

# On the Randers Spaces of Second Order

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## Abstract

The geometry of Randers spaces of order one has been investigated since fifties by G. Randers [9], using Riemannian techniques, only. A Finsler approach of this Randers space was given by R. Ingarden [4]. A fully Finslerian treatment was developed by R. Miron in [6,7]. In this paper we define and study the geometrical theory of a Randers space of order two, using the theory of the bundle of accelerations of order two,  $T^2M$ .

**Mathematics Subject Classification:** 53C60

**Key words:** Randers spaces, nonlinear connection,  $N$ -connection

## 1 Introduction

The first examples of regular Lagrangians of order greater than one were given by R. Miron. These are the prolongations of order  $k$ , ( $k > 1$ ) of a Riemannian, or a Finslerian metric. Using the prolongation of order two of a Riemannian metric  $\gamma_{ij}$  and a globally defined covector field  $A_i$  on the base manifold, we define the fundamental function  $F$  of a Randers space  $RF^{(2)} = (M, \alpha + \beta)$ . Then we determine the fundamental tensor  $g_{ij}$  of  $RF^{(2)}$  and the relation between  $d$ -tensors  $\gamma_{ij}$  and  $g_{ij}$ , that is similar with the  $k=1$  case. As this Randers space is a Finsler space of order two, we use techniques of Finslerian geometries of higher order to determine the canonical nonlinear connection, the canonical metrical  $N$ -connection. We may notice here that all the geometrical objects depend on the electromagnetic tensor  $F_{ij}$  of the space  $RF^{(2)}$ .

## 2 The notion of Randers space of order two

Let  $M$  be an  $n$ -dimensional, smooth manifold and  $\mathcal{R}^n = (M, \gamma_{ij}(x))$  be a Riemannian space. We denote by  $ProI^2\mathcal{R}^n = (T^2\widetilde{M}, G)$  the prolongation of order two of the space  $\mathcal{R}^n$ , [8]. The nonlinear connection  $\overset{\circ}{N}$  of the space  $ProI^2\mathcal{R}^n$ , [8], has the dual coefficients

$$(1.1) \quad \begin{cases} \overset{\circ}{M}_{(1)j}^i = \gamma_{jh}^i(x) y^{(1)h}, \\ \overset{\circ}{M}_{(2)j}^i = \frac{1}{2} (\Gamma_{(1)} \overset{\circ}{M}_{(1)j}^i + \overset{\circ}{M}_{(1)m}^i \overset{\circ}{M}_{(1)j}^m), \end{cases}$$

where  $\Gamma$  is the operator

$$(1.2) \quad \Gamma = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}}$$

and  $\gamma_{jk}^i(x)$  are the Christoffel symbols of the Riemannian space  $\mathcal{R}^n$ . Between the coefficients of the nonlinear connection  $\overset{\circ}{N}_{(1)j}^i, \overset{\circ}{N}_{(2)j}^i$  and the dual coefficients  $\overset{\circ}{M}_{(1)j}^i, \overset{\circ}{M}_{(2)j}^i$  we have the relations:

$$(1.3) \quad \begin{cases} \overset{\circ}{N}_{(1)j}^i = \overset{\circ}{M}_{(1)j}^i, \\ \overset{\circ}{N}_{(2)j}^i = \overset{\circ}{M}_{(2)j}^i - \overset{\circ}{M}_{(1)m}^i \overset{\circ}{M}_{(1)j}^m. \end{cases}$$

Having the coefficients of the nonlinear connection  $\overset{\circ}{N}$  we can write its adapted basis,  $\left\{ \frac{\overset{\circ}{\delta}}{\delta x^i}, \frac{\overset{\circ}{\delta}}{\delta y^{(1)i}}, \frac{\overset{\circ}{\delta}}{\delta y^{(2)i}} \right\}$ , given by

$$(1.4) \quad \begin{cases} \frac{\overset{\circ}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \overset{\circ}{N}_{(1)i}^j \frac{\partial}{\partial y^{(1)j}} - \overset{\circ}{N}_{(2)i}^j \frac{\partial}{\partial y^{(2)j}}, \\ \frac{\overset{\circ}{\delta}}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \overset{\circ}{N}_{(1)i}^j \frac{\partial}{\partial y^{(2)j}}, \\ \frac{\overset{\circ}{\delta}}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}. \end{cases}$$

The dual basis of the previous adapted basis is given by  $\{dx^i, \overset{\circ}{\delta} y^{(1)i}, \overset{\circ}{\delta} y^{(2)i}\}$ , where:

$$\begin{cases} \overset{\circ}{\delta} y^{(1)i} = dy^{(1)i} + \overset{\circ}{M}_{(1)j}^i dx^j, \\ \overset{\circ}{\delta} y^{(2)i} = dy^{(2)i} + \overset{\circ}{M}_{(1)j}^i dy^{(1)j} + \overset{\circ}{M}_{(2)j}^i dx^j. \end{cases}$$

We consider the Liouville  $d$ -vector fields, [8]

$$(1.5) \quad \begin{aligned} z^{(1)m} &= y^{(1)m}, \\ z^{(2)m} &= y^{(2)m} + \frac{1}{2} \gamma_{ij}^m(x) y^{(1)i} y^{(1)j}. \end{aligned}$$

The following theorem is known

**Theorem 2.1** *The function  $\alpha^2 = \gamma_{ij}(x)z^{(2)i}z^{(2)j}$  is a differential Lagrangian. It has the properties:*

- 1)  $\alpha^2$  is global defined on  $T^2\widetilde{M} = T^2M \setminus \{0\}$ ;
- 2)  $\alpha^2$  is a regular Lagrangian;
- 3)  $\alpha^2$  depends on the metric  $\gamma_{ij}$ , only;
- 4) The Lagrangian  $\alpha^2$  is homogeneous of order 4 on the fibres of  $T^2M$ ;
- 5) The fundamental tensor field of  $\alpha^2$  is given by

$$(1.6) \quad \frac{1}{2} \frac{\partial^2 \alpha^2}{\partial y^{(2)i} \partial y^{(2)j}} = \gamma_{ij}(x).$$

Let us consider the functions

$$(1.7) \quad \beta(x, y^{(1)}, y^{(2)}) = A_i(x)z^{(2)i}$$

where  $A_i(x)$  are the electromagnetic potentials on the base manifold  $M$ . Clearly, the function  $\beta$  has a physical means.

Let us consider the function  $F : T^2M \rightarrow \mathbb{R}$  which is given by  $F = \alpha + \beta$ , i.e.

$$(1.8) \quad F(x, y^{(1)}, y^{(2)}) = \sqrt{\gamma_{ij}(x)z^{(2)i}z^{(2)j}} + A_i(x)z^{(2)i}$$

and the square of this function

$$(1.8)' \quad L(x, y^{(1)}, y^{(2)}) = (\alpha + \beta)^2.$$

We can formulate

**Theorem 2.2** *a) The function  $L = (\alpha + \beta)^2$  is a differentiable Lagrangian of order 2;*

- b)  $F$  is 2-homogeneous and  $L$  is 4-homogeneous on the fibers of  $T^2M$ ;
- c) The fundamental tensor field of the Lagrangian  $L = (\alpha + \beta)^2$  is given by

$$(1.9) \quad g_{ij} = (p\gamma_{ij} + l_i l_j) - p \overset{\circ}{l}_i \overset{\circ}{l}_j$$

where  $\overset{\circ}{l}_i = \frac{\partial \alpha}{\partial y^{(2)i}}$ ,  $l_i = \overset{\circ}{l}_i + A_i$ ,  $p = \frac{\alpha + \beta}{\alpha}$ .

**Proof.** a)  $L$  is of  $C^\infty$ -class on  $T^2\widetilde{M}$  and continuous on the null section of the projection  $\pi : T^2M \rightarrow M$  because  $\alpha$  and  $\beta$  have these properties.

b) It is known that  $z^{(2)i}$  is 2-homogeneous of the fibres of  $T^2\widetilde{M}$ .

c) By a straightforward calculus we obtain the formula (1.9). □

**Theorem 2.3** *The pair  $F^{(2)n} = (M, F = \alpha + \beta)$  is a Finsler space of order 2.*

**Proof.** We must show the following properties:

1.  $F$  is of  $C^\infty$ -class on  $\widetilde{T^2M}$  and continuous on the null section;
2.  $F$  is positive on an open set, where  $\beta \geq 0$ ;
3.  $F$  is 2-homogeneous on the fibres of  $T^2M$ ;
4. The Hessian of  $F^2$  with the elements:

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(2)i} \partial y^{(2)j}}$$

is positively defined.

Since the properties 1,2,3 holds in virtue of the properties of  $\alpha$  and  $\beta$  we have to prove the property 4. In order to do it, we calculate the contravariant  $g^{ij}$  of the fundamental tensor field of the Lagrangian  $L$ .

The contravariant  $g^{ij}$  of  $g_{ij}$  is given by:

$$(1.10) \quad g^{ij} = \frac{1}{p} \gamma^{ij} - \frac{1}{p^2} [l^i \overset{\circ}{l}^j (1 - \tilde{l}^2) + \overset{\circ}{l}^i A^j + \overset{\circ}{l}^j A^i],$$

where  $\tilde{l}^2 = \frac{1}{p} \gamma^{ij} l_i l_j$ .

Moreover,  $\det \|g_{ij}\| = p^{n+1} \det \|\gamma_{ij}\|$  where  $p = \frac{\alpha + \beta}{\beta} > 0$ . □

The Finsler space  $RF^{(2)n} = (M, F)$  is called *the Randers space of order two*.

**Theorem 2.4** *The nonlinear connection  $\overset{\circ}{N}$  of the space  $ProI^2\mathcal{R}^n$ , from (1.1), is a nonlinear connection for the Randers space of order 2  $RF^{(2)n}$  determined only by the fundamental function  $F = \alpha + \beta$ .*

### 3 Nonlinear connection of the space $RF^{(2)n}$

Let us consider the mixed form  $F_j^i(x) = \gamma^{im}(x) F_{mj}(x)$  of the electromagnetic tensor field  $F_{mj}(x) = \frac{\partial A_j}{\partial x^m} - \frac{\partial A_m}{\partial x^j}$ .

**Theorem 3.1** *The Randers space  $RF^{(2)n} = (M, \alpha + \beta)$  has a nonlinear connection  $N$ , whose dual coefficients are given by*

$$(2.1) \quad \begin{cases} M_j^i = \overset{\circ}{M}_j^i - F_j^i \|y^{(1)}\|, \\ M_j^i = \overset{\circ}{M}_j^i - \frac{1}{2} \Gamma(F_j^i \|y^{(1)}\|) + \frac{1}{2} (M_m^i \overset{\circ}{F}_j^m + M_j^m \overset{\circ}{F}_m^i) \|y^{(1)}\| - \frac{1}{2} \|y^{(1)}\|^2 F_m^i F_j^m. \end{cases}$$

where  $\|y^{(1)}\| = \sqrt{\gamma_{ij}(x) y^{(1)i} y^{(1)j}}$ .

Indeed,  $M_j^i$  has the same rule of transformation with respect to changes of coordinates on  $\widetilde{T^2M}$  like  $\overset{\circ}{M}_j^i$  and  $M_j^i = \frac{1}{2} (\Gamma M_j^i + M_m^i M_j^m)$ .

The nonlinear connection (2.1) will be called *canonical nonlinear connection* of the Randers space  $RF^{(2)n}$ .

The relations between coefficients  $(N_j^i, N_j^i)$  and dual coefficients of the nonlinear connection  $N$  are given by:

$$(2.2) \quad \begin{cases} N_{(1)}^i = M_{(1)}^i, \\ N_{(2)}^i = \frac{1}{2}(\Gamma_{(1)} M_{(1)}^i + M_{(1)}^i M_{(1)}^m) - M_{(1)}^i M_{(1)}^m = \frac{1}{2}(\Gamma_{(1)} N_{(1)}^i - N_{(1)}^i N_{(1)}^m). \end{cases}$$

We use this canonical nonlinear connection in study of the canonical metrical  $N$ -connection.

### 4 The canonical $N$ -linear connection

Let us consider  $d$ -tensor field

$$(3.1) \quad T_i^j = \Gamma(F_i^j || y^{(1)} ||) + F_j^i || y^{(1)} || - (M_m^i F_j^m + M_j^m F_m^i) || y^{(1)} || + || y^{(1)} ||^2 F_m^i F_j^m$$

The adapted basis  $\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}} \right\}$  of the canonical nonlinear connection can be expressed in the following form

$$(3.2) \quad \begin{cases} \frac{\delta}{\delta x^i} = \frac{\overset{\circ}{\delta}}{\delta x^i} + F_i^j || y^{(1)} || \frac{\overset{\circ}{\delta}}{\delta y^{(1)j}} + \frac{1}{2} T_i^j \frac{\partial}{\partial y^{(2)j}}, \\ \frac{\delta}{\delta y^{(1)i}} = \frac{\overset{\circ}{\delta}}{\delta y^{(1)i}} + F_i^j || y^{(1)} || \frac{\partial}{\partial y^{(2)j}} \\ \frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}. \end{cases}$$

Of course, this adapted basis depend only on the fundamental function  $F = \alpha + \beta$  of the Randers spaces of order two,  $RF^{(2)n}$ .

Using a very known method we can determine the canonical  $N$ -linear connection  $D$  on  $T^2M$  metrical with respect to fundamental tensor  $g_{ij}$ , which depend only on the fundamental function  $F$  of the space  $RF^{(2)n}$ .

We have:

**Theorem 4.1** 1) *There exists an unique  $N$ -linear connection  $D$  on  $T^2M$  which verifies the following axioms:*

1° *The nonlinear connection  $N$  is specified by (2.2).*

2°  $g_{ij|k} = 0, \quad g_{(1)ij|k} = 0, \quad g_{(2)ij|k} = 0,$

3°  $T_{jk}^i = S_{jk}^i = S_{jk}^i = 0.$

2) This connection has the coefficients given by the generalized Christoffel symbols:

$$(3.3) \quad \begin{cases} L_{jm}^i = \frac{1}{2}g^{is} \left\{ \frac{\delta g_{sm}}{\delta x^j} + \frac{\delta g_{sj}}{\delta x^m} - \frac{\delta g_{jm}}{\delta x^s} \right\}, \\ C_{(1)jm}^i = \frac{1}{2}g^{is} \left\{ \frac{\delta g_{sm}}{\delta y^{(1)j}} + \frac{\delta g_{sj}}{\delta y^{(1)m}} - \frac{\delta g_{jm}}{\delta y^{(1)s}} \right\}, \\ C_{(2)jm}^i = \frac{1}{2}g^{is} \left\{ \frac{\partial g_{sm}}{\partial y^{(2)j}} + \frac{\partial g_{sj}}{\partial y^{(2)m}} - \frac{\partial g_{jm}}{\partial y^{(2)s}} \right\}. \end{cases}$$

3) This connection depend only on Randers space  $RF^{(2)n}$ .

The metrical connection from the previous Theorem is the canonical connection of the Randers space  $RF^{(2)n} = (M, \alpha + \beta)$ .

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