

# Integral inequalities for maximal space-like submanifolds in the indefinite space form

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## Abstract

In this note, we give two intrinsic integral inequalities for compact maximal space-like submanifolds in the indefinite space form and a sufficient and necessary condition for such submanifolds to be totally geodesic.

**Mathematics Subject Classification:** 53C40, 53C42, 53C50.

**Key words:** maximal space-like submanifold, indefinite space form, flat normal bundle.

## 1 Introduction

Let  $M_p^{n+p}(c)$  be an  $(n+p)$ -dimensional connected semi-Riemannian manifold of constant curvature  $c$  whose index is  $p$ . It is called an indefinite space form of index  $p$  and simply a space form when  $p = 0$ . If  $c > 0$ , we call it as a de Sitter space of index  $p$ . Let  $M^n$  be an  $n$ -dimensional Riemannian manifold immersed in  $M_p^{n+p}(c)$ . As the semi-Riemannian metric of  $M_p^{n+p}(c)$  induces the Riemannian metric of  $M^n$ ,  $M^n$  is called a space-like submanifold. A space-like submanifold with vanishing mean curvature is called a maximal space-like submanifold. Kobayashi [5] gave the Weierstrass-Enneper representation formulas for maximal space-like surfaces in 3-dimensional Minkowski space and exhibited various examples. In particular, he determined the maximal space-like surfaces which are rotation surfaces or ruled surfaces. Montiel [6] give an integral inequality for compact space-like hypersurfaces in the de Sitter space and by use of this integral inequality, he studied the constant mean curvature space-like hypersurfaces. Also, Akutagawa [1] and Ramanathan [8] investigated space-like hypersurfaces in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature  $H$  satisfies  $H^2 \leq c$  when  $n = 2$  and  $n^2 H^2 < 4(n-1)c$  when  $n \geq 3$ . Later, Cheng [3] generalized this result to general submanifolds in a de Sitter space.

In this paper, we study compact maximal space-like submanifolds in the indefinite space form with flat normal bundle and obtain two intrinsic integral inequalities for such submanifolds. We also give a sufficient and necessary condition for such submanifolds to be totally geodesic. We will prove the following

**Theorem 1.** Let  $M^n$  be an  $n$ -dimensional compact maximal space-like submanifold in  $M_p^{n+p}(c)$  with flat normal bundle, then

$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mj}^2 - ncR \right\} * 1 \leq 0.$$

**Theorem 2.** Let  $M^n$  be an  $n$ -dimensional compact maximal space-like submanifold in  $M_p^{n+p}(c)$  with flat normal bundle, then

$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \frac{1}{n} S^2 + (n-2)cS - n(n-1)c^2 \right\} * 1 \leq 0.$$

**Theorem 3.** Let  $M^n$  be an  $n$ -dimensional compact maximal space-like submanifold in  $M_p^{n+p}(c)$  with flat normal bundle, then  $M^n$  is totally geodesic if and only if

$$\int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + (n-2)cS - n(n-1)c^2 \right\} * 1 = 0.$$

In the above theorems,  $\sum R_{mijk}^2$  is the square length of the Riemannian curvature tensor,  $\sum R_{mj}^2$  the square length of the Ricci curvature tensor,  $S$  the square length of the second fundamental form,  $R$  the scalar curvature. All these are intrinsic properties of  $M^n$ .

## 2 Preliminaries

Let  $M_p^{n+p}(c)$  be an  $(n+p)$ -dimensional semi-Riemannian manifold of constant curvature  $c$  whose index is  $p$ . Let  $M^n$  be an  $n$ -dimensional Riemannian manifold immersed in  $M_p^{n+p}(c)$ . We choose a local field of semi-Riemannian orthonormal frames  $e_1, \dots, e_{n+p}$  in  $M_p^{n+p}(c)$  such that at each point of  $M^n$ ,  $e_1, \dots, e_n$  span the tangent space of  $M^n$  and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+p; \quad 1 \leq i, j, k, \dots \leq n; \quad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

Let  $\omega_1, \dots, \omega_{n+p}$  be its dual frame field so that the semi-Riemannian metric of  $M_p^{n+p}(c)$  is given by  $d\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \epsilon_A \omega_A^2$ , where  $\epsilon_i = 1$  and  $\epsilon_\alpha = -1$ .

Then the structure equations of  $M_p^{n+p}(c)$  are given by

$$(1) \quad d\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$(2) \quad d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D,$$

$$(3) \quad K_{ABCD} = c \epsilon_A \epsilon_B (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}).$$

By restricting these forms to  $M^n$ , we have

$$(4) \quad \omega_\alpha = 0, \quad n+1 \leq \alpha \leq n+p,$$

the Riemannian metric of  $M^n$  is written as  $ds^2 = \sum_i \omega_i^2$ . From Cartan's lemma we can write

$$(5) \quad \omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

From these formulas, we obtain the structure equations of  $M^n$ :

$$(6) \quad d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$(7) \quad d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} K_{ijkl} \omega_k \wedge \omega_l,$$

$$(8) \quad R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

where  $R_{ijkl}$  are the components of the curvature tensor of  $M^n$ .

For indefinite Riemannian manifolds in detail, refer to O'Neill [7].

We call

$$(9) \quad h = \sum_\alpha h_\alpha e_\alpha = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$$

the second fundamental form of  $M^n$  and the square length of the second fundamental form is defined by

$$(10) \quad S = \sum_\alpha \text{tr}(h_\alpha)^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2.$$

The mean curvature vector  $N$  of  $M^n$  is defined by

$$(11) \quad N = \frac{1}{n} \sum_\alpha \text{tr}(h_\alpha) e_\alpha = \frac{1}{n} \sum_\alpha \left( \sum_i h_{ii}^\alpha \right) e_\alpha,$$

and it is well known that  $N$  is independent of the choice of unit normal vectors  $e_{n+1}, \dots, e_{n+p}$  to  $M^n$ . The length of the mean curvature vector is called the mean curvature of  $M^n$ , denoted by  $H$ .

If  $M^n$  is maximal, then

$$(12) \quad \sum_i h_{ii}^\alpha = 0, \quad \alpha = n+1, \dots, n+p.$$

Define the first and the second covariant derivatives of  $\{h_{ij}^\alpha\}$ , say  $\{h_{ijk}^\alpha\}$  and  $\{h_{ijkl}^\alpha\}$  by

$$(13) \quad \sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_k h_{kj}^\alpha \omega_{ki} + \sum_k h_{ik}^\alpha \omega_{kj} + \sum_\beta h_{ij}^\beta \omega_{\beta\alpha},$$

$$(14) \quad \begin{aligned} \sum_l h_{ijkl}^\alpha \omega_l &= dh_{ijk}^\alpha + \sum_m h_{mjk}^\alpha \omega_{mi} + \sum_m h_{imk}^\alpha \omega_{mj} \\ &+ \sum_m h_{ijm}^\alpha \omega_{mk} + \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha}. \end{aligned}$$

Then we have

$$(15) \quad h_{ijk}^\alpha = h_{ikj}^\alpha,$$

$$(16) \quad h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{jm}^\alpha R_{mikl} + \sum_\beta h_{ij}^\beta R_{\alpha\beta kl},$$

where  $R_{\alpha\beta kl}$  are the components of the normal curvature tensor of  $M^n$ , that is

$$(17) \quad R_{\alpha\beta kl} = \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta).$$

If  $R_{\alpha\beta kl} = 0$  at point  $x$  of  $M^n$  we say that the normal connection of  $M^n$  is flat at  $x$  and it is well known [2] that  $R_{\alpha\beta kl} = 0$  at  $x$  if and only if  $h_\alpha$  are simultaneously diagonalizable at  $x$ .

The Laplacian  $\Delta h_{ij}^\alpha$  of the fundamental form  $h_{ij}^\alpha$  is defined to be  $\sum_k h_{ijkk}^\alpha$ , and hence, if  $M^n$  has flat normal bundle, from (15) and (16) we have

$$(18) \quad \begin{aligned} \Delta h_{ij}^\alpha &= \sum_k (h_{ijkk}^\alpha - h_{ikjk}^\alpha) + \sum_k (h_{ikjk}^\alpha - h_{ikkj}^\alpha) + \sum_k (h_{ikkj}^\alpha - h_{kkij}^\alpha) \\ &= \sum_{m,k} h_{im}^\alpha R_{mkjk} + \sum_{m,k} h_{mk}^\alpha R_{mijk} \end{aligned}$$

### 3 Proofs of the Theorems

**Proof of Theorem 1.** From (8), (12) and (18), we have

$$(19) \quad \begin{aligned} \sum h_{ij}^\alpha \Delta h_{ij}^\alpha &= \sum h_{ij}^\alpha h_{mk}^\alpha R_{mijk} + \sum h_{ij}^\alpha h_{im}^\alpha R_{mkjk} \\ &= \frac{1}{2} \sum (h_{ij}^\alpha h_{mk}^\alpha - h_{mj}^\alpha h_{ik}^\alpha) R_{mijk} + \sum (h_{ij}^\alpha h_{im}^\alpha - h_{ii}^\alpha h_{jm}^\alpha) R_{mj} \\ &= \frac{1}{2} \sum [c(\delta_{ij}\delta_{mk} - \delta_{mj}\delta_{ik}) - R_{imjk}] R_{mijk} \\ &\quad + \sum [c(\delta_{ij}\delta_{im} - \delta_{ii}\delta_{jm}) + R_{ijim}] R_{mj} \\ &= \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mj}^2 - ncR. \end{aligned}$$

Since  $\int_{M^n} \{ \sum h_{ij}^\alpha \Delta h_{ij}^\alpha \} * 1 \leq 0$ , we have

$$(20) \quad \int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \sum R_{mj}^2 - ncR \right\} * 1 \leq 0.$$

Theorem 1 is proved.

In order to prove Theorem 2, we need the following algebraic lemma

**Lemma.** Let  $a_1, \dots, a_n$  be real numbers, then

$$(21) \quad \sum (a_i)^2 \geq \frac{1}{n} (\sum a_i)^2,$$

and the equality holds if and only if  $a_1 = \dots = a_n$ .

In fact,

$$(22) \quad n \sum (a_i)^2 - \left( \sum a_i \right)^2 = \sum (a_i - a_j)^2,$$

then Lemma follows immediately from (22).

**Proof of Theorem 2.** From (8), we have

$$(23) \quad R_{mj} = (n-1)c\delta_{mj} + \sum h_{mi}^\alpha h_{ij}^\alpha,$$

$$(24) \quad R = n(n-1)c + S.$$

Since  $M^n$  has flat normal bundle, so we can diagonalize the second fundamental form simultaneously so that  $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$ , then from (21), we have

$$(25) \quad R_{mj} = (n-1)c\delta_{mj} + \sum (\lambda_j^\alpha)^2 \delta_{mj},$$

$$(26) \quad \begin{aligned} \sum R_{mj}^2 &= n(n-1)^2 c^2 + 2(n-1)cS + \sum (\lambda_j^\alpha)^4 \\ &\geq n(n-1)^2 c^2 + 2(n-1)cS + \frac{1}{n} \left\{ \sum (\lambda_j^\alpha)^2 \right\}^2 \\ &= n(n-1)^2 c^2 + 2(n-1)cS + \frac{1}{n} S^2, \end{aligned}$$

therefore from (20), we have

$$(27) \quad \int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + \frac{1}{n} S^2 + (n-2)cS - n(n-1)c^2 \right\} * 1 \leq 0.$$

Theorem 2 is proved.

**Proof of Theorem 3.** From (27), we have

$$(28) \quad \int_{M^n} \left\{ \frac{1}{2} \sum R_{mijk}^2 + (n-2)cS - n(n-1)c^2 \right\} * 1 \leq 0.$$

If  $M^n$  is totally geodesic, i.e.,  $S = 0$ ,  $h_{ij}^\alpha = 0$ , then from (8), we have

$$(29) \quad R_{mijk} = c(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}), \quad \sum R_{mijk}^2 = 2n(n-1)c^2,$$

in this case, (28) becomes an equality.

Inversely, if (28) becomes an equality, then  $S = 0$ ,  $M^n$  is totally geodesic.

**Acknowledgements.** This note was written during the author's stay at the Max-Planck-Institut für Mathematik in Bonn. The author would like to express his thanks to the institute for its hospitality and support.

This work is partially supported by the National Natural Science Foundation of China.

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