On Pseudo Ricci-Symmetric Manifolds

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Abstract

In the present study we consider pseudo Ricci-symmetric manifolds in the sense of M. C. Chaki. We show that pseudo Ricci-symmetric manifolds satisfying $divR = 0$ (resp.$divC = 0$) property are Einstein (resp.Ricci flat) manifolds.

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1 Introduction

Let M be an n-dimensional ($n \geq 3$) Riemannian manifold. For the vector fields $X, Y, Z \in \chi(M)$ and the the Levi-Civita connection ∇ of M the *curvature tensor* $\mathcal{R}$ and the *Ricci operator* $\mathcal{S}$ of M are defined by

$$\mathcal{R}(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z,$$

and

$$S(X,Y) = g(\mathcal{S}X, Y)$$

respectively. Furthermore for the vector field W the Riemannian Christoffel curvature tensor R of M is defined by $R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z, W)$ [4].

Let Π be a non-degenerate tangent plane to M at $p \in M$ given by $X, Y \in \chi(M)$. Then the *sectional curvature* $K(\Pi)$ of Π defined by

$$K(X,Y)\{g(X,X)g(Y,Y) - g(X,Y)^2\} = g(\mathcal{R}(X,Y)Y, X)$$

which is independent of the choice of the basis X, Y for Π.

A tensor field R of type (1, 2) on M is called *algebraic curvature tensor field* if it has symmetric properties of the curvature tensor field of Riemannian manifolds.

The curvature tensor $\mathcal{R}$ satisfies the second *Bianchi identity* if

$$(\nabla_X\mathcal{R})(Y,Z,W) + (\nabla_Y\mathcal{R})(X,Z,W) + (\nabla_Z\mathcal{R})(X,Y,W) = 0. \tag{1.1}$$

Let R be an algebraic curvature tensor field which satisfies the second Bianchi identity. If S is the associated Ricci tensor field then

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$$(divR)(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{1.2}$$

For a Riemannian manifold M if the Ricci tensor S is of the form $S = \lambda g$ then it is called *Einstein space* [4]. If $S = 0$ then M is called *Ricci-flat.*

The Weyl conformal curvature tensor C is defined by

$$\begin{aligned} C(X,Y,Z,W) &= R(X,Y,Z,W) - \frac{1}{n-2}\{g(X,W)S(Y,Z) + g(Y,Z)S(X,W) - \\ &- g(X,Z)S(Y,W) - g(Y,W)S(X,Z)\} + \\ &+ \frac{\tau}{(n-1)(n-2)}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\}. \end{aligned}$$

An algebraic curvature tensor field R is harmonic (or Codazzi type in the sense of [7]) if $(divR)(X,Y,Z) = 0$. A Riemannian manifold M is called R-harmonic if its curvature tensor field R is harmonic [1]

The divergence $divC$ of the Weyl conformal curvature tensor C is defined by

$$\begin{aligned} (divC)(X,Y,Z) &= \frac{n-3}{n-2}\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\} - \\ &- \frac{1}{2(n-1)}\{g(Y,Z)\nabla_X \tau - g(X,Z)\nabla_Y \tau\}. \end{aligned} \tag{1.3}$$

In the present study we consider pseudo Ricci-symmetric submanifolds and also hypersurfaces.

The notion of pseudo Ricci-symmetric (PRS) manifolds were introduced by M. C. Chaki in 1987. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called *pseudo Ricci-symmetric* if its Ricci tensor S is not identically zero and satisfies

$$(\nabla_X S)(Y,Z) = 2\alpha(X)S(Y,Z) + \alpha(Y)S(X,Z) + \alpha(Z)S(Y,X), \tag{1.4}$$

where α is a 1-form which is non-zero for every $X, Y, Z \in \chi(M)$ and ∇ being operator of covariant differentiation with respect to the metric g [2]. In [3] the authors considered conformally flat pseudo Ricci-symmetric manifolds, see also [5] for the case M is a contact manifold.

In the present study we consider pseudo Ricci-symmetric manifolds. We show that pseudo Ricci-symmetric manifolds satisfying $divR = 0$ (resp.$divC = 0$) property are Einstein (resp.Ricci flat) manifolds.

2 Pseudo Ricci-Symmetric manifolds

Let (M, g), $(n \geq 3)$, be an n-dimensional Riemannian manifold and e_i, e_j $(1 \leq i, j \leq n)$ orthonormal vector fields tangent to M and K_{ij} is the sectional curvature of a plane spanned by the vectors e_i and e_j. Then by definition of S we have

$$S(e_i, e_i) = \sum_{k=1}^{n} g(\mathcal{R}(e_k, e_i)e_i, e_k) = \sum_{k=1}^{n} K_{ik}, \tag{2.1}$$

$$S(e_j, e_j) = \sum_{k=1}^{n} K_{jk},\ S(e_i, e_j) = 0. \tag{2.2}$$

First we prove the following result.

Proposition. *Let M^n be a n-dimensional Riemannian manifold. If M is pseudo Ricci-symmetric then*

$$\sum_{k=1}^{n}(K_{ik} - K_{jk})g(e_j, \nabla_{e_i}e_i) = \alpha(e_j)\sum_{k=1}^{n} K_{ik}, \tag{2.3}$$

$$e_i\left[\sum_{k=1}^{n} K_{ik}\right] = 4\alpha(e_i)\sum_{k=1}^{n} K_{ik}, \tag{2.4}$$

$$e_i\left[\sum_{k=1}^{n} K_{jk}\right] = 2\alpha(e_i)\sum_{k=1}^{n} K_{jk}. \tag{2.5}$$

Proof. Let e_i, e_j be orthonormal vector fields tangent to M. Combining (2.1)-(2.2) and (1.4) we find

$$(\nabla_{e_i}S)(e_i, e_j) = \alpha(e_j)S(e_i, e_i), \tag{2.6}$$

$$(\nabla_{e_i}S)(e_i, e_i) = 4\alpha(e_i)S(e_i, e_i), \tag{2.7}$$

$$(\nabla_{e_i}S)(e_j, e_j) = 2\alpha(e_i)S(e_j, e_j). \tag{2.8}$$

Moreover, from the covariant differentiation of S we have

$$(\nabla_{e_i}S)(e_i, e_j) = -S(\nabla_{e_i}e_i, e_j) - S(e_i, \nabla_{e_i}e_j), \tag{2.9}$$

$$(\nabla_{e_i}S)(e_i, e_i) = \nabla_{e_i}S(e_i, e_i), \tag{2.10}$$

$$(\nabla_{e_i}S)(e_j, e_j) = \nabla_{e_i}S(e_j, e_j) - 2S(\nabla_{e_i}e_j, e_j). \tag{2.11}$$

By the use of (2.1)-(2.2) we get

$$S(\nabla_{e_i}e_i, e_j) = \sum_{k=1}^{n} g(\mathcal{R}(e_k, \nabla_{e_i}e_i)e_j, e_k) = \sum_{k=1}^{n} K_{jk}g(e_j, \nabla_{e_i}e_i), \tag{2.12}$$

$$S(e_i, \nabla_{e_i}e_j) = \sum_{k=1}^{n} K_{jk}g(e_j, \nabla_{e_i}e_i),\ S(\nabla_{e_i}e_i, e_i) = 0. \tag{2.13}$$

Combining (2.13), (2.13) and (2.9) we obtain

$$(\nabla_{e_i} S)(e_i, e_j) = \left(\sum_{k=1}^{n} (K_{ik} - K_{jk}) \right) g(e_j, \nabla_{e_i} e_i). \tag{2.14}$$

Forthermore differentiating (2.1) and (2.2) covariantly we have

$$(\nabla_{e_i} S)(e_i, e_i) = e_i \left[\sum_{k=1}^{n} K_{ik} \right], \tag{2.15}$$

$$(\nabla_{e_i} S)(e_j, e_j) = e_i \left[\sum_{k=1}^{n} K_{jk} \right] \left(\text{or } (\nabla_{e_j} S)(e_i, e_i) = e_j \left[\sum_{k=1}^{n} K_{ik} \right] \right). \tag{2.16}$$

Since the left hand sides of the equations (2.6)-(2.8) are equal to the left hand sides of (2.14)-(2.16) we get the result. □

Theorem 2.2. *Let M be a n-dimensional pseudo Ricci-symmetric manifold. If M is R -harmonic (i.e $divR = 0$) then it is Ricci-flat.*

Proof. Let M is R-harmonic so $divR = 0$. Using (1.2) we have

$$(\nabla_{e_i} S)(e_i, e_j) - (\nabla_{e_j} S)(e_i, e_i) = 0. \tag{2.17}$$

Making use of (2.7), (2.8) and (2.1) the equation (2.17) reduces to $\alpha(e_j)S(e_i, e_i) = 0$. Since α is a non-zero one form then $S(e_i, e_i) = 0$. Thus M is Ricci flat this complates the proof. □

Theorem. *Let M be a n-dimensional pseudo Ricci-symmetric manifold. If $divC = 0$ then M^n is an Einstein manifold.*

Proof. Suppose $divC = 0$. Then by (1.3) we have

$$(\nabla_{e_i} S)(e_i, e_j) - (\nabla_{e_j} S)(e_i, e_i) + \frac{1}{2} \frac{n-2}{(n-1)(n-3)} e_j[\tau] = 0. \tag{2.18}$$

where $\nabla_{e_j} \tau = e_j[\tau]$. Substituting (2.14) and (2.16) into (2.18) we obtain

$$\sum_{k=1}^{n} (K_{ik} - K_{jk}) g(e_j, \nabla_{e_i} e_i) - e_j [\sum_{k=1}^{n} K_{ik}] + \frac{1}{2} \frac{n-2}{(n-1)(n-3)} e_j[\tau] = 0. \tag{2.19}$$

Moreover, substituting (2.3) and (2.5) into (2.19) we get

$$-\alpha(e_j) \sum_{k=1}^{n} K_{ik} + \frac{1}{2} \frac{n-2}{(n-1)(n-3)} e_j[\tau] = 0.$$

By the use of (2.1) the above equation becomes

$$\alpha(e_j) S(e_i, e_i) = \frac{1}{2} \frac{n-2}{(n-1)(n-3)} e_j[\tau]. \tag{2.20}$$

On the other hand

$$\sum_{i=1}^{n} \alpha(e_j) S(e_i, e_i) = \frac{1}{2} \frac{n-2}{(n-1)(n-3)} e_j[\tau] \sum_{i=1}^{n} g(e_i, e_i), \tag{2.21}$$

which implies

$$e_j[\tau] = \frac{2(n-1)(n-3)}{n(n-2)}\alpha(e_j)\tau. \tag{2.22}$$

However combining (2.22) and (2.21) one can get $S(e_i, e_i) = \frac{1}{n}\tau$. Thus M is an Einstein manifold. □

Theorem 2.4. *Let M be a n-dimensional pseudo Ricci-symmetric manifold whose one of family of curvature lines consists of geodesic (i.e. $\nabla_{e_i} e_i = 0$). Then M^n is Ricci flat.*

Proof. Putting $\nabla_{e_i} e_i = 0$ into (2.3) we get $\alpha(e_i)\sum_{k=1}^{n} K_{ik} = 0$. Since the one form α is non-zero then by (2.1) one can get $S(e_i, e_i) = 0$, which means that M is Ricci flat. □

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