Weighting Patterns and the Controllability and Observability of Time-Variable 2D Continuous-Discrete Linear Systems

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Abstract

A class of 2D systems is considered, which is the continuous-discrete analogue of Attasi's 2D discret model, generalized to the time-variable framework. The input-output map of these systems is established and main properties like controllability and observability are examined in detail. The weighting pattern is defined and the connection of the minimality of a realization and its controllability and observability is emphasized.

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1 Introduction

In [1] Attasi introduced a class of 2D discrete-time time-invariant models which is the closest to the usual 1D systems. The continuous-discrete counterpart of this class was studied in [6]. Other such time-invariant 2D models were examined by Kaczorek [4]. The continuous-discrete models appear in many problems like the iterative learning control synthesis [5] or repetitive processes [3].

The aim of this paper is the study of time-variable 2D continuous-discrete linear systems. By using the (continuous and discrete) fundamental matrices of the drift matrices the form of the input-output map of such systems is established. The concepts of controllability and observability are defined and the characterization of completely controllable and completely observable systems is performed by means of some extensions of the usual controllability and observability Gramians.

A weighting pattern is associated to this class of systems and its realizability is discussed. It is shown that a system is a minimal realization of a weighting pattern iff it is both completely controllable and completely observable on some interval.

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2 State space representation

A 2D *continuous-discrete linear system* is a quintuplet

$$\Sigma = (A_1(t), A_2(k), B(t,k), C(t,k), D(t,k)),$$

where $t \in \mathbf{R}_+$ is the continuous time, $k \in \mathbf{Z}_+$ is the discrete time, $A_1(t)$ and $A_2(k)$ are commutative $n \times n$ real matrices for any t and k, $B(t,k)$, $C(t,k)$, $D(t,k)$ are respectively $n \times m$, $p \times n$ and $p \times m$ real matrices. The following equalities represent the *state* and the *output* equation of the system Σ:

$$\begin{aligned} \dot{x}(t,k+1) &= A_1(t)x(t,k+1) + A_2(k)\dot{x}(t,k) - \\ &- A_1(t)A_2(k)x(t,k) + B(t,k)u(t,k) \end{aligned} \tag{2.1}$$

$$y(t,k) = C(t,k)x(t,k) + D(t,k)u(t,k), \tag{2.2}$$

where $x(t,k) \in \mathbf{R}^n$, $u(t,k) \in \mathbf{R}^m$ and $y(t,k) \in \mathbf{R}^p$ are respectively the *state*, the *input* and the *output* of Σ at the moment $(t,k) \in \mathbf{R}_+ \times \mathbf{Z}_+$ and $\dot{x}(t,k) = \dfrac{\partial x}{\partial t}(t,k)$; the number n is said to be the *dimension* of Σ and it is denoted $\dim \Sigma$.

We denote by $\Phi(t,t_0)$ the fundamental matrix of $A_1(t)$, i.e., the unique matrix solution of the system

i) $\dfrac{d\Phi}{dt}(t,t_0) = A_1(t)\Phi(t,t_0)$,

ii) $\Phi(t_0,t_0) = I$.
It is well known that $\Phi(t,t_0)$ has the following properties:

iii) $\Phi(t,t_1)\Phi(t_1,t_0) = \Phi(t,t_0)$ (the semigroup property);

iv) $\Phi(t,t_0)^{-1} = \Phi(t_0,t)$ and v) the solution of the initial value problem $\dot{x}(t) = A_1(t)x(t) + b(t)$, $x(t_0) = x_0$ is given by the *variation-of-parameters* formula

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)b(s)ds.$$

Let us define the *discrete-time fundamental matrix* of $A_2(k)$ by

$$F(k,l) = \begin{cases} A_2(k-1)A_2(k-2)\dots A_2(l+1)A_2(l) & \text{if} \quad k > l \\ I & \text{if} \quad k = l. \end{cases} \tag{2.3}$$

Under the hypothesis (H): "All matrices $A_2(k)$, $k \in \mathbf{Z}_+$, are nonsingular", we can define $F(k,l)$ for $k < l$ as

$$F(k,l) = [A_2(l-1)A_2(l-2)\dots A_2(k+1)A_2(k)]^{-1} \quad \text{if} \quad k < l. \tag{2.4}$$

The discrete-time fundamental matrix has the following properties:

i') $F(k+1,l) = A_2(k)F(k,l)$;

ii') $F(l,l) = I$;

iii') $F(k,k_1)F(k_1,k_0) = F(k,k_0)$ for any $k \geq k_1 \geq k_0 \geq 0$ (and for any $k, k_1, k_0 \in \mathbf{Z}_+$ under the hypothesis (H);

iv') under the hypothesis (H) $F(l,k) = F(k,l)^{-1}, \ \forall k,l \in \mathbf{Z}_+$;

v') the solution of the initial value problem $x(k+1) = A_2(k)x(k) + b(k)$, $x(k_0) = x_0$ is given by the *discrete-time variation-of-parameters* formula

$$x(k) = F(k,k_0)x_0 + \sum_{l=k_0}^{k-1} F(k,l+1)b(l).$$

Since $A_1(t)A_2(k) = A_2(k)A_1(t)$ for any $t \in \mathbf{R}_+$ and $k \in \mathbf{Z}_+$, by using Peano-Baker formula for $\Phi(t,t_0)$ and the definition (2.3) of $F(k,k_0)$ we can prove that

$$\Phi(t,t_0)F(k,k_0) = F(k,k_0)\Phi(t,t_0), \ \ \forall t,t_0 \in \mathbf{R}_+, \ \ k,k_0 \in \mathbf{Z}_+;$$

the commutativity of the fundamental matrices will be very useful in some of the calculi below.

Definition 2.1. If the state x of Σ verifies the boundary conditions

$$\begin{aligned} x(t,k_0) &= \Phi(t,t_0)x_0 \quad && \forall t,t_0 \in \mathbf{R}_+,, \ t \geq t_0 \\ x(t_0,k) &= F(k,k_0)x_0 \quad && \forall k,k_0 \in \mathbf{Z}_+,, \ k \geq k_0 \end{aligned} \tag{2.5}$$

for some $x_0 \in \mathbf{R}^n$, then x_0 is called the *initial state of* Σ *at the moment* (t_0,k_0).

Now, if x_0 is the initial state of Σ at (t_0,k_0), the state equation (2.1) splits in two state equations, one continuous and one discrete, by introducing a vector $\tilde{x}(t,k) \in \mathbf{R}^n$, $t \in \mathbf{R}_+$, $k \in \mathbf{Z}_+$:

$$\tilde{x}(t,k) = x(t,k+1) - A_2(k)x(t,k). \tag{2.6}$$

From (2.5) we get, for any $k \geq k_0$

$$\tilde{x}(t_0,k) = 0. \tag{2.7}$$

Indeed, (2.5), (2.6) and i') imply

$$\tilde{x}(t_0,k) = x(t_0,k+1) - A_2(k)x(t_0,k) = F(k+1,k_0)x_0 - A_2(k)F(k,k_0)x_0 = 0.$$

By replacing $\tilde{x}(t,k)$ in (2.1), this equation becomes the state equation of the 1D continuous system

$$\frac{d\tilde{x}}{dt}(t,k) = A_1(t)\tilde{x}(t,k) + B(t,k)u(t,k)$$

and, taking into account the initial condition (2.7), the variation-of-parameters formula gives the solution

$$\tilde{x}(t,k) = \int_{t_0}^{t} \Phi(t,s)B(s,k)u(s,k)ds. \tag{2.8}$$

We can write (2.6) as the state equation of the 1D discrete linear system

$$x(t,k+1) = A_2(t)x(t,k) + \tilde{x}(t,k)$$

and by using the discrete-time variation-of parametrs formula we get the solution

$$x(t,k) = F(k,k_0)x(t,k_0) + \sum_{l=k_0}^{k-1} F(k,l+1)\tilde{x}(t,l);$$

hence, by replacing $\tilde{x}(t,l)$ and $x(t,k_0)$ given respectively by (2.8) and (2.5) we obtain the formula of the state of the 2D system Σ at the moment $(t,k) \in \mathbf{R}_+ \times \mathbf{Z}_+$:

$$x(t,k) = \Phi(t,t_0)F(k,k_0)x_0 + \int_{t_0}^{t} \sum_{l=k_0}^{k-1} \Phi(t,s)F(k,l+1)B(s,l)u(s,l)ds. \tag{2.9}$$

The *input-output map* of the system Σ results by replacing (2.9) into the output equation (2.2):

$$\begin{aligned} y(t,k) &= C(t,k)\Phi(t,t_0)F(k,k_0)x_0 + \\ &+ \int_{t_0}^{t} \sum_{l=k_0}^{k-1} C(t,k)\Phi(t,s)F(k,l+1)B(s,l)u(s,l)ds + D(t,k)u(t,k) \end{aligned} \tag{2.10}$$

3 Controllability

In this section we need only the state equation (2.1) of the system Σ, such that we shall consider 2D systems of the form $\Sigma = (A_1(t), A_2(t), B(t,k))$.

The triplet $(t,k,x) \in \mathbf{R}_+ \times \mathbf{Z}_+ \times \mathbf{R}^n$ is called a *phase* of Σ if $x(t,k) = x$, i.e. if Σ is in the state x at time (t,k).

Definition 3.1. A phase (t,k,x) of Σ is said to be *controllable* (or *controllable on* $[t,t_1] \times [k,k_1]$) if there exist some moments $t_1 > t$, $k_1 > k$ and some control u which transfers the phase (t,k,x) to $(t_1,k_1,0)$.

A phase (t,k,x) of Σ is said to be *reachable* (or *reachable on* $[t_0,t] \times [k_0,k]$) if there exist some moments $t_0 < t$, $k_0 < k$ and some control u which transfers the phase $(t_0,k_0,0)$ to (t,k,x).

If, for some fixed $t_0, t \in \mathbf{R}_+$, $k_0, k \in \mathbf{Z}_+$, any phase is controllable (reachable) on $[t_0,t] \times [k_0,k]$, the system Σ is said to be *completely controllable* (*completely reachable*) *on* $[t_0,t] \times [k_0,k]$.

Let us replace in the state equation (2.9) $x(t,k) = \bar{x}$ and $x_0 = 0$. It results that the phase $(t,k,\bar{x})$ is reachable on $[t_0,t] \times [k_0,k] \subset \mathbf{R}_+ \times \mathbf{Z}_+$ iff there exists some control $u(\cdot,\cdot)$ defined on this interval such that

$$\bar{x} = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} \Phi(t,s)\ F(k,l+1)\ B(s,l)\ u(s,l)\ ds. \tag{3.1}$$

Similarly, for $x(t,k)=0$ in (2.6), it results that the phase (t_0,k_0,x_0) is controllable on $[t_0,t]\times[k_0,k]\subset \mathbb{R}_+\times \mathbb{Z}_+$ iff there exists some control $u(\,\cdot\,,\,\cdot\,)$ defined on this interval such that

$$\Phi(t,t_0)F(k,k_0)x_0=-\int_{t_0}^{t}\sum_{l=k_0}^{k-1}\Phi(t,s)F(k,l+1)B(s,l)u(s,l)ds. \tag{3.2}$$

We associate to Σ the *first controllability Gramian*

$$\mathbf{C}_1(t_0,t;k_0,k)=\int_{t_0}^{t}\sum_{l=k_0}^{k-1}\Phi(t_0,s)F(k,l+1)B(s,l)B(s,l)^TF(k,l+1)^T\Phi(t_0,s)^Tds.$$

It can be proved that $\mathbf{C}_1=\mathbf{C}_1(t_0,t;k_0,k)$ is a symmetrical non-negative $n\times n$ matrix.
Theorem 3.2. it It is possible to transfer the phase (t_0,k_0,x_0) to the phase (t,k,x) iff

$$\Phi(t_0,t)x-F(k,k_0)x_0\in Im\mathbf{C}_1(t_0,t;k_0,k). \tag{3.3}$$

Proof. *Sufficiency.* From (3.3) it results that there exists some $v\in\mathbf{R}^n$ such that $\Phi(t_0,t)x-F(k,k_0)x_0=\mathbf{C}_1(t_0,t;k_0,k)v$. By premultiplying this equality by $\Phi(t,t_0)$ and by considering the control

$$u(s,l)=B(s,l)^TF(k,l+1)^T\Phi(t_0,s)^Tv \tag{3.4}$$

we obtain

$$x=\Phi(t,t_0)F(k,k_0)x_0+\int_{t_0}^{t}\sum_{l=k_0}^{k-1}\Phi(t,s)F(k,l+1)B(s,l)u(s,l)ds \tag{3.5}$$

hence by (2.9) we have $x(t,k)=x$, i.e. $u(s,l)$ realizes the desired transfer.

Necessity. Let us denote $x_1=\Phi(t_0,t)x-F(k,k_0)x_0$ and let us assume that a control u_1 realizes the transfer, hence (3.5) holds with u_1 instead of u. Then, by premultiplying this equality by $\Phi(t_0,t)$ we obtain

$$x_1=\int_{t_0}^{t}\sum_{l=k_0}^{k-1}\Phi(t_0,s)F(k,l+1)B(s,l)u_1(s,l)ds. \tag{3.6}$$

Since the controllability Gramian is a symmetrical matrix, the state space $\mathbf{R}^n$ splits in the direct-sum decomposition $\mathbf{R}^n=Im\mathbf{C}_1\oplus Ker\mathbf{C}_1$, hence $\exists! x_2,x_3$ with $x_2\in Im\mathbf{C}_1$, $x_3\in Ker\mathbf{C}_1$ and $x_1=x_2+x_3$. Since $x_2\in Im\mathbf{C}_1$, as in the sufficieny part we obtain a control u_2 of the form (3.4) such that

$$x_2=\int_{t_0}^{t}\sum_{l=k_0}^{k-1}\Phi(t_0,s)F(k,l+1)B(s,l)u_2(s,l)ds. \tag{3.7}$$

Then, the difference between equations (3.6) and (3.7) gives, for $u_3=u_1-u_2$

(3.8) $$x_3 = x_1 - x_2 = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} \Phi(t_0, s) F(k, l+1) B(s, l) u_3(s, l) ds.$$

Now, since $x_3 \in Ker\, \mathbf{C}_1$, we have $x_3^T \mathbf{C}_1 x_3 = 0$, relation which can be written as

$$\int_{t_0}^{t} \sum_{l=k_0}^{k-1} \left\| B(s,l)^T F(k, l+1)^T \Phi(t_0, s)^T x_3 \right\|^2 ds = 0.$$

The integrand and each term of the sum being non-negative, we get

(3.9) $$B(s,l)^T F(k,l+1)^T \Phi(t_0,s)^T x_3 = 0 \text{ a.e. on } [t_0, t] \times [k_0, k] \subset \mathbf{R}_+ \times \mathbf{Z}_+ .$$

From (3.8) and (3.9) we obtain

$$||x_3||^2 = x_3^T x_3 = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} u_3(s,l)^T B(s,l)^T F(k,l+1)^T \Phi(t_0,s)^T x_3 ds = 0,$$

hence $x_3 = 0$ and $x_1 = x_2 \in Im\mathbf{C}_1$. □

Under the hypothesis (H): "All matrices $A_2(l)$, $l \in \mathbf{Z}_+$ are non-singular" we can define the *second controllability Gramian*

$$\mathbf{C}_2(t_0, t; k_0, k) = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} \Phi(t_0, s) F(k_0, l+1) B(s,l) B(s,l)^T F(k_0, l+1)^T \Phi(t_0, s)^T ds$$

and we can prove

Theorem 3.3. *Under the hypothesis (H), it is possible to transfer the phase* (t_0, k_0, x_0) *to* (t, k, x) *iff*

(3.10) $$\Phi(t_0, t) F(k_0, k) x - x_0 \in Im\mathbf{C}_2(t_0, t; k_0, k).$$

Proof. Since $F(k_0, k)$ is well defined for $k_0 < k$ (as $F(k, k_0)^{-1}$), premultiplication of (3.3) by $F(k_0, k)$ shows the equivalence of (3.3) and (3.10). □

From (3.10) with $x = 0$ it results that the phase (t_0, k_0, x_0) is controllable on $[t_0, t] \times [k_0, k] \subset \mathbf{R}_+ \times \mathbf{Z}_+$, iff $x_0 \in Im\mathbf{C}_2(t_0, t; k_0, k)$, hence we have:

Corollary 3.4. *Under the hypothesis (H), the set of all controllable states on* $[t_0, t] \times [k_0, k]$ *is the space* $X_c = Im\,\mathbf{C}_2(t_0, t; k_0, k)$.

Since Σ is completely controllable iff $X_c = \mathbf{R}^n$, we obtain

Theorem 3.5. *Under the hypothesis (H),* Σ *is completely controllable on* $[t_0, t] \times [k_0, k]$ *iff rank* $\mathbf{C}_2(t_0, t; k_0, k) = n$.

In a similar way we can introduce the *reachability Gramian* of the system Σ

$$R(t_0, t; k_0, k) = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} \Phi(t, s) F(k, l+1) B(s,l) B(s,l)^T F(k, l+1)^T \Phi(t, s)^T ds$$

and we can prove

Theorem 3.6. Σ *is completely reachable on* $[t_0, t] \times [k_0, k]$ *iff*

$$rank\; R(t_0, t; k_0, k) = n.$$

4 Observability

Since the concept of observability involves only the drift and the output matrices, in this paragraph a system will be a triplet $\Sigma = (A_1(t), A_2(k), C(k,l))$.

Definition 4.1. The phase $(t_0, k_0, x) \in \mathbf{R}_+ \times \mathbf{Z}_+ \times (\mathbf{R}^n \setminus \{0\})$ is said to be *unobservable* (*unobservable on* $[t_0,t]\times[k_0,k]$), if for any control u it provides the same output $y(s,l)$, $s \geq t_0$, $l \geq k_0$ ($(s,l) \in [t_0,t] \times [k_0,k]$) as the phase $(t_0, k_0, 0)$.

The system Σ is said to be *completely observable at time* (t_0, k_0) if there is no unobservable phase (t_0, k_0, x). Σ is said to be *completely observable on* $[t_0,t] \times [k_0,k]$ if there is no phase (t_0, k_0, x) unobservable on $[t_0,t] \times [k_0,k]$.

A characterization of the unobservable states us given by

Proposition 4.2. The phase (t_0, k_0, x) is unobservable (unobservable on $[t_0,t] \times [k_0,k]$) iff

$$C(s,l)\Phi(s,t_0)F(l,k_0)x = 0 \tag{4.1}$$

for any time $(s,l) \geq (t_0,k_0)$ (for any $(s,l) \in [t_0,t] \times [k_0,k]$).

Proof. By replacing in the expression of the input-output map (2.10) x_0 succesively by x and 0, we get that the initial states x and 0 provide the same output for some control u iff

$$C(t,k)\Phi(t,t_0)F(k,k_0)x + \int_{t_0}^{t} \sum_{l=k_0}^{k-1} C(t,k)\Phi(t,s)F(k,l+1)B(s,l)u(s,l)ds +$$

$$+D(t,k)u(t,k) = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} C(t,k)\Phi(t,s)F(k,l+1)B(s,l)u(s,l)ds + D(t,k)u(t,k)$$

and obviously this equality is equivalent to (4.1). □

Similarly with controllability, a completely observable system can be fully characterized by using a suitable Gramian.

By definition, the *observability Gramian* of Σ is the matrix

$$O(t_0,t,k_0,k) = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} F(l,k_0)^T \Phi(s,t_0)^T C(s,l)^T C(s,l)\Phi(s,t_0)F(l,k_0)ds.$$

Obviously, $O = O(t_0, t; k_0, k)$ is a symmetrical non-negative $n \times n$ matrix.

Theorem 4.3. *The phase* (t_0, k_0, x) *is unobservable iff* $x \in Ker\, O(t_0, t; k_0, k)$ *for any* $(t,k) > (t_0, k_0)$.

Proof. *Necessity.* By Proposition 4.2 it results that if (t_0, k_0, x) is unobservable then $Ox = 0$.

Sufficiency. If $x \in Ker\, O$, then $x^T O x = 0$, hence

$$0 = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} x^T F(l,k_0)^T \Phi(s,t_0)^T C(s,l)^T C(s,l)\Phi(s,t_0)F(l,k_0)x ds =$$

$$= \int_{t_0}^{t} \sum_{l=k_0}^{k-1} ||C(s,l)\Phi(s,t_0)F(l,k_0)x||^2 ds.$$

The integrand being non-negative, it results that the sum (of non-negative terms) is zero, hence each term is zero, i.e.

$$C(s,l)\Phi(s,t_0)F(l,k_0)x = 0, \ \ \forall x \geq t_0, \ \ \forall l \geq k_0.$$

From Proposition 4.2 we conclude that the phase (t_0, k_0, x) is unobservable. □

A similar proof gives

Corollary 4.4. *The set of all states x such that the phase (t_0, k_0, x) is unobsevable on $[t_0, t] \times [k_0, k]$ is the subspace $Ker\ O(t_0, t; , k_0, k)$.*

It results that the system Σ is completely observable on $[t_0, t] \times [k_0, k]$ iff the subspace $Ker\ O(t_0, t; k_0, k)$ of $\mathbf{R}^n$ reduces to $\{0\}$; therefore we proved

Theorem 4.5. *Σ is completely observable on $[t_0, t] \times [k_0, k]$ iff*

$$rank\ O(t_0, t; k_0, k) = n.$$

5 Weighting patterns

As it was noticed in [2], in the design of systems for control or communication purposes it is much natural to specify a weighting pattern that it is to specify the complete system Σ. We associate to the system $\Sigma = (A_1(t), A_2(k), B(t,k), C(t,k))$ (hence $D(t,k) = 0$) the $p \times m$ matrix W, called the *weighting pattern of* Σ, defined by

$$W = W(t,k,s,l) = C(t,k)\Phi(t,s)F(k,l+1)B(s,l) \tag{5.1}$$

From the input-output map formula (2.10), we obtain for $x_0 = 0$ the general response of Σ, expressed by the means of the weighting pattern:

$$y(t,k) = \int_{t_0}^{t} \sum_{l=k_0}^{k-1} W(t,k,s,l)u(s,l)ds \tag{5.2}$$

hence, for zero initial condition, the weighting pattern completely determines the input-output behaviour of the system.

Definition 5.1. A $p \times m$ matrix $W(\cdot,\cdot,\cdot,\cdot)$ defined on $\mathbf{R}_+ \times \mathbf{Z}_+ \times \mathbf{R}_+ \times \mathbf{Z}_+$ is said to be *realizable as a weighting pattern* if there exists a 2D system Σ such that (5.1) holds. Σ is called a *realization* of the weighting pattern W.

If $\dim \Sigma \leq \dim \tilde{\Sigma}$ for any realization $\tilde{\Sigma}$ of W, then Σ is said to be a *minimal realization* of W.

Although some of the following results are true in the general case, in the sequel we shall consider only realizations fulfilling the hypothesis (H) i.e. realizations Σ with all matrices $A_2(k)$ nonsingular, $k \in \mathbf{Z}_+$; therefore we can use the discrete-time fundamental matrix $F(k,l)$ even for $k < l$.

Theorem 5.2. *A matrix W is realizable as a weighting pattern iff there exist two matrix functions G and H defined on $\boldsymbol{R}_+ \times \boldsymbol{Z}_+$ such that, for any $t, s \in \boldsymbol{R}_+$ and $k, l \in \boldsymbol{Z}_+$, W has the factorization*

$$W(t,k,s,l) = G(t,k)H(s,l). \tag{5.3}$$

Proof. *Sufficiency.* If there exist G and H, let us consider the system $\Sigma^0 = (0, I, H, G)$ (i.e. $A_1 = 0$, $A_2 = I$, $B = H$, $C = G$). Then $\Phi(t,s) = I$, $F(k,l) = I$ and the equality (5.1) follows from (5.3).

Necessity. If $\Sigma = (A_1(t), A_2(k), B(t,k), C(t,k))$ is a realization of W then, for fixed $s_0 \in \mathbf{R}_+$ and $k_0 \in \mathbf{Z}_+$, (5.1) can be written as

$$W(t,k,s,l) = C(t,k)\Phi(t,t_0)F(k,k_0)\Phi(t_0,s)F(k_0,l+1)B(s,l),$$

hence (5.3) holds with $G(t,k) = C(t,k)\Phi(t,t_0)F(k,k_0)$ and
$H(s,l) = \Phi(t_0,s)F(k_0,l+1)B(s,l)$. □

Definition 5.3. The weighting pattern W factorized as in (5.3) is said *to be in reduced form* if the columns of G and the rows of H are both linearly independent sets of functions on $\mathbf{R}_+ \times \mathbf{Z}_+$. Otherwise W is called *reducible.*

If W is in reduced form, the number of columns of G (equal to the number of rows of H) is called the *order* of W.

Proposition 5.4. *If W is in reduced form and it has the order n, then $n = rank\, W(t,k,s,l)$.*

Proof. By Sylvester's Inequality we get from (5.3):

$$rank\, G(t,k) + rank\, H(s,l) - n \le rank\, W(t,k,s,l) \le \min(rank\, G(t,k), rank\, H(s,l)).$$

Since $rank\, G(t,k) = rank\, H(s,l) = n$ it results $n \le rank\, W(t,k,s,l) \le n$. □

Proposition 5.5. If the weighting pattern W is in reduced form then the dimension of any minimal realization is the same as the order of W.

Proof. Let n be the order of W and let us assume that W has a realization Σ with $\dim \Sigma = \tilde{n} < n$. Then W has the factorization determined in Theorem 5.2 in which the number of columns of $G(t,k)$ and the number of rows of $H(s,l)$ is at most $\tilde{n}$; this contradicts the assumption that W has the order n. □

Theorem 5.6. *A weighting pattern W always admits a minimal realization.*

Proof. Let $W = GH$ be a factorization of W. By elementary row operations we can transform the matrix H into a matrix $\tilde{H} = MH = \begin{bmatrix} H_1 \\ 0 \end{bmatrix}$ where the rows of H_1 are linearly independent and M is a unimodular matrix. Let us partition the matrix $\tilde{G} = GM^{-1}$ as $\tilde{G} = [G_1\ G_{12}]$ correspondingly. Then we have $W = GH = \tilde{G}\tilde{H} = G_1H_1$. Now G_1 can be transformed by elementary column operations into a matrix $\tilde{G}_1 = G_1P = [G_2\ 0]$ where the columns of G_2 are linearly independent and P is unimodular. Let us partition the matrix $\tilde{H}_1 = P^{-1}H_1$ as $\tilde{H}_1 = \begin{bmatrix} H_2 \\ H_{21} \end{bmatrix}$. We obtain $W = G_1H_1 = \tilde{G}_1\tilde{H}_1 = G_2H_2$, hence $W = G_2H_2$ is now in reduced form. By Theorem 5.2 and Proposition 5.5 it results that W has the minimal realization $\Sigma^0 = (0, I, H_2, G_2)$. □

The following theorem emphasizes the connection between controllability, observability and minimality. This result is similar to that concerning 1D systems, see [2, §15, Theorem 2].

Theorem 5.7. *The system* $\Sigma = (A_1(t), A_2(k), B(t,k), C(t,k))$ *is a minimal realization of the weighting pattern* $W(t,k,s,l)$ *iff* Σ *is both completely controllable and completely observable on some interval* $[t_0,t_1]\times[k_0,k_1] \in \boldsymbol{R}_+ \times \boldsymbol{Z}_+$.

Proof. *Sufficiency.* Let us assume that Σ is not minimal and let

$$\tilde{\Sigma} = (\tilde{A}_1(t), \tilde{A}_2(k), \tilde{B}(t,k), \tilde{C}(t,k))$$

be a realization of dimension $\tilde{n}$ with $\tilde{n} < n = \dim \Sigma$. Let us consider four arbitrary fixed numbers $t_0, t_1 \in \mathbf{R}_+$, $k_0, k_1 \in \mathbf{Z}_+$. We denote by $G(t,k), \tilde{G}(t,k), H(s,l), \tilde{H}(s,l)$ respectively the matrices

$$G(t,k) = C(t,k)\Phi(t,t_0)F(k,k_0), \qquad \tilde{G}(t,k) = \tilde{C}(t,k)\Phi(t,t_0)\tilde{F}(k,k_0),$$

$$H(s,l) = \Phi(t_0,s)F(k_0,l+1)B(s,l), \qquad \tilde{H}(s,l) = \tilde{\Phi}(t_0,s)\tilde{F}(k_0,l+1)\tilde{B}(s,l),$$

where $\tilde{\Phi}(t_0,s)$ and $\tilde{F}(k_0,l+1)$ are the (continuous-time and discrete-time) fundamental matrices of $\tilde{A}_1(t)$ and $\tilde{A}_2(k)$ respectively. Since

$$W(t,k,s,l) = G(t,k)H(s,l) = \tilde{G}(t,k)\tilde{H}(s,l),$$

we have

$$C(t,k)\Phi(t,t_0)F(k,k_0)\Phi(t_0,s)F(k_0,l+1)B(s,l) =$$
$$= \tilde{C}(t,k)\tilde{\Phi}(t,t_0)\tilde{F}(k,k_0)\tilde{\Phi}(t_0,s)\tilde{F}(k_0,l+1)\tilde{B}(s,l).$$

Now let us premultiply and postmultiply this equality by $\Phi(t,t_0)^T F(k,k_0)^T\ C(t,k)^T$ and $B(s,l)^T F(k_0,l+1)^T \Phi(t_0,s)^T$ respectively, then let us integrate the obtained equality with respect to t and s over the square $[t_0,t_1]\times[t_0,t_1]$ and take the summation with respect to k and l over $[k_0,k_1-1]\times[k_0,k_1-1]$. It results

$$O(t_0,t_1;k_0,k_1)\ \mathbf{C}_1(t_0,t_1;k_0,k_1) =$$
$$= \int_{t_0}^{t} \sum_{k=k_0}^{k_1-1} \Phi(t,t_0)^T F(k,k_0)^T C(t,k)^T \tilde{F}(k,k_0)\tilde{\Phi}(t,t_0)dt \times$$
$$\times \int_{t_0}^{t} \sum_{l=k_0}^{k_1-1} \tilde{\Phi}(t_0,s)\tilde{F}(k_0,l+1)\tilde{B}(s,l)B(s,l)^T F(k_0,l+1)^T \Phi(t_0,s)^T ds.$$

Since the two integrals in the right hand member are constant matrices with $\tilde{n}$ columns and $\tilde{n}$ rows respectively, it results from Sylvester's Inequality that the rank of their product is less than or equal to $\tilde{n}$, $\tilde{n} < n$; then at least one of the matrices $O(t_0,t_1;k_0,k_1)$ and $\mathbf{C}_1(t_0,t_1;k_0,k_1)$ has the rank less than n. Therefore, by Theorems 3.5 and 4.5 it results that Σ is not both completely controllable and completely observable on $[t_0,t_1]\times[k_0,k_1]$.

Necessity. Assume that for any $t_0, t_1, k_0, k_1, \Sigma$ is not both completely controllable and completely observable. By using the notations of the sufficiency part and the notations O and $\mathbf{C}$ for $O(t_0,t_1;k_0,k_1)$ and $\mathbf{C}_1(t_0,t_1;k_0,k_1)$ respectively, we obtain

$$O = \int_{t_0}^{t_1} \sum_{l=k_0}^{k_1-1} G(s,l)^T G(s,l) ds \text{ and } \mathbf{C} = \int_{t_0}^{t_1} \sum_{l=k_0}^{k_1-1} H(s,l) H(s,l)^T ds.$$

Since these matrices are non-negative definite, there exist two nonsingular matrices U and V and two signature matrices S_1, S_2 with $S_1^2 = S_1$, $S_2^2 = S_2$, at least one of S_1 and S_2 having the rank less than n, such that $US_1U^T = \mathbf{C}$, $V^T S_2 V = O$. Then

$$\begin{aligned} 0 &\leq \int_{t_0}^{t_1} \sum_{l=k_0}^{k_1-1} [US_1U^{-1}H(s,l) - H(s,l)][US_1U^{-1}H(s,l) - H(s,l)]^T ds = \\ &= US_1U^{-1}\mathbf{C}U^{-T}S_1U^T - US_1U^{-1}\mathbf{C} - \\ &- \mathbf{C}U^{-T}S_1U^T + \mathbf{C} = US_1U^T - US_1U^T - US_1U^T + US_1U^t = 0, \end{aligned}$$

hence $US_1U^{-1}H(s,l) = H(s,l)$ a.e. In the same manner we can prove that the equality $G(t,k)V^{-1}S_2V = G(t,k)$ holds a.e. on $[t_0,t_1] \times [k_0,k_1]$. Then the weighting pattern $W(t,k,s,l) = G(t,k)H(t,k)$ can be factorized as

$$W(t,k,s,l) = G(t,k)V^{-1}S_2VUS_1U^{-1}H(s,l). \tag{5.4}$$

Let us assume that $rank\ S_2 < n$, hence $S_2 = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}$ where $q < n$. If we partition the following matrices as

$$G(t,k)V^{-1} = [G_1(t,k)\ G_2(t,k)],\ VUS_1U^{-1}H(s,l) = \begin{bmatrix} H_1(s,l) \\ H_2(s,l) \end{bmatrix},$$

where $G_1(t,l)$ and $H_1(s,l)$ have respectively q columns and q rows, then from (5.4) we obtain the factorization

$$W(t,k,s,l) = G_1(t,k)H_1(s,l).$$

It results that the order of W is less than n, hence by Proposition 5.4 Σ is not minimal.

6 Conclusion

We have presented a class of time-variable continuous-discrete 2D linear sysyem. Its state space representation allowed the development of a theory which contains the most important notions and results belonging to system theory, concerning reachability and controllability, observability, weighting patterns and minimality. This study can be continued by including further results referring for instance to stability, feedback or optimal control.

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