

An Expanded Class of Parametric Partial Linear Complex Vector Functional Equations

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Abstract

In this paper one expanded class of parametric complex vector partial linear functional equations is solved. The results presented here generalize the results given in [3].

Mathematics Subject Classification 2000: 39B32, 39B52

Key words: expanded parametric complex vector functional equations

0 Introduction

First we introduce the following notations. Let \mathcal{V} , \mathcal{V}' be finite dimensional complex vector spaces and \mathbf{Z}_i , $i \in \mathbf{N}$, be vectors in \mathcal{V} . We may assume that $\mathbf{Z}_i = (z_{i1}(t), \dots, z_{in}(t))^T$, where $z_{ij}(t)$ ($1 \leq j \leq n$) are complex functions and $\mathbf{O} = (0, \dots, 0)^T$ is the zero-vector in \mathcal{V} or \mathcal{V}' . We also denote by \mathcal{V}^0 the subspace of all real vectors in \mathcal{V} (thus $\mathcal{V} = \mathcal{V}^0 \oplus i\mathcal{V}^0$), and by $\mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ the space of linear mappings $\mathcal{V}^0 \rightarrow \mathcal{V}'$.

In the present paper we will solve the following simple complex vector functional equation

$$(1) \quad \sum_{i=1}^{m+n+k} f \left(\sum_{j=0}^{m-1} a^{m-1-j} \mathbf{Z}_{i+j}, \sum_{j=0}^{n-1} a^{n-1-j} \mathbf{Z}_{i+m+j}, \sum_{j=0}^{k-1} a^{k-1-j} \mathbf{Z}_{i+m+n+j} \right) = \mathbf{O},$$

$$(\mathbf{Z}_{m+n+k+i} \equiv \mathbf{Z}_i, \quad a \in \mathbf{C}),$$

where \mathbf{C} is the field of complex numbers and $f : \mathcal{V}^3 \rightarrow \mathcal{V}'$ is an unknown complex vector function.

The above equation for $k = 0$ was solved in [3]. Also, the functional equation (1) for $a = 1$ was solved in [1] under the hypothesis that the function and variables are real.

The generalized functional equation

$$(2) \quad \sum_{i=1}^{m+n+k} f_i \left(\sum_{j=0}^{m-1} a^{m-1-j} \mathbf{Z}_{i+j}, \sum_{j=0}^{n-1} a^{n-1-j} \mathbf{Z}_{i+m+j}, \sum_{j=0}^{k-1} a^{k-1-j} \mathbf{Z}_{i+m+n+j} \right) = \mathbf{O}$$

$$(f_{m+n+k+i} \equiv f_i, \quad \mathbf{Z}_{m+n+k+i} \equiv \mathbf{Z}_i, \quad a \in \mathbf{C})$$

will be also solved here.

1 Solution of the Simple Functional Equation

Now we will solve the functional equation (1) in the following cases.

I. Let $a = 1$. If we introduce the notations

$$(3) \quad \mathbf{S} = \sum_{j=1}^{m+n+k} \mathbf{Z}_j$$

and

$$(4) \quad f(\mathbf{U}, \mathbf{V}, \mathbf{S}) = g(\mathbf{U}, \mathbf{V}, \mathbf{S} + \mathbf{U} + \mathbf{V}),$$

then the equation (1) becomes

$$(5) \quad \sum_{j=1}^{m+n+k} g \left(\sum_{j=0}^{m-1} \mathbf{Z}_{i+j}, \sum_{j=0}^{n-1} \mathbf{Z}_{m+i+j}, \mathbf{S} \right) = \mathbf{O}.$$

If we introduce new variables \mathbf{T}_i ($1 \leq i \leq m+n+k-1$) by the equalities

$$\mathbf{T}_i = \mathbf{Z}_i - \frac{\mathbf{S}}{m+n+k} \quad (1 \leq i \leq m+n+k-1),$$

i.e.

$$\mathbf{Z}_{m+n+k} = \frac{\mathbf{S}}{m+n+k} - \sum_{j=1}^{m+n+k-1} \mathbf{T}_j,$$

then the equation (5) becomes

$$(6) \quad \sum_{i=1}^k g \left(\frac{m\mathbf{S}}{m+n+k} + \sum_{j=0}^{m-1} \mathbf{T}_{i+j}, \frac{n\mathbf{S}}{m+n+k} + \sum_{j=0}^{n-1} \mathbf{T}_{m+i+j}, \mathbf{S} \right) +$$

$$+ \sum_{i=k+1}^{n+k} g \left(\frac{m\mathbf{S}}{m+n+k} + \sum_{j=0}^{m-1} \mathbf{T}_{i+j}, \frac{n\mathbf{S}}{m+n+k} - \sum_{j=0}^{m+k-1} \mathbf{T}_{m+n+i+j}, \mathbf{S} \right) +$$

$$+ \sum_{i=n+k+1}^{m+n+k} g \left(\frac{m\mathbf{S}}{m+n+k} - \sum_{j=0}^{n+k-1} \mathbf{T}_{m+i+j}, \frac{n\mathbf{S}}{m+n+k} + \sum_{j=0}^{n-1} \mathbf{T}_{m+i+j}, \mathbf{S} \right) = \mathbf{O}.$$

By putting

$$(7) \quad g \left(\frac{m\mathbf{S}}{m+n+k} + \mathbf{U}, \frac{n\mathbf{S}}{m+n+k} + \mathbf{V}, \mathbf{S} \right) = H(\mathbf{U}, \mathbf{V}, \mathbf{S}),$$

the equations (6) becomes

$$\begin{aligned}
 & \sum_{i=1}^k H\left(\sum_{j=0}^{m-1} \mathbf{T}_{i+j}, \sum_{j=0}^{n-1} \mathbf{T}_{m+i+j}, \mathbf{S}\right) + \\
 (8) \quad & + \sum_{i=k+1}^{k+n} H\left(\sum_{j=0}^{m-1} \mathbf{T}_{i+j}, -\sum_{j=0}^{m+k-1} \mathbf{T}_{m+n+i+j}, \mathbf{S}\right) + \\
 & + \sum_{i=n+k+1}^{m+n+k} H\left(-\sum_{j=0}^{n+k-1} \mathbf{T}_{m+i+j}, \sum_{j=0}^{n-1} \mathbf{T}_{m+i+j}, \mathbf{S}\right) = \mathbf{O}.
 \end{aligned}$$

Now, we may suppose that the variable \mathbf{S} is fixed, and we may put

$$(9) \quad H(\mathbf{U}, \mathbf{V}, \mathbf{S}) = h(\mathbf{U}, \mathbf{V}).$$

In this case, the functional equation (8) takes the following form

$$\begin{aligned}
 & \sum_{i=1}^k h\left(\sum_{j=0}^{m-1} \mathbf{T}_{i+j}, \sum_{j=0}^{n-1} \mathbf{T}_{m+i+j}\right) + \\
 (10) \quad & + \sum_{i=k+1}^{m+k} h\left(\sum_{j=0}^{m-1} \mathbf{T}_{i+j}, -\sum_{j=0}^{m+k-1} \mathbf{T}_{m+n+i+j}\right) + \\
 & + \sum_{i=n+k+1}^{m+n+k} h\left(-\sum_{j=0}^{n+k-1} \mathbf{T}_{m+i+j}, \sum_{j=0}^{n-1} \mathbf{T}_{m+i+j}\right) = \mathbf{O}.
 \end{aligned}$$

The complex vector function h has the following properties:

1°. If $m < n < k$ and $m + n < k$, then the following relations hold

$$(11) \quad h(\mathbf{O}, \mathbf{O}) = \mathbf{O},$$

$$(12) \quad h(\mathbf{U}, -\mathbf{U}) + h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{U}) = \mathbf{O},$$

$$(13) \quad h(\mathbf{O}, \mathbf{U}) + h(\mathbf{O}, -\mathbf{U}) = \mathbf{O},$$

$$(14) \quad h(\mathbf{U}, \mathbf{V}) = h(\mathbf{U} + \mathbf{V}, \mathbf{O}) - h(\mathbf{V}, \mathbf{O}) + h(\mathbf{O}, \mathbf{V}),$$

$$(15) \quad h(\mathbf{U}, \mathbf{O}) + h(-\mathbf{U}, \mathbf{O}) = \mathbf{O},$$

$$(16) \quad h(\mathbf{U}, \mathbf{V}) = h(\mathbf{O}, \mathbf{U} + \mathbf{V}) - h(\mathbf{O}, \mathbf{U}) + h(\mathbf{U}, \mathbf{O}).$$

2°. If $m < n < k$ and $m + n = k$, then (11), (12), (13), (14), (15) and (16) hold.

3°. If $m < n < k$ and $m + n > k$, then (11), (12), (13), (14), (15) hold and

$$(17) \quad h(\mathbf{U}, \mathbf{V}) + h(\mathbf{V}, -\mathbf{U} - \mathbf{V}) + h(-\mathbf{U} - \mathbf{V}, \mathbf{U}) = \mathbf{O}.$$

4°. If $m < n = k$, then (11), (12), (13), (14) and (17) hold.

5°. If $m = n$ and $2m < k$, then (11), (12) hold and

$$(18) \quad h(\mathbf{U}, \mathbf{V}) + h(\mathbf{V}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{U} - \mathbf{V}) + h(-\mathbf{U} - \mathbf{V}, \mathbf{U}) = \mathbf{O}.$$

6°. If $m = n$ and $2m = k$, then (11), (12) and (18) hold.

7°. If $m = n < k$ and $2m > k$, then (11), (12) and (18) hold.

8°. If $n < m < k$ and $m + n < k$, then (11), (12), (15), (16), (13) and (14) hold.

9°. If $n < m < k$ and $m + n = k$, then (11), (12), (15), (16), (13) and (14) hold.

10°. If $n < m < k$ and $m + n > k$, then (11), (12), (15), (16), (13) and (17) hold.

11°. If $n < m = k$, then (11), (12), (15), (16) and (17) hold.

12°. If $m = n = k$, then (11), (12) and (17) hold.

Now, we will prove the theorems which treat the functional equation (1).

Theorem 1. *If $a = 1$, $m, n < k$, $m \neq n$ and $m + n \neq k$, then the general continuous solution of the functional equation (1) is*

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= G_1(\mathbf{U} + \mathbf{V} + \mathbf{W}) \operatorname{Re} \left(\mathbf{U} - m \frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k} \right) + \\ &+ G_2(\mathbf{U} + \mathbf{V} + \mathbf{W}) \operatorname{Im} \left(\mathbf{U} - m \frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k} \right) + \\ &+ G_3(\mathbf{U} + \mathbf{V} + \mathbf{W}) \operatorname{Re} \left(\mathbf{V} - n \frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k} \right) + \\ &+ G_4(\mathbf{U} + \mathbf{V} + \mathbf{W}) \operatorname{Im} \left(\mathbf{V} - n \frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k} \right), \end{aligned}$$

where $G_i : \mathcal{V} \rightarrow \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ ($1 \leq i \leq 4$) are arbitrary continuous complex vector functions and $\operatorname{Re} \mathbf{U}$ resp. $\operatorname{Im} \mathbf{U}$ denotes the real resp. imaginary part of \mathbf{U} .

Proof. Let $m < n < k$ and $m + n < k$. On the basis of the expressions (14) and (16), we obtain that the complex vector function h satisfies the equation

$$h(\mathbf{U} + \mathbf{V}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U} + \mathbf{V}) = h(\mathbf{U}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U}) + h(\mathbf{V}, \mathbf{O}) - h(\mathbf{O}, \mathbf{V}),$$

and hence we get

$$h(\mathbf{U}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U}) = G' \operatorname{Re} \mathbf{U} + G'' \operatorname{Im} \mathbf{U},$$

where $G', G'' \in \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ are arbitrary continuous functions of \mathbf{S} .

Thus, the equation (16) has the form

$$(19) \quad h(\mathbf{U}, \mathbf{V}) = h(\mathbf{O}, \mathbf{U} + \mathbf{V}) + G' \operatorname{Re} \mathbf{U} + G'' \operatorname{Im} \mathbf{U}.$$

If we substitute the function h determined by (19) into (10), on the basis of the expression (13) we obtain

$$(20) \quad \sum_{i=1}^k h \left(\mathbf{O}, \sum_{j=1}^{m+n-1} \mathbf{T}_{i+j} \right) - \sum_{i=k+1}^{k-1} h \left(\mathbf{O}, \sum_{j=0}^{k-1} \mathbf{T}_{m+n+i+j} \right) = \mathbf{O}.$$

If we put $\mathbf{T}_1 = \mathbf{U}$, $\mathbf{T}_k = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ ($j \neq 1, k$) into (20), then we obtain

$$h(\mathbf{O}, \mathbf{U} + \mathbf{V}) = h(\mathbf{O}, \mathbf{U}) + h(\mathbf{O}, \mathbf{V}),$$

and hence we deduce

$$(21) \quad h(\mathbf{O}, \mathbf{U}) = G_3 Re \mathbf{U} + G_4 Im \mathbf{U},$$

where $G_3, G_4 \in \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ are arbitrary complex vector functions of \mathbf{S} .

On the basis of the expressions (21) and (19), we obtain

$$(22) \quad h(\mathbf{U}, \mathbf{V}) = G_1 Re \mathbf{U} + G_2 Im \mathbf{U} + G_3 Re \mathbf{V} + G_4 Im \mathbf{V},$$

where $G_1 = G' + G_3$ and $G_2 = G'' + G_4$.

Let $m < n < k$ and $m + n > k$. On the basis of the expressions (14) and (17), we get

$$\begin{aligned} & h(\mathbf{U} + \mathbf{V}, \mathbf{O}) - h(-\mathbf{U} - \mathbf{V}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U} + \mathbf{V}) = \\ & = h(\mathbf{U}, \mathbf{O}) - h(-\mathbf{U}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U}) + h(\mathbf{V}, \mathbf{O}) - h(-\mathbf{V}, \mathbf{O}) - h(\mathbf{O}, \mathbf{V}), \end{aligned}$$

and hence we can calculate that

$$(23) \quad h(\mathbf{U}, \mathbf{O}) - h(-\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U}) = G_3 Re \mathbf{U} + G_4 Im \mathbf{U},$$

where $G_3, G_4 \in \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ are arbitrary complex vector functions of \mathbf{S} .

On the basis of the identity (23), the equation (14) becomes

$$(24) \quad h(\mathbf{U}, \mathbf{V}) = h(\mathbf{U} + \mathbf{V}, \mathbf{O}) - h(-\mathbf{U}, \mathbf{O}) + G_3 Re \mathbf{V} + G_4 Im \mathbf{V}.$$

If we substitute the function h determined by (24) into (10), we obtain

$$(25) \quad \begin{aligned} & \sum_{i=1}^k h\left(\sum_{i=1}^{m+n-1} \mathbf{T}_{i+j}, \mathbf{O}\right) + \sum_{i=k+1}^{m+n+k} h\left(-\sum_{j=0}^{k-1} \mathbf{T}_{m+n+i+j}, \mathbf{O}\right) - \\ & - \sum_{i=k+1}^{m+n+k} h\left(\sum_{j=0}^{m+k-1} \mathbf{T}_{m+n+i+j}, \mathbf{O}\right) - \sum_{i=1}^{m+k} h\left(-\sum_{j=0}^{n-1} \mathbf{T}_{i+j}, \mathbf{O}\right) = \mathbf{O}. \end{aligned}$$

For $\mathbf{T}_1 = \mathbf{U}$, $\mathbf{T}_{m+k} = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ ($j \neq 1, m+k$) from (25) we have

$$h(\mathbf{U} + \mathbf{U}, \mathbf{O}) = h(\mathbf{U}, \mathbf{O}) + h(\mathbf{V}, \mathbf{O}),$$

where

$$h(\mathbf{U}, \mathbf{O}) = G_1 Re \mathbf{U} + G_2 Im \mathbf{U}.$$

On the basis of this, the equation (24) becomes (25).

Let $n < m < k$ and $m + n < k$. On the basis of the equations (16) and (14) we obtain

$$h(\mathbf{U} + \mathbf{V}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U} + \mathbf{V}) = h(\mathbf{U}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U}) + h(\mathbf{V}, \mathbf{O}) - h(\mathbf{O}, \mathbf{V}),$$

from where it follows that

$$(26) \quad h(\mathbf{U}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U}) = G'_1 Re \mathbf{U} + G''_2 Im \mathbf{U}.$$

On the basis of the equality (26), the equation (16) becomes

$$(27) \quad h(\mathbf{U}, \mathbf{V}) = h(\mathbf{O}, \mathbf{U} + \mathbf{V}) + G' Re \mathbf{U} + G'' Im \mathbf{U}.$$

If we substitute the function h determined by (27) into (10), we obtain the following equation

$$(28) \quad \sum_{i=1}^k h\left(\mathbf{O}, \sum_{j=0}^{m+n+1} \mathbf{T}_{i+j}\right) = \sum_{i=k+1}^{m+n+k} h\left(\mathbf{O}, \sum_{j=0}^{k-1} \mathbf{T}_{m+n+i+j}\right).$$

By putting $\mathbf{T}_1 = \mathbf{U}$, $\mathbf{T}_{m+n} = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ ($j \neq 1, m+n$) in (28) we have

$$h(\mathbf{O}, \mathbf{U}, \mathbf{V}) = h(\mathbf{O}, \mathbf{U}) + h(\mathbf{O}, \mathbf{V}),$$

from where we conclude that

$$h(\mathbf{O}, \mathbf{U}) = G_3 Re \mathbf{U} + G_4 Im \mathbf{U}.$$

On the basis of this, the equation (27) becomes (22).

Let $n < m < k$ and $m+n > k$. On the basis of (16) and (17), we obtain that the function h satisfies the functional equation

$$\begin{aligned} h(\mathbf{U} + \mathbf{V}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{U} - \mathbf{V}) - h(\mathbf{O}, \mathbf{U} + \mathbf{V}) = \\ = h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{U}) - h(\mathbf{O}, \mathbf{U}) + h(\mathbf{V}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{V}) - h(\mathbf{O}, \mathbf{V}). \end{aligned}$$

Therefore, the function $h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{U}) - h(\mathbf{O}, \mathbf{U})$ is determined by

$$(29) \quad h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{U}) - h(\mathbf{O}, \mathbf{V}) = G_1 Re \mathbf{U} + G_2 Im \mathbf{U}.$$

The equation (16), on the basis of the equality (29), becomes

$$(30) \quad h(\mathbf{U}, \mathbf{V}) = h(\mathbf{O}, \mathbf{U} + \mathbf{V}) - h(\mathbf{O}, -\mathbf{U}) + G' Re \mathbf{U} + G'' Im \mathbf{U}.$$

The function h needs to satisfy the equation (10). If we substitute (30) into (10), we get

$$(31) \quad \begin{aligned} \sum_{i=1}^k h\left(\mathbf{O}, \sum_{j=0}^{m+n-1} \mathbf{T}_{i+j}\right) + \sum_{i=k+1}^{m+n+k} h\left(\mathbf{O}, -\sum_{j=0}^{k-1} \mathbf{T}_{m+n+i+j}\right) - \\ - \sum_{i=1}^{k+1} h\left(\mathbf{O}, -\sum_{j=0}^{m-1} \mathbf{T}_{i+j}\right) - \sum_{i=n+k+1}^{m+n+k} h\left(\mathbf{O}, \sum_{j=0}^{m-1} \mathbf{T}_{m+n+i+j}\right) = \mathbf{O}. \end{aligned}$$

If we put $\mathbf{T}_1 = \mathbf{U}$, $\mathbf{T}_{n+k} = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ ($j \neq 1, n+k$) into (31), we obtain

$$h(\mathbf{O}, \mathbf{U} + \mathbf{V}) = h(\mathbf{O}, \mathbf{U}) + h(\mathbf{O}, \mathbf{V}),$$

from where it follows that

$$h(\mathbf{O}, \mathbf{U}) = G_3 Re \mathbf{U} + G_4 Im \mathbf{U}.$$

On the basis of the last equality, the equality (30) has just the form (22). Consequently, in all four cases the function h is determined by (22).

We denote the arbitrary complex functions G_i ($1 \leq i \leq n$) of \mathbf{S} by $G_i(\mathbf{S})$ ($1 \leq i \leq 4$). On the basis of that, the equality (22) and the transformations (9), (7), (4) and (3) there follows the proof of the theorem. \square

Theorem 2. *If $a = 1$, $m \neq n$ and $m + n = k$, then the general continuous solution of the functional equation (1) is determined by*

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F\left(\mathbf{U} + \mathbf{V} - k \frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m + n + k}, \mathbf{U} + \mathbf{V} + \mathbf{W}\right) - \\ &- F\left(-\mathbf{U} - \mathbf{V} + k \frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m + n + k}, \mathbf{U} + \mathbf{V} + \mathbf{W}\right) + \\ &+ G_1(\mathbf{U} + \mathbf{V} + \mathbf{W}) \operatorname{Re} \left(\mathbf{U} - m \frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m + n + k} \right) + \\ &+ G_2(\mathbf{U} + \mathbf{V} + \mathbf{W}) \operatorname{Im} \left(\mathbf{U} - m \frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m + n + k} \right), \end{aligned}$$

where $F : \mathcal{V}^2 \rightarrow \mathcal{V}'$ and $G_i : \mathcal{V} \rightarrow \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ ($i = 1, 2$) are arbitrary continuous complex vector functions.

Proof. Let $m < n$. As in the proof of the previous theorem, on the basis of the expressions (14) and (16), we may show that the function h has the form determined by (19). If we substitute the function h determined as previously into (1), on the basis of the expression (13), we conclude that the equality is valid and consequently, the function $h(\mathbf{O}, \mathbf{U})$ may be arbitrary. According to (13), for $h(\mathbf{O}, \mathbf{U})$ we can put

$$h(\mathbf{O}, \mathbf{U}) = F(\mathbf{U}) - F(-\mathbf{U}),$$

where F is an arbitrary continuous function.

On the basis of this, the equality (19) becomes

$$(32) \quad h(\mathbf{U}, \mathbf{V}) = F(\mathbf{U} + \mathbf{V}) - F(-\mathbf{U} - \mathbf{V}) + G_1 \operatorname{Re} \mathbf{U} + G_2 \operatorname{Im} \mathbf{U}.$$

Let $n < m$. As in this case the equalities (14) and (16) hold and the function h has the form determined by (19). If we substitute (19) into (10), we obtain that the equation (10) holds, which means that $h(\mathbf{O}, \mathbf{U})$ is an arbitrary function for which (13) holds.

On the basis of this, we obtain that the function h in this case has the form determined by (32).

If we put that the arbitrary continuous complex vector functions G_1 and G_2 are $G_1(\mathbf{S})$ and $G_2(\mathbf{S})$ respectively, on the basis of the expressions (9), (7), (4) and (3), there follows the proof of the theorem. \square

Theorem 3. *If $a = 1$ and $m < n = k$, the functional equation (1) has a general continuous solution determined by*

$$\begin{aligned}
f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F\left(\mathbf{U} + \mathbf{V} - (m+n)\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k}, \mathbf{U} + \mathbf{V} + \mathbf{W}\right) - \\
&- F\left(-\mathbf{V} + n\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k}, \mathbf{U} + \mathbf{V} + \mathbf{W}\right) + \\
&+ G_1(\mathbf{U} + \mathbf{V} + \mathbf{W})\operatorname{Re}\left(\mathbf{V} - n\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k}\right) + \\
&+ G_2(\mathbf{U} + \mathbf{V} + \mathbf{W})\operatorname{Im}\left(\mathbf{V} - n\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k}\right),
\end{aligned}$$

where $F : \mathcal{V}^2 \rightarrow \mathcal{V}'$ and $G_i : \mathcal{V} \rightarrow \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ ($i = 1, 2$) are arbitrary continuous complex vector functions.

Proof. On the basis of the equalities (14) and (17), as in the proof of Theorem 1, we conclude that (23) holds, i.e. (24) holds. If we substitute (24) into (10), since $n = k$, we obtain an identity which means that for $h(\mathbf{U}, \mathbf{O})$ we may take an arbitrary continuous function.

On the basis of this, for the function $f(\mathbf{U}, \mathbf{V})$ we obtain

$$(33) \quad h(\mathbf{U}, \mathbf{V}) = F(\mathbf{U} + \mathbf{V}) - F(-\mathbf{V}) + G_1 \operatorname{Re} \mathbf{V} + G_2 \operatorname{Im} \mathbf{V}.$$

From (33), on the basis of the transformations (9), (7), (4) and (3) and putting that G_1 and G_2 are arbitrary continuous functions of \mathbf{S} , we obtain that the function f has the form given in the Theorem 3. \square

Theorem 4. If $a = 1$ and $n < m = k$, then the general continuous solution of the functional equation (1) is

$$\begin{aligned}
f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F\left(\mathbf{U} + \mathbf{V} - (m+n)\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k}, \mathbf{U} + \mathbf{V} + \mathbf{W}\right) - \\
&- F\left(-\mathbf{U} + m\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k}, \mathbf{U} + \mathbf{V} + \mathbf{W}\right) + \\
&+ G_1(\mathbf{U} + \mathbf{V} + \mathbf{W})\operatorname{Re}\left(\mathbf{U} - m\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k}\right) + \\
&+ G_2(\mathbf{U} + \mathbf{V} + \mathbf{W})\operatorname{Im}\left(\mathbf{U} - m\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m+n+k}\right),
\end{aligned}$$

where $F : \mathcal{V}^2 \rightarrow \mathcal{V}'$ and $G_i : \mathcal{V} \rightarrow \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ ($i = 1, 2$) are arbitrary continuous complex vector functions.

Proof. On the basis of (16), (17) and (29), we obtain that the function h satisfies the equation (30).

If the function h determined by (30) is substituted into (10), then the equation (10) becomes an identity, which means that $h(\mathbf{O}, \mathbf{U})$ can be substituted by an arbitrary continuous function $F(\mathbf{U})$.

On the basis of this, the equality (30) has the form

$$(34) \quad h(\mathbf{U}, \mathbf{V}) = F(\mathbf{U} + \mathbf{V}) - F(-\mathbf{U}) + G_1 \operatorname{Re} \mathbf{U} + G_2 \operatorname{Im} \mathbf{U}.$$

According to the transformations (9), (7), (4) and (3), from the equation (34), by putting $G_1 = G_1(\mathbf{S})$ and $G_2 = G_2(\mathbf{S})$, it follows that the function f has the form which is given in this theorem. \square

Theorem 5. *If $a = 1$, $m = n < k$ and $2m \neq k$, then the general continuous solution of the functional equation (1) is determined by*

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F\left(\mathbf{U} - m\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m + n + k}, \mathbf{U} + \mathbf{V} + \mathbf{W}\right) - \\ &- F\left(\mathbf{V} - n\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m + n + k}, \mathbf{U} + \mathbf{V} + \mathbf{W}\right) + \\ &+ G_1(\mathbf{U} + \mathbf{V} + \mathbf{W})\operatorname{Re}\left(\mathbf{U} - m\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m + n + k}\right) + \\ &+ G_2(\mathbf{U} + \mathbf{V} + \mathbf{W})\operatorname{Im}\left(\mathbf{U} - m\frac{\mathbf{U} + \mathbf{V} + \mathbf{W}}{m + n + k}\right), \end{aligned}$$

where $F : \mathcal{V}^2 \rightarrow \mathcal{V}'$ and $G_i : \mathcal{V} \rightarrow \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ ($i = 1, 2$) are arbitrary continuous complex vector functions.

Proof. Let $2m < k$. On the basis of the expression (18), we obtain

$$\begin{aligned} h(\mathbf{U}, \mathbf{V}) + h(\mathbf{V}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{U} - \mathbf{V}) + h(-\mathbf{U} - \mathbf{V}, \mathbf{U}) &= \mathbf{O}, \\ h(-\mathbf{U} - \mathbf{V}, \mathbf{U}) + h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, \mathbf{V}) + h(\mathbf{V}, -\mathbf{U} - \mathbf{V}) &= \mathbf{O}, \\ h(\mathbf{V}, -\mathbf{U} - \mathbf{V}) + h(-\mathbf{U} - \mathbf{V}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U}) + h(\mathbf{U}, \mathbf{V}) &= \mathbf{O}. \end{aligned}$$

If from these three equalities we eliminate $h(-\mathbf{U} - \mathbf{V}, \mathbf{U})$ and $h(\mathbf{V}, -\mathbf{U} - \mathbf{V})$, we obtain

$$\begin{aligned} 2h(\mathbf{U}, \mathbf{V}) &= \\ -h(-\mathbf{U} - \mathbf{V}, \mathbf{O}) - h(\mathbf{O}, -\mathbf{U} - \mathbf{V}) + h(\mathbf{U}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U}) + h(\mathbf{O}, \mathbf{V}) - h(\mathbf{V}, \mathbf{O}). \end{aligned}$$

If we substitute this value of h into (10), we get

$$\begin{aligned} (35) \quad &\sum_{i=1}^k h\left(-\sum_{j=0}^{m+n-1} \mathbf{T}_{i+j}, \mathbf{O}\right) + \sum_{i=1}^k h\left(\mathbf{O}, -\sum_{j=0}^{m+n-1} \mathbf{T}_{i+j}\right) + \\ &+ \sum_{i=k+1}^{m+n+k} h\left(\sum_{j=0}^{k-1} \mathbf{T}_{m+n+i+j}, \mathbf{O}\right) + \sum_{i=k+1}^{m+n+k} h\left(\mathbf{O}, \sum_{j=0}^{k-1} \mathbf{T}_{m+n+i+j}\right) = \mathbf{O}. \end{aligned}$$

If we put $\mathbf{T}_1 = \mathbf{U}$, $\mathbf{T}_{m+n} = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ ($j \neq 1, m+n$) into (35), we have

$$\begin{aligned} (36) \quad &h(\mathbf{U} + \mathbf{V}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U} + \mathbf{V}) + (k-1)[h(\mathbf{V}, \mathbf{O}) + h(\mathbf{O}, \mathbf{V})] + \\ &+ h(-\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{U}) + k[h(-\mathbf{V}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{V})] = \mathbf{O}. \end{aligned}$$

For $\mathbf{V} = \mathbf{O}$, from (36) we obtain

$$(37) \quad h(-\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{U}) + h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U}) = \mathbf{O},$$

i.e. the equality (36) has the form

$$h(\mathbf{U} + \mathbf{V}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U} + \mathbf{V}) = h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U}) + h(\mathbf{V}, \mathbf{O}) + h(\mathbf{O}, \mathbf{V}).$$

Hence, it follows that

$$h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U}) = G_1 \operatorname{Re} \mathbf{U} + G_2 \operatorname{Im} \mathbf{U}.$$

Thus we obtain that the function $h(\mathbf{U}, \mathbf{V})$ is determined by

$$h(\mathbf{U}, \mathbf{V}) = h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, \mathbf{V}),$$

or

$$(38) \quad h(\mathbf{U}, \mathbf{V}) = F(\mathbf{U}) - F(\mathbf{V}) + G_1 \operatorname{Re} \mathbf{U} + G_2 \operatorname{Im} \mathbf{V},$$

where we have put $F(\mathbf{U}) = -h(\mathbf{O}, \mathbf{U})$.

Let $2m > k$. If we put $\mathbf{T}_1 = \mathbf{U}$, $\mathbf{T}_k = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ ($j \neq 1, k$) into (35), we get

$$(39) \quad \begin{aligned} & h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U}) + (m+n)[h(\mathbf{V}, \mathbf{O}) + h(\mathbf{O}, \mathbf{V})] + \\ & + h(-\mathbf{U} - \mathbf{V}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U} - \mathbf{V}) + (m+n-1)[h(-\mathbf{V}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{V})] = \mathbf{O}, \end{aligned}$$

and hence for $\mathbf{V} = \mathbf{O}$ we obtain (37), i.e. the equality (39) has the following form

$$h(\mathbf{U} + \mathbf{V}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U} + \mathbf{V}) = h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U}) + h(\mathbf{V}, \mathbf{O}) + h(\mathbf{O}, \mathbf{V}).$$

Hence it follows that the function h has the form (38).

Thus, on the basis of the expressions (38), (9), (7), (4) and (3), we conclude that the function f has the form given in the theorem. \square

Theorem 6. *If $a = 1$ and $2m = 2n = k$, then the general continuous solution of the functional equation (1) is given by*

$$f(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{U}, \mathbf{V} + \mathbf{W}) - F(\mathbf{V}, \mathbf{W} + \mathbf{U}) + G(\mathbf{U} + \mathbf{V}, \mathbf{W}) - G(\mathbf{W}, \mathbf{U} + \mathbf{V}),$$

where $F, G : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

Proof. From the equation (35), which holds in this case, there immediately follows the equality

$$h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U}) + h(-\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, -\mathbf{U}) = \mathbf{O},$$

i.e. we may put

$$(40) \quad h(\mathbf{U}, \mathbf{O}) + h(\mathbf{O}, \mathbf{U}) = 2G(\mathbf{U}) - 2G(-\mathbf{U}),$$

where G is an arbitrary complex vector function.

Since by virtue of (15) it holds

$$2h(\mathbf{U}, \mathbf{V}) =$$

$$-h(-\mathbf{U} - \mathbf{V}, \mathbf{O}) - h(\mathbf{O}, -\mathbf{U} - \mathbf{V}) + h(\mathbf{U}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U}) + h(\mathbf{O}, \mathbf{V}) - h(\mathbf{V}, \mathbf{O}),$$

on the basis of the expression (40) we have

$$(41) \quad h(\mathbf{U}, \mathbf{V}) = F(\mathbf{U}) - F(\mathbf{V}) + G(\mathbf{U} + \mathbf{V}) - G(-\mathbf{U} - \mathbf{V}),$$

where we introduced the notation

$$2F(\mathbf{U}) = h(\mathbf{U}, \mathbf{O}) - h(\mathbf{O}, \mathbf{U}).$$

From the equalities (41), (9), (7), (4) and (3) there follows the proof of the theorem.

□

Theorem 7. *If $a = 1$ and $m = n = k$, then the general continuous solution of the functional equation (1) is given by*

$$f(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{U}, \mathbf{V}, \mathbf{W}) - F(\mathbf{V}, \mathbf{W}, \mathbf{U}),$$

where $F : \mathcal{V}^3 \rightarrow \mathcal{V}'$ is an arbitrary complex vector function.

Proof. If we put $\mathbf{T}_1 = \mathbf{U}$, $\mathbf{T}_{m+1} = \mathbf{V}$ and $\mathbf{T}_j = \mathbf{O}$ ($j \neq 1, m+1$) into (10), then we obtain

$$h(\mathbf{U}, \mathbf{V}) + h(\mathbf{V}, -\mathbf{U} - \mathbf{V}) + h(-\mathbf{U} - \mathbf{V}, \mathbf{U}) = \mathbf{O}.$$

According to [2] the general solution of this equation is given by

$$(42) \quad h(\mathbf{U}, \mathbf{V}) = F(\mathbf{U}, \mathbf{V}) - F(\mathbf{V}, -\mathbf{U} - \mathbf{V}),$$

where $F : \mathcal{V}^2 \rightarrow \mathcal{V}'$ is an arbitrary complex vector function.

On the basis of the expressions (42), (9), (7), (4) and (3), we conclude that the function f is determined by the form given in the theorem. □

II. Let $a^{m+n+k} \neq 1$. If we put

$$(43) \quad f(\mathbf{U}, \mathbf{V}, \mathbf{W}) = g(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}, \mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}),$$

then the functional equation (1) becomes

$$\begin{aligned} & \sum_{i=1}^{m+n+k} g \left(\sum_{j=0}^{m-1} a^{m-1-j} \mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{m+n+k-1-j} \mathbf{Z}_{m+i+j} + \sum_{j=0}^{k-1} a^{m+k-1-j} \mathbf{Z}_{m+n+i+j}, \right. \\ & \quad \sum_{j=0}^{n-1} a^{n-1-j} \mathbf{Z}_{m+i+j} + \sum_{j=0}^{k-1} a^{m+n+k-1-j} \mathbf{Z}_{m+n+i+j} + \sum_{j=0}^{m-1} a^{m+n-1-j} \mathbf{Z}_{i+j}, \\ & \quad \left. \sum_{j=0}^{k-1} a^{k-1-j} \mathbf{Z}_{m+n+i+j} + \sum_{j=0}^{n-1} a^{m+n+k-1-j} \mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{n+k-1-j} \mathbf{Z}_{m+i+j} \right) = \mathbf{O}, \end{aligned}$$

i.e.

$$\begin{aligned} & \sum_{i=0}^{m+n+k} g \left(\sum_{j=0}^{m+n+k-1} a^j \mathbf{Z}_{m-1+i-j}, \sum_{j=0}^{m+n+k-1} a^j \mathbf{Z}_{m+n-1+i-j}, \right. \\ & \quad \left. \sum_{j=0}^{m+n+k-1} a^j \mathbf{Z}_{m+n+k-1+i-j} \right) = \mathbf{O}. \end{aligned}$$

This transformation of the equation (1) is possible since $a^{m+n+k} \neq 1$. If we introduce new variables \mathbf{T}_i by the relations

$$\mathbf{T}_i = \sum_{j=0}^{m+n+k-1} a^j \mathbf{Z}_{m-1+i-j} \quad (1 \leq i \leq m+n+k),$$

then the previous equation becomes

$$(44) \quad \sum_{i=1}^{m+n+k} g(\mathbf{T}_i, \mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) = \mathbf{O}.$$

Now we will give the following results.

Lemma 1. *The general solution of the functional equation (44) is determined by*

$$1^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{U}) - F(\mathbf{W}) - H(\mathbf{V}) - H(\mathbf{W})$$

$$(m \neq n \neq k \neq m, \quad m \neq n+k, \quad n \neq m+k, \quad k \neq m+n),$$

$$2^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{U}) - F(\mathbf{V}) - G(\mathbf{U}, \mathbf{W}) - G(\mathbf{W}, \mathbf{U})$$

$$(m \neq n \neq k \neq m, \quad m = n+k),$$

$$3^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{W}) - F(\mathbf{U}) - G(\mathbf{U}, \mathbf{V}) - G(\mathbf{V}, \mathbf{U})$$

$$(m \neq n \neq k \neq m, \quad n = m+k),$$

$$4^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{U}) - F(\mathbf{V}) + G(\mathbf{V}, \mathbf{W}) - G(\mathbf{W}, \mathbf{V})$$

$$(m \neq n \neq k \neq m, \quad k = m+n),$$

$$5^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = G(\mathbf{U}, \mathbf{V}) - G(\mathbf{V}, \mathbf{W})$$

$$(m \neq n = k, \quad m \neq n+k),$$

$$6^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = K(\mathbf{U}, \mathbf{V}) - K(\mathbf{V}, \mathbf{W}) + G(\mathbf{W}, \mathbf{U}) - G(\mathbf{U}, \mathbf{W})$$

$$(m \neq n = k, \quad m = n+k),$$

$$7^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = G(\mathbf{U}, \mathbf{V}) - G(\mathbf{W}, \mathbf{U})$$

$$(m = n \neq k, \quad k \neq m+n),$$

$$8^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = K(\mathbf{U}, \mathbf{V}) - K(\mathbf{W}, \mathbf{U}) + G(\mathbf{V}, \mathbf{W}) - G(\mathbf{W}, \mathbf{V})$$

$$(m = n \neq k, \quad k = m+n),$$

$$9^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = G(\mathbf{V}, \mathbf{W}) - G(\mathbf{W}, \mathbf{U})$$

$$(m = k \neq n, \quad n \neq k+m),$$

$$10^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = K(\mathbf{U}, \mathbf{V}) - K(\mathbf{V}, \mathbf{U}) + G(\mathbf{V}, \mathbf{W}) - G(\mathbf{W}, \mathbf{U})$$

$$(m = k \neq n, \quad n = k+m),$$

$$11^0. \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = L(\mathbf{U}, \mathbf{V}, \mathbf{W}) - L(\mathbf{V}, \mathbf{W}, \mathbf{U})$$

$$(m = n = k),$$

where $F, H : \mathcal{V} \rightarrow \mathcal{V}'$, $G, K : \mathcal{V}^2 \rightarrow \mathcal{V}'$ and $L : \mathcal{V}^3 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

Proof. Since the proofs of the particular cases are similar or completely the same, we will prove the lemma only in the cases 3⁰ and 8⁰.

3⁰. If all variables in (44), except for \mathbf{Z}_i , \mathbf{Z}_{i+n} and \mathbf{Z}_{i+n+k} , are equal to some constant, we obtain

$$g(\mathbf{Z}_i, \mathbf{Z}_{i+n}, \mathbf{Z}_{i+n+k}) = F(\mathbf{Z}_{i+n+k}) + K(\mathbf{Z}_i, \mathbf{Z}_{i+n}),$$

i.e.

$$(45) \quad g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{W}) + K(\mathbf{U}, \mathbf{V}).$$

Hence, the equation (44) becomes

$$\sum_{i=1}^{m+n+k} F(\mathbf{Z}_i) + \sum_{i=1}^{m+n+k} K(\mathbf{Z}_i, \mathbf{Z}_{i+n}) = \mathbf{O}.$$

If we put $\mathbf{Z}_r = \mathbf{O}$ ($r \neq i, i+n$) in the above equation, we have

$$F(\mathbf{Z}_i) + F(\mathbf{Z}_{i+n}) + K(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + K(\mathbf{Z}_{i+n}, \mathbf{Z}_i) = \mathbf{O},$$

and thus it follows that

$$K(\mathbf{U}, \mathbf{V}) = G(\mathbf{U}, \mathbf{V}) - G(\mathbf{V}, \mathbf{U}) - F(\mathbf{U}).$$

Now, the equality (45) becomes

$$g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{W}) - F(\mathbf{U}) + G(\mathbf{U}, \mathbf{V}) - G(\mathbf{V}, \mathbf{U}).$$

8⁰. If we put $\mathbf{Z}_r = \mathbf{O}$ ($r \neq i, i+n, i+n+k$) into (44), then the equation (44) becomes

$$g(\mathbf{Z}_i, \mathbf{Z}_{i+n}, \mathbf{Z}_{i+n+k}) = K(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + K_1(\mathbf{Z}_{i+n}, \mathbf{Z}_{i+n+k}) + K_2(\mathbf{Z}_{i+n+k}, \mathbf{Z}_i),$$

i.e.

$$g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = K(\mathbf{U}, \mathbf{V}) + K_1(\mathbf{V}, \mathbf{W}) + K_2(\mathbf{W}, \mathbf{U}),$$

where $K, K_1, K_2 : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

By a substitution of the value obtained for g into (44), we get

$$(46) \quad \sum_{i=1}^{m+n+k} K(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + \sum_{i=1}^{m+n+k} K_1(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + \sum_{i=1}^{m+n+k} K_2(\mathbf{Z}_i, \mathbf{Z}_{i+n}) = \mathbf{O}.$$

Hence, for \mathbf{Z}_r ($r \neq i, i+k$) we obtain

$$K_2(\mathbf{Z}_i, \mathbf{Z}_{i+k}) + K_2(\mathbf{Z}_{i+k}, \mathbf{Z}_i) + H(\mathbf{Z}_i) + H(\mathbf{Z}_{i+k}) = \mathbf{O},$$

where $H : \mathcal{V} \rightarrow \mathcal{V}'$ is an arbitrary complex vector function.

From the above equation it follows that the function K_2 has the form

$$(47) \quad K_2(\mathbf{U}, \mathbf{V}) = G(\mathbf{U}, \mathbf{V}) - G(\mathbf{V}, \mathbf{U}) - H(\mathbf{U}),$$

where $G : \mathcal{V}^2 \rightarrow \mathcal{V}'$ is an arbitrary complex vector function.

If we put (47) into (46), we obtain

$$(48) \quad \sum_{i=1}^{m+n+k} [K(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + K_1(\mathbf{Z}_i, \mathbf{Z}_{i+n}) + H(\mathbf{Z}_i)] = \mathbf{O}.$$

For $\mathbf{Z}_r = \mathbf{O}$ ($r \neq i, i+n$) from the above equation we obtain

$$K_1(\mathbf{U}, \mathbf{V}) = -K(\mathbf{U}, \mathbf{V}) - 2H(\mathbf{U}) - 2H(\mathbf{V}).$$

By putting the above obtained value of K_1 into (48), we have

$$\sum_{i=1}^{m+n+k} H(\mathbf{Z}_i) = \mathbf{O},$$

i.e.

$$H(\mathbf{U}) = \mathbf{O}.$$

On the basis of the previous equalities, the function g is determined by

$$g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = K(\mathbf{U}, \mathbf{V}) - K(\mathbf{W}, \mathbf{U}) + G(\mathbf{V}, \mathbf{W}) - G(\mathbf{W}, \mathbf{V}). \quad \square$$

According to the previous lemma and the transformation (43), we obtain the following theorems.

Theorem 8. *If $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$, $m \neq n+k$, $n \neq m+k$ and $k \neq m+n$, then the general solution of the functional equation (1) is*

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}) - F(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) + \\ &+ G(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) - G(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}), \end{aligned}$$

where $F, G : \mathcal{V} \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 9. *If $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ and $m = n+k$, then the general solution of the functional equation (1) is*

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}) - F(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) + \\ &+ G(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) - \\ &- G(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}, \mathbf{U} + a^{k+n}\mathbf{V} + a^m\mathbf{W}), \end{aligned}$$

where $F : \mathcal{V} \rightarrow \mathcal{V}'$ and $G : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 10. *If $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ and $n = m+k$, then the general solution of the functional equation (1) is*

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) - F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}) + \\ &+ G(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) - \\ &- G(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}, \mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}), \end{aligned}$$

where $F : \mathcal{V} \rightarrow \mathcal{V}'$ and $G : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 11. If $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ and $k = m + n$, then the functional equation (1) has a general solution

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}) - F(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) + \\ &\quad + G(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}, \mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) - \\ &\quad - G(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}, \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}), \end{aligned}$$

where $F : \mathcal{V} \rightarrow \mathcal{V}'$ and $G : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 12. If $a^{m+n+k} \neq 1$ and $m \neq n + k$, then the general solution of the functional equation (1) is

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) - \\ &\quad - F(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}, \mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}), \end{aligned}$$

where $F : \mathcal{V}^2 \rightarrow \mathcal{V}'$ is an arbitrary complex vector function.

Theorem 13. If $a^{m+n+k} \neq 1$, $m \neq n = k$ and $m = 2n$, then the functional equation (1) has a general solution determined by

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) - \\ &\quad - F(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}, \mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) + \\ &\quad + G(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}, \mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}) - \\ &\quad - G(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}), \end{aligned}$$

where $F, G : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 14. If $a^{m+n+k} \neq 1$, $m = n \neq k$ and $k \neq m + n$, then the functional equation (1) has a general solution given by

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) - \\ &\quad - F(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}, \mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}), \end{aligned}$$

where $F : \mathcal{V}^2 \rightarrow \mathcal{V}'$ is an arbitrary complex vector function.

Theorem 15. If $a^{m+n+k} \neq 1$, $m = n \neq k$ and $k = m + n$, then the functional equation (1) has a general solution

$$\begin{aligned} f(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F(\mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}) - \\ &\quad - F(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}, \mathbf{U} + a^{k+m}\mathbf{V} + a^m\mathbf{W}) + \\ &\quad + G(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}, \mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) - \\ &\quad - G(\mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}, \mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}), \end{aligned}$$

where $F, G : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 16. If $a^{m+n+k} \neq 1$, $m = k \neq n$ and $n \neq m + k$, then the functional equation (1) has general solution

$$f(\mathbf{U}, \mathbf{V}, \mathbf{W}) = F(\mathbf{V} + a^{m+n}\mathbf{W} + a^n\mathbf{U}, \mathbf{W} + a^{n+k}\mathbf{U} + a^k\mathbf{V}) -$$

$$-F(\mathbf{W} + a^{n+k}U + a^k\mathbf{V}, U + a^{k+m}\mathbf{V} + a^m\mathbf{W}),$$

where $F : \mathcal{V}^2 \rightarrow \mathcal{V}'$ is an arbitrary complex vector function.

Theorem 17. If $a^{m+n+k} \neq 1$, $m = k \neq n$ and $n = m + k$, then the functional equation (1) has a solution given by

$$\begin{aligned} f(U, \mathbf{V}, \mathbf{W}) = & F(U + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{V} + a^{m+n}\mathbf{W} + a^nU) - \\ & -F(\mathbf{V} + a^{m+n}\mathbf{W} + a^nU, U + a^{k+m}\mathbf{V} + a^m\mathbf{W}) + \\ & +G(\mathbf{V} + a^{m+n}\mathbf{W} + a^nU, \mathbf{W} + a^{n+k}U + a^k\mathbf{V}) - \\ & -G(\mathbf{W} + a^{n+k}U + a^k\mathbf{V}, U + a^{m+n}\mathbf{V} + a^n\mathbf{W}), \end{aligned}$$

where $F, G : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 18. If $a^{m+n+k} \neq 1$, $m = n = k$, then the general solution of the functional equation (1) is

$$\begin{aligned} f(U, \mathbf{V}, \mathbf{W}) = & F(U + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{V} + a^{m+n}\mathbf{W} + a^nU, \mathbf{W} + a^{n+k}U + a^k\mathbf{V}) - \\ & -F(\mathbf{V} + a^{m+n}\mathbf{W} + a^nU, \mathbf{W} + a^{n+k}U + a^k\mathbf{V}, U + a^{k+m}\mathbf{V} + a^m\mathbf{W}), \end{aligned}$$

where $F : \mathcal{V}^3 \rightarrow \mathcal{V}'$ is an arbitrary complex vector function.

III. If $a^{m+n+k} = 1$, this case is very difficult and up to now we are not able to solve the functional equation (1).

2 Solution of the Generalized Functional Equation

Further, we will consider the solving of the functional equation (2). For the equation (2) we will determine the general solution only if $a^{m+n+k} \neq 1$.

Now, we transform the equation (2). If we put

$$(49) \quad \begin{aligned} & f_i(U, \mathbf{V}, \mathbf{W}) \\ & = g_i(U + a^{k+m}\mathbf{V} + a^m\mathbf{W}, \mathbf{V} + a^{m+n}\mathbf{W} + a^nU, \mathbf{W} + a^{n+k}U + a^k\mathbf{V}), \end{aligned}$$

then the equation (2) becomes

$$\begin{aligned} & \sum_{i=1}^{m+n+k} g_i \left(\sum_{j=0}^{m-1} a^{m-1-j} \mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{m+n+k-1-j} \mathbf{Z}_{m+i+j} + \sum_{j=0}^{k-1} a^{m+k-1-j} \mathbf{Z}_{m+n+i+j}, \right. \\ & \quad \sum_{j=0}^{n-1} a^{n-1-j} \mathbf{Z}_{m+i+j} + \sum_{j=0}^{k-1} a^{m+n+k-1-j} \mathbf{Z}_{m+n+i+j} + \sum_{j=0}^{m-1} a^{m+n-1-j} \mathbf{Z}_{i+j}, \\ & \quad \left. \sum_{j=0}^{k-1} a^{k-1-j} \mathbf{Z}_{m+n+i+j} + \sum_{j=0}^{m-1} a^{m+n+k-1-j} \mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{n+k-1-j} \mathbf{Z}_{m+i+j} \right) = \mathbf{O}, \end{aligned}$$

i.e.

$$(50) \quad \sum_{i=1}^{m+n+k} g_i \left(\sum_{j=0}^{m+n+k-1} a^j \mathbf{Z}_{m-1+i-j}, \sum_{j=0}^{m+n+k-1} a^j \mathbf{Z}_{m+n-1+i-j}, \right)$$

$$\sum_{j=0}^{m+n+k-1} a^j \mathbf{Z}_{m+n+k-1+i-j} = \mathbf{O}.$$

Since the linear forms

$$\mathbf{T}_i = \sum_{j=0}^{m+n+k-1} a^j \mathbf{Z}_{m-1+i-j} \quad (1 \leq i \leq m+n+k)$$

are linearly independent, it is possible to introduce new variables \mathbf{T}_i which are determined by the above relations.

Therefore, the functional equation (50) takes the following form

$$(51) \quad \sum_{i=1}^{m+n+k} g_i(\mathbf{T}_i, \mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) = \mathbf{O}.$$

For the last equation the following lemma holds.

Lemma 2. *The general solution of the functional equation (51) is given by the equalities*

$$1^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) - F_{i+n+k}(\mathbf{W}) - H_i(\mathbf{V}) - H_{i+k}(\mathbf{W}) - A_i,$$

$$\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$$

$$(m \neq n \neq k \neq m, \quad m \neq n+k, \quad n \neq m+k, \quad k \neq m+n),$$

$$2^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = F_i(\mathbf{V}) - F_{i+k}(\mathbf{W}) + G_i(U, \mathbf{W}) \quad (1 \leq i \leq m),$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(\mathbf{V}) - F_{i+k}(\mathbf{W}) - G_{i+m}(\mathbf{W}, U) - A_{i+m} \quad (m+1 \leq i \leq 2m),$$

$$\sum_{i=1}^m A_i = \mathbf{O}$$

$$(m \neq n \neq k \neq m, \quad m \neq n+k),$$

$$3^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = F_i(\mathbf{W}) - F_{i+m}(U) + G_i(U, \mathbf{V}) \quad (1 \leq i \leq n),$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(\mathbf{W}) - F_{i+m}(U) - G_{i+m}(\mathbf{V}, U) - A_{i+m} \quad (n+1 \leq i \leq 2n),$$

$$\sum_{i=1}^n A_i = \mathbf{O}$$

$$(m \neq n \neq k \neq m, \quad n = m+k),$$

$$4^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) - F_{i+n}(\mathbf{V}) + G_i(\mathbf{V}, \mathbf{W}) \quad (1 \leq i \leq k),$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) - F_{i+n}(\mathbf{V}) - G_{i+k}(\mathbf{W}, \mathbf{V}) - A_{i+k} \quad (k+1 \leq i \leq 2k),$$

$$\sum_{i=1}^k A_i = \mathbf{O}$$

$$(m \neq n \neq k \neq m, \quad k = m + n),$$

$$5^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = G_i(U, \mathbf{V}) - G_{i+n}(\mathbf{V}, \mathbf{W}) + A_i \quad (1 \leq i \leq m + n + k),$$

$$\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$$

$$(m \neq n = k, \quad m \neq n + k),$$

$$6^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = K_i(U, \mathbf{V}) - K_{i+n}(\mathbf{V}, \mathbf{W}) + G_i(\mathbf{W}, U) \quad (1 \leq i \leq m),$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = K_i(U, \mathbf{V}) - K_{i+n}(\mathbf{V}, \mathbf{W}) - G_{i+m}(U, \mathbf{W}) \quad (m + 1 \leq i \leq 2m),$$

$$(m \neq n = k, \quad m = n + k),$$

$$7^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = G_i(U, \mathbf{V}) - G_{i+n+k}(\mathbf{W}, U) + A_i \quad (1 \leq i \leq m + n + k),$$

$$\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$$

$$(m = n \neq k, \quad k \neq m + n),$$

$$8^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = K_i(U, \mathbf{V}) - K_{i+n+k}(\mathbf{W}, U) + G_i(\mathbf{V}, \mathbf{W}) + A_i \quad (1 \leq i \leq k),$$

$$\sum_{i=1}^k A_i = \mathbf{O},$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = K_i(U, \mathbf{V}) - K_{i+n+k}(\mathbf{W}, U) - G_i(\mathbf{V}, \mathbf{W}) \quad (k + 1 \leq i \leq 2k),$$

$$(m = n \neq k, \quad k = m + n),$$

$$9^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = G_i(U, \mathbf{V}) - G_{i+m+n}(\mathbf{W}, \mathbf{V}) + A_i \quad (1 \leq i \leq m + n + k),$$

$$\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$$

$$(m = k \neq n, \quad n \neq m + k),$$

$$10^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = K_i(\mathbf{V}, \mathbf{W}) - K_{i+k}(\mathbf{W}, U) + G_i(U, \mathbf{V}) + A_i \quad (1 \leq i \leq n),$$

$$\sum_{i=1}^n A_i = \mathbf{O},$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = K_i(\mathbf{V}, \mathbf{W}) - K_{i+k}(\mathbf{W}, U) - G_{i+n}(\mathbf{V}, U) \quad (n + 1 \leq i \leq 2n),$$

$$(m = k \neq n, \quad n = m + k),$$

$$11^0. \quad g_i(U, \mathbf{V}, \mathbf{W}) = L_i(U, \mathbf{V}, \mathbf{W}) \quad (1 \leq i \leq 2m),$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = -L_i(\mathbf{V}, \mathbf{W}, U) - L_{i+2m}(\mathbf{W}, U, \mathbf{V}) \quad (2m + 1 \leq i \leq 3m),$$

$$(m = n = k).$$

In all cases the functions $F_i, H_i : \mathcal{V} \rightarrow \mathcal{V}'$, $G_i, K_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ and $L_i : \mathcal{V}^3 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions such that $F_{i+n+m+k} = F_i$, $G_{i+n+m+k} = G_i$, \dots . Also, for the arbitrary constant complex vectors A_i the equalities $A_{i+n+m+k} = A_i$ hold.

Proof. Since the statements of the lemma for particular cases may be proven in a similar way, therefore it is sufficient to prove the lemma only for some cases, for example 4^0 and 5^0 .

4^0 . If in (51) all variables, except \mathbf{T}_i , \mathbf{T}_{i+n} and \mathbf{T}_{i+n+k} , are equal to some constant, we obtain

$$g_i(\mathbf{T}_i, \mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) = F_i(\mathbf{T}_i) + K_i(\mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}),$$

i.e.

$$(52) \quad g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) + K_i(\mathbf{V}, \mathbf{W}),$$

where F_i and K_i ($1 \leq i \leq m+n+k$) are arbitrary complex vector functions.

On the basis of the equation (52), the equation (51) becomes

$$\sum_{i=1}^{m+n+k} F_i(\mathbf{T}_i) + \sum_{i=1}^{m+n+k} K_i(\mathbf{T}_i, \mathbf{T}_{i+n+k}) = \mathbf{O},$$

i.e.

$$\sum_{i=1}^{m+n+k} F_{i+n}(\mathbf{T}_{i+n}) + \sum_{i=1}^{m+n+k} K_i(\mathbf{T}_i, \mathbf{T}_{i+n+k}) = \mathbf{O}.$$

If we put $\mathbf{T}_r = \mathbf{O}$ ($r = i+n, i+n+k$) into the previous equality, we obtain

$$(53) \quad F_{i+n}(\mathbf{T}_{i+n}) + F_{i+n+k}(\mathbf{T}_{i+n+k}) + K_i(\mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) \\ + K_{i+k}(\mathbf{T}_{i+n+k}, \mathbf{T}_{i+n}) + A_{i+k} = \mathbf{O},$$

where A_i are arbitrary constant complex vectors.

From (53) we obtain

$$K_i(\mathbf{V}, \mathbf{W}) = -K_{i+m}(\mathbf{W}, \mathbf{V}) - F_{i+n}(\mathbf{V}) - F_{i+n+k}(\mathbf{W}) - A_{i+k},$$

i.e. the equation (52) takes the form

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) + K_i(\mathbf{V}, \mathbf{W}) \quad (1 \leq i \leq k),$$

$$g_i(U, \mathbf{V}, \mathbf{W}) = F_i(U) - K_{i+k}(\mathbf{W}, \mathbf{V}) - F_{i+n}(\mathbf{V}) - F_{i+n+k}(\mathbf{W}) - A_{i+k} \\ (k+1 \leq i \leq 2k),$$

such that

$$\sum_{i=1}^k A_i = \mathbf{O}.$$

If we introduce new functions G_i by

$$G_i(\mathbf{V}, \mathbf{W}) = K_i(\mathbf{V}, \mathbf{W}) + F_{i+n}(\mathbf{V}),$$

the previous equations take the following form

$$\begin{aligned} g_i(U, \mathbf{V}, \mathbf{W}) &= F_i(U) - F_{i+n}(\mathbf{V}) + G_i(\mathbf{V}, \mathbf{W}) \quad (1 \leq i \leq k), \\ g_i(U, \mathbf{V}, \mathbf{W}) &= F_i(U) - F_{i+n}(\mathbf{V}) - G_{i+k}(\mathbf{W}, \mathbf{V}) - A_{i+k} \quad (k+1 \leq i \leq 2k), \end{aligned}$$

where

$$\sum_{i=1}^k A_i = \mathbf{O}.$$

5^o. If $\mathbf{T}_j = \mathbf{O}$ ($j \neq i, i+n, i+n+k$), from the equation (51) we obtain

$$g_i(\mathbf{T}_i, \mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) = K_i(\mathbf{T}_i, \mathbf{T}_{i+n}) + M_i(\mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}),$$

i.e.

$$(54) \quad g_i(U, \mathbf{V}, \mathbf{W}) = K_i(U, \mathbf{V}) + M_i(\mathbf{V}, \mathbf{W}),$$

where K_i and M_i are arbitrary complex vector functions.

On the basis of the relation (54), the functional equation (51) becomes

$$\sum_{i=1}^{m+n+k} K_i(\mathbf{T}_i, \mathbf{T}_{i+n}) + \sum_{i=1}^{m+n+k} M_i(\mathbf{T}_{i+n}, \mathbf{T}_{i+n+k}) = \mathbf{O},$$

i.e.

$$(55) \quad \sum_{i=1}^{m+n+k} K_i(\mathbf{T}_i, \mathbf{T}_{i+n}) + \sum_{i=1}^{m+n+k} M_{i+m+n}(\mathbf{T}_i, \mathbf{T}_{i+n}) = \mathbf{O}.$$

If we put $\mathbf{T}_r = \mathbf{O}$ ($r \neq i, i+n$) into (55), then

$$(56) \quad K_i(\mathbf{T}_i, \mathbf{T}_{i+n}) + M_{i+m+n}(\mathbf{T}_i, \mathbf{T}_{i+n}) + P_i(\mathbf{T}_i) - Q_{i+n}(\mathbf{T}_{i+n}) = \mathbf{O},$$

where P_i and Q_i are arbitrary complex vector functions.

On the basis of the above equation (65), the equation (55) becomes

$$\sum_{i=1}^{m+n+k} P_i(\mathbf{T}_i) - \sum_{i=1}^{m+n+k} Q_i(\mathbf{T}_i) = \mathbf{O},$$

from which we obtain

$$Q_i(\mathbf{T}_i) = P_i(\mathbf{T}_i) + B_i, \quad \sum_{i=1}^{m+n+k} B_i = \mathbf{O}.$$

Therefore, the equality (56) takes the form

$$K_i(\mathbf{T}_i, \mathbf{T}_{i+n}) + M_{i+m+k}(\mathbf{T}_i, \mathbf{T}_{i+n}) + P_i(\mathbf{T}_i) - P_{i+n}(\mathbf{T}_{i+n}) - B_{i+n} = \mathbf{O}.$$

On the basis of this equality and (54), we have

$$g_i(U, \mathbf{V}, \mathbf{W}) = K_i(U, \mathbf{V}) - K_{i+n}(\mathbf{V}, \mathbf{W}) + P_{i+n}(U) - P_{i+2n}(\mathbf{V}) + B_{i+2n},$$

i.e.

$$g_i(U, \mathbf{V}, \mathbf{W}) = G_i(U, \mathbf{V}) - G_{i+n}(\mathbf{V}, \mathbf{W}) + A_i, \quad \sum_{i=1}^{m+n+k} A_i = \mathbf{O},$$

where we put

$$G_i(U, \mathbf{V}) = K_i(U, \mathbf{V}) + P_{i+n}(U), \quad A_i = B_{i+2n}.$$

On the basis of Lemma 2 and the transformations (49), there follow the theorems which treat the functional equation (2).

Theorem 19. *If $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$, $m \neq n+k$, $n \neq m+k$ and $k \neq m+n$, then the general solution of the functional equation (2) is*

$$\begin{aligned} f_i(U, V, W) &= F_i(V + a^{m+n}W + a^nU) - F_{i+k}(W + a^{n+k}U + a^kV) + \\ &+ G_i(U + a^{k+m}V + a^mW, W + a^{n+k}U + a^kV) \quad (1 \leq i \leq m), \\ f_i(U, V, W) &= F_i(V + a^{m+n}W + a^nU) - F_{i+k}(W + a^{n+k}U + a^kV) - \\ &- G_{i+m}(W + a^{n+k}U + a^kV, U + a^{k+m}V + a^mW) - A_{i+m} \quad (m+1 \leq i \leq 2m), \\ &\sum_{i=1}^m A_i = \mathbf{O}, \end{aligned}$$

where $F_i : \mathcal{V} \rightarrow \mathcal{V}'$ and $G_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions, and A_i are arbitrary constant complex vectors such that $\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$.

Theorem 20. *If $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ and $m = n+k$, then the general solution of the functional equation (2) is*

$$\begin{aligned} f_i(U, V, W) &= F_i(V + a^{m+n}W + a^nU) - F_{i+k}(W + a^{n+k}U + a^kV) + \\ &+ G_i(U + a^{k+m}V + a^mW, W + a^{n+k}U + a^kV) \quad (1 \leq i \leq m), \\ f_i(U, V, W) &= F_i(V + a^{m+n}W + a^nU) - F_{i+k}(W + a^{n+k}U + a^kV) - \\ &- G_{i+m}(W + a^{n+k}U + a^kV, U + a^{k+m}V + a^mW) - A_{i+m} \quad (m+1 \leq i \leq 2m), \end{aligned}$$

where $F_i : \mathcal{V} \rightarrow \mathcal{V}'$ and $G_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions, and A_i are arbitrary constant complex vectors such that $\sum_{i=1}^m A_i = \mathbf{O}$.

Theorem 21. *If $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ and $n = m+k$, then the general solution of the functional equation (2) is*

$$\begin{aligned} f_i(U, V, W) &= F_i(W + a^{n+k}U + a^kV) - F_{i+m}(U + a^{k+m}V + a^mW) + \\ &+ G_i(U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU) \quad (1 \leq i \leq n), \\ f_i(U, V, W) &= F_i(W + a^{n+k}U + a^kV) - F_{i+m}(U + a^{k+m}V + a^mW) - \\ &- G_{i+n}(V + a^{m+n}W + a^nU, U + a^{k+m}V + a^mW) - A_{i+n} \quad (n+1 \leq i \leq 2n), \end{aligned}$$

where $F_i : \mathcal{V} \rightarrow \mathcal{V}'$ and $G_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions, A_i are arbitrary constant complex vectors such that $\sum_{i=1}^n A_i = \mathbf{O}$.

Theorem 22. If $a^{m+n+k} \neq 1$, $m \neq n \neq k \neq m$ and $k = m + n$, then the general solution of the functional equation (2) is given by

$$\begin{aligned} f_i(U, V, W) &= F_i(U + a^{k+m}V + a^mW) - F_{i+n}(V + a^{m+n}W + a^nU) + \\ &+ G_i(V + a^{m+n}W + a^nU, W + a^{n+k}U + a^kV) \quad (1 \leq i \leq k), \\ f_i(U, V, W) &= F_i(U + a^{k+m}V + a^mW) - F_{i+n}(V + a^{m+n}W + a^nU) - \\ &- G_{i+k}(W + a^{n+k}U + a^kV, V + a^{m+n}W + a^nU) - A_{i+k} \quad (k+1 \leq i \leq 2k), \end{aligned}$$

where $F_i : \mathcal{V} \rightarrow \mathcal{V}'$ and $G_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions, A_i are arbitrary constant complex vectors such that $\sum_{i=1}^k A_i = \mathbf{O}$.

Theorem 23. If $a^{m+n+k} \neq 1$, $m \neq n = k$ and $m \neq n + k$, then the general solution of the functional equation (2) is

$$\begin{aligned} f_i(U, V, W) &= F_i(U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU) - \\ &- F_{i+n}(V + a^{m+n}W + a^nU, W + a^{n+k}U + a^kV) + A_i \quad (1 \leq i \leq m + n + k), \end{aligned}$$

where $F_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions, A_i are arbitrary constant complex vectors such that $\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$.

Theorem 24. If $a^{m+n+k} \neq 1$, $m \neq n = k$ and $m = n + k$, then the general solution of the functional equation (2) is determined by

$$\begin{aligned} f_i(U, V, W) &= F_i(U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU) - \\ &- F_{i+n}(V + a^{m+n}W + a^nU, W + a^{n+k}U + a^kV) + \\ &+ G_i(W + a^{n+k}U + a^kV, U + a^{k+m}V + a^mW) \quad (1 \leq i \leq m), \\ f_i(U, V, W) &= F_i(U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU) - \\ &- F_{i+n}(V + a^{m+n}W + a^nU, W + a^{n+k}U + a^kV) - \\ &- G_{i+m}(U + a^{k+m}V + a^mW, W + a^{n+k}U + a^kV) \quad (m+1 \leq i \leq 2m), \end{aligned}$$

where $F_i, G_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

Theorem 25. If $a^{m+n+k} \neq 1$, $m = n \neq k$ and $k \neq m + n$, then the general solution of the functional equation (2) is

$$\begin{aligned} f_i(U, V, W) &= F_i(U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU) - \\ &- F_{i+n+k}(W + a^{n+k}U + a^kV, U + a^{k+m}V + a^mW) + A_i \quad (1 \leq i \leq m + n + k), \end{aligned}$$

where $F_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions, A_i are arbitrary constant complex vectors such that $\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$.

Theorem 26. *If $a^{m+n+k} \neq 1$ and $k = m + n$, then the general solution of the functional equation (2) is*

$$\begin{aligned} f_i(U, V, W) &= F_i(U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU) - \\ &- F_{i+n+k}(W + a^{n+k}U + a^kV, U + a^{k+m}V + a^mW) + \\ &+ G_i(V + a^{m+n}W + a^nU, W + a^{n+k}U + a^kV) + A_i \quad (1 \leq i \leq k), \\ f_i(U, V, W) &= F_i(U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU) - \\ &- F_{i+n+k}(W + a^{n+k}U + a^kV, U + a^{k+m}V + a^mW) - \\ &- G_{i+k}(W + a^{n+k}U + a^kV, V + a^{m+n}W + a^nU) \quad (k + 1 \leq i \leq 2k), \end{aligned}$$

where $F_i, G_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions, and A_i are arbitrary constant complex vectors such that $\sum_{i=1}^k A_i = \mathbf{O}$.

Theorem 27. *If $a^{m+n+k} \neq 1$, $m = k \neq n$ and $n \neq m + k$, then the general solution of the functional equation (2) is*

$$\begin{aligned} f_i(U, V, W) &= F_i(U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU) - \\ &- F_{i+m+n}(W + a^{n+k}U + a^kV, V + a^{m+n}W + a^nU) + A_i \quad (1 \leq i \leq m + n + k), \end{aligned}$$

where $F_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions, and A_i are arbitrary constant complex vectors such that $\sum_{i=1}^{m+n+k} A_i = \mathbf{O}$.

Theorem 28. *If $a^{m+n+k} \neq 1$, $m = k \neq n$ and $n = m + k$, then the general solution of the functional equation (2) is determined by*

$$\begin{aligned} f_i(U, V, W) &= F_i(V + a^{m+n}W + a^nU, W + a^{n+k}U + a^kV) - \\ &- F_{i+k}(W + a^{n+k}U + a^kV, U + a^{k+m}V + a^mW) + \\ &+ G_i(U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU) + A_i \quad (1 \leq i \leq n), \\ f_i(U, V, W) &= F_i(V + a^{m+n}W + a^nU, W + a^{n+k}U + a^kV) - \\ &- F_{i+k}(W + a^{n+k}U + a^kV, U + a^{k+m}V + a^mW) - \\ &- G_{i+n}(V + a^{m+n}W + a^nU, U + a^{k+m}V + a^mW) \quad (n + 1 \leq i \leq 2n), \end{aligned}$$

where $F_i, G_i : \mathcal{V}^2 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions, and A_i are arbitrary constant complex vectors such that $\sum_{i=1}^n A_i = \mathbf{O}$.

Theorem 29. *If $a^{m+n+k} \neq 1$, $m = n = k$, then the general solution of the functional equation (2) is*

$$\begin{aligned} f_i(U, V, W) &= F_i(U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU, W + a^{n+k}U + a^kV) \\ &(1 \leq i \leq 2m), \end{aligned}$$

$$f_i(U, V, W) = -F_i(V + a^{m+n}W + a^nU, W + a^{n+k}U + a^kV, U + a^{k+m}V + a^mW) - \\ -F_{i+2m}(W + a^{n+k}U + a^kV, U + a^{k+m}V + a^mW, V + a^{m+n}W + a^nU) \\ (2m + 1 \leq i \leq 3m),$$

where $F_i : \mathcal{V}^3 \rightarrow \mathcal{V}'$ are arbitrary complex vector functions.

If $a = 1$ or $a^{m+n+k} = 1$ ($a \neq 1$), then the solution of the functional equation (2) is very complicated and up to now we cannot obtain it.

Acknowledgements. Supported in part by the Bulgarian Science Fund under Grant MM-706.

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