude the proof of existence it suffices to show that Ψ^∞ is Gâteaux differentiable, i.e., $\partial\Psi^\infty$ is single valued.

To this end we define the operator $\Gamma : H \rightarrow H$, $\Gamma y = -p(0)$ where $p \in C([0, T]; H)$ is the solution to the system (T is fixed)

$$(14) \quad \begin{cases} x' = Ax + Bu, & 0 \leq t \leq T \\ p' + A^*p \in \partial g(x), & 0 \leq t \leq T \\ x(0) = y, p(T) \in -\partial\Psi^\infty(x(T)) \end{cases}$$

which, by virtue of the maximum principles is equivalent with the control problem

$$(15) \quad \inf \left\{ \int_0^T (g(x) + h(u))dt + \Psi^\infty(x(T)); x' = Ax + Bu, x(0) = y \right\}.$$

Since F and F^* are strictly convex, so is h^* and therefore there exists a unique optimal pair (x, u) for the problem (15). Since F is Gâteaux differentiable, ∂F^* and therefore ∂h is single valued. Thus there exists a unique $B^*p = \partial h(u)$ which satisfies the system (14) and therefore Γ is single valued. On the other hand, we see that $\Gamma y \in \partial\Psi^\infty(y)$. To prove that $\Gamma = \partial\Psi^\infty$ it suffices to show that the range $R(I + \Gamma)$ is all of H . To this end we consider the equation $y + \Gamma y = w$ which is equivalent to

$$(16) \quad \begin{cases} x' = Ax + Bu, & 0 \leq t \leq T \\ p' + A^*p \in \partial g(x), & 0 \leq t \leq T \\ x(0) = y, p(0) = y - w \\ p(T) \in -\partial \Psi^\infty(x(T)) \\ B^*p \in \partial h(u), & 0 \leq t \leq T \end{cases}$$

and again by virtue of the maximum principle it is equivalent to the control problem

$$\inf \left\{ \int_0^T (g(x) + h(u))dt + \Psi^\infty(x(T)) + \frac{1}{2}|x(0) - w|^2; x' = Ax + Bu \right\}$$

which clearly admits at least one solution which is also a solution to (16). Hence $\Gamma = \partial \Psi^\infty$, and $\partial \Psi^\infty = (\Psi^\infty)'$ is single valued as claimed.

As for uniqueness we consider the differential equation

$$(17) \quad \begin{cases} x' = Ax - B\nabla h^*(B^*\Psi_x^0(x)), & t \geq 0 \\ x(0) = y \end{cases}$$

where Ψ^0 is any solution to Eq. (10). For each $y \in D(A)$, Eq. (17) has a unique solution $\tilde{x} \in W_{loc}^{1,\infty}(R^+; H)$. We set $\tilde{u} = -\nabla h^*(B^*\Psi_x^0(x))$ and take the inner product of (17) by $-\Psi_x^0(\tilde{x})$. Since Ψ^0 is a solution to Eq. (10) we get

$$\frac{d}{dt} \Psi^0(\tilde{x}(t)) - F(B^*\Psi_x^0(\tilde{x}(t))) + g(\tilde{x}(t)) + (\tilde{u}(t), -B^*\Psi_x^0(x^*(t))) = 0$$

a.e. $t \geq 0$. By using the conjugacy formula, the latter becomes

$$\frac{d}{dt} \Psi^0(\tilde{x}(t)) + g(\tilde{x}(t)) + h(\tilde{u}(t)) = 0 \quad \text{a.e. } t \geq 0$$

and therefore

$$(18) \quad \Psi^0(y) = \Psi^0(\tilde{x}(t)) + \int_0^t (g(\tilde{x}(s)) + h(\tilde{u}(s)))ds, \quad \forall t \geq 0.$$

Finally

$$(19) \quad \Psi^0(y) = \int_0^\infty (g(\tilde{x}(s)) + h(\tilde{u}(s)))ds \geq \Psi^\infty(y)$$

where Ψ^∞ is given by (1). (To get (19) we have used the fact that $\tilde{x}(t_n) \rightarrow 0$ for some $t_n \rightarrow \infty$ because $g(\tilde{x}) \in L^1(R^+)$ and g satisfies condition (2)).

Now let (x^*, u^*) be the optimal pair for Problem (1) ($y \in D(A)$). Again by Eq. (10) we have

$$\begin{aligned} \frac{d}{dt} \Psi^0(x^*(t)) &= (x'^*(t), \Psi_x^0(x^*(t))) = (Ax^*(t) + Bu^*(t), \Psi_x^0(x^*(t))) = \\ &= F(B^*\Psi_x^0(x^*(t))) - g(x^*(t)) - (u^*(t), -B^*\Psi_x^0(x^*(t))) \geq \\ &\geq -h(u^*(t)) - g(x^*(t)) \quad \text{a.e. } t > 0 \end{aligned}$$

and therefore

$$\Psi^0(x^*(t)) + \int_0^t (h(u^*(s)) + g^*(x^*(s)))ds \geq \Psi^0(y), \quad \forall t \geq 0.$$

Since $g(x^*) \in L^1(R^+)$ it follows that $\lim_{t_n \rightarrow \infty} x^*(t_n) \rightarrow 0$ for some sequence $t_n \rightarrow \infty$ and therefore $\lim_{t_n \rightarrow \infty} \Psi^0(x^*(t_n)) = \Psi^0(0) = 0$. Hence $\Psi^0(y) \leq \Psi^\infty(y)$ for all $y \in D(A)$.

Along with (19) the latter implies that $\Psi^\infty = \Psi^0$ as claimed. \square

Now we shall briefly discuss a Galerkin approximation of the stationary Hamilton-Jacobi equation (10). Assume that the hypotheses (i)-(iii) hold. It will be more convenient to regard A as a linear continuous operator from the space $V \subset H$ to its dual $V' \supset H$. Following a well-known scheme (see for instance [1], [2], [3]), the internal approximations of the spaces V and U , of the operators A and B^* and of the functions F and g will be defined. Thus on the finite dimensional space V_h , Eq. (10) becomes

$$(20) \quad F_h(B_h^* \Psi_y^h(y_h)) - (A_h y_h, \Psi_y^h(y_h))_h = g_h(y_h) \text{ for all } y_h \in V_h.$$

By Theorem 2, Eq. (20) has a generalized solution $\Psi^h : V_h \rightarrow R$ given by

$$\Psi^h(y_h) = \inf \left\{ \int_0^\infty (g_h(x_h(t)) + F_h^*(-u_h(t)))dt \right\}$$

where x_h, u_h satisfy the system

$$\begin{cases} x_h' = A_h x_h + B_h u_h \text{ a.e. } t > 0 \\ x_h(0) = y_h. \end{cases}$$

As regards the convergence of the solutions Ψ^h to a solution Ψ to Eq. (10), we have the following result

Theorem 3. *For $h \rightarrow 0$ we have $\Psi^h(y_h) \rightarrow \Psi(y)$ for all $y \in V$, where Ψ is the variational solution to Eq. (10) given by (1).*

The proof is similar (making obvious changes) to the proofs of the analogous results in [1] and [3].

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