Curvature Tensors on A-Einstein Sasakian Manifolds

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Abstract

The author and Mishra [1] have introduced some curvature tensors to study their physical and geometric properties. In this paper, W_2-curvature tensor, its associated symmetric and skew-symmetric tensors are studied in an A-Einstein Sasakian manifold.

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1 Introduction

Let us consider an n-dimensional real differential manifold M_n. Let there exist a vector valued linear function F, a 1-form A and a vector field T, satisfying

$$(1.1) \qquad (a) \quad \overline{\overline{X}} + X = A(X)T, \quad (b) \quad \overline{X} \underset{=}{def} F(X)$$

for any arbitrary vector field X. Then M_n is called an *almost contact manifold*, and the structure (F, T, A) is called an *almost contact structure*.

From (1.1) we have, $rank(F) = n - 1$, n is odd, say $2m + 1$, and that $\overline{T} = 0$,

$$(1.2) \qquad A(\overline{X}) = 0, \qquad A(T) = 1.$$

In addition, if in M_n there exists a metric tensor g satisfying

$$(1.3a) \qquad g(\overline{X}, \overline{Y}) = g(X, Y) - A(X)A(Y),$$

then M_n is called an *almost Grayan manifold*. From (1.1) and (1.3)a, we have

$$(1.3b) \qquad g(X, T) = A(X).$$

Putting $'F(X, Y) = g(\overline{X}, Y)$ then, we have

$$(1.4) \qquad \begin{array}{ll} (a) & 'F(\overline{X}, \overline{Y}) = -g(X, \overline{Y}) = g(\overline{X}, Y) = \ 'F(X, Y) \\ (b) & 'F(X, Y) +' F(Y, X) = 0 \end{array}$$

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If in an almost Grayan Manifold,

$$'F(X,Y) = (D_X A)(Y) - (D_Y A)(X) = (dA)(X,Y),$$

where D is a Riemannian connexion, then M_n is called an *almost Sasakian manifold.* In a Sasakian manifold, F is closed. An almost Sasakian manifold is said to be *Sasakian manifold*, if T is killing vector:

$$(D_X A)(Y) + (D_Y A)(X) = 0.$$

Thus in a Sasakian manifold

$$'F(X,Y) = (D_X A)(Y) \quad \text{and} \quad (D_X\ 'F)(Y,Z) = \ 'R(X,Y,Z,T),$$

where R is the curvature tensor of the type $(0,4)$ of M_n. In a Sasakian manifold, we have [2]

$$(1.5)\qquad \begin{aligned} &(a)\quad 'R(T,X,Y,T) = g(\overline{X},\overline{Y}) = g(X,Y) - A(X)A(Y)\\ &(b)\quad 'R(X,Y,Z,T) = A[R(X,Y,Z)] = A(X)g(Y,Z) - A(Y)g(X,Z)\\ &(c)\quad 'R(T,X,Y,Z) = A(Z)g(X,Y) - g(X,Z)A(Y)\\ &(d)\quad 'R(T,Y,Z) = g(Y,Z)T - A(Z)Y\\ &(e)\quad 'R(X,Y,T) = A(Y)X - A(X)Y, \end{aligned}$$

where $'R(X,Y,Z,U) = g[R(X,Y,Z),U]$ and

$$(1.6)\qquad \begin{aligned} &(a)\quad Ric(X,T) = g(r(X),T) = A(r(X)) = (n-1)A(X)\\ &(b)\quad Ric(\overline{X},Y) + Ric(X,\overline{Y}) = 0, \end{aligned}$$

where Ric is the Ricci tensor.

The Sasakian manifold M_n is called an *A-Einstein Sasakian manifold*, if the Ricci tensor satisfies [3]

$$(1.7)\qquad Ric(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y)$$

for some scalar fields α and β.

The author and Mishra [1] have defined a tensor

(1.8)

$$'W_2(X,Y,Z,U) = \ 'R(X,Y,Z,U) + \frac{1}{n-1}\big[g(X,Z)Ric(Y,U) - g(Y,Z)Ric(X,U)\big]$$

This tensor is skew symmetric in X and Y, therefore, breaking it into skew symmetric part and symmetric part in Z and U , we obtained

$$(1.9)\qquad \begin{aligned} 'E(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{2(n-1)}\Big[&g(X,Z)Ric(Y,U)-\\ -g(Y,Z)Ric(X,U) - g(X,U)Ric(Y,Z) + &g(Y,U)Ric(X,Z)\Big] \end{aligned}$$

$$(1.10)\qquad \begin{aligned} 'F(X,Y,Z,U) = \frac{1}{2(n-1)}[&g(X,Z)Ric(Y,U)-\\ -g(Y,Z)Ric(X,U) + g(X,U)Ric(Y,Z) - &g(Y,U)Ric(X,Z)] \end{aligned}$$

The tensor $'E(X,Y,Z,U)$ possesses all the symmetric and skew-symmetric properties of $'R(X,Y,Z,U)$ as well as the cyclic property. The tensor $'F(X,Y,Z,U)$ satisfies only the cyclic property with fixed U.

2 Properties of tensors

In this section we study the properties of W_2, E and F curvature tensors in A-Einstein Sasakian manifolds.

Theorem 2.1. *In an A-Einstein Sasakian manifold, we have*
(2.1)

$$(a)\quad 'W_2(T,Y,Z,U) = \frac{A(Z)}{n-1}[(\alpha-1)g(Y,U)+\beta A(Y)A(U)],$$

$$(b)\quad 'W_2(T,\overline{Y},Z,\overline{U}) = [\frac{\alpha-1}{n-1}][g(Y,U)-A(Y)A(U)]A(Z),$$

$$(c)\quad 'W_2(\overline{X},\overline{Y},\overline{Z},\overline{U}) -' W_2(X,Y,Z,U) = (\frac{\alpha}{n-1}-1)A(Z)A(Y)g(X,U).$$

Proof. From (1.8) we have

$$'W_2(T,Y,Z,U) =' R(T,Y,Z,U) + \frac{1}{n-1}\big[g(T,Z)Ric(Y,U) - g(Y,Z)Ric(T,U)\big].$$

Using (1.3)b,(1.5)c,(1.6)a, and (1.7), we get (2.1)a. Barring Y and U in (2.1)a and using (1.2), we get (2.1)b. Barring all the vector fields in (1.8) and using (1.2),(1.3),(1.5) and (1.7), we get (2.1)c. □

Corollary 2.1. *For A-Einstein Sasakian manifold, we have*

$$'W_2(T,Y,\overline{Z},U) = 0,$$

$$'W_2(T,\overline{Y},Z,U) = \left[\frac{\alpha-1}{n-1}\right]A(Z)g(\overline{Y},U),$$

$$'W_2(T,Y,Z,\overline{U}) = \left[\frac{\alpha-1}{n-1}\right]A(Z)g(Y,\overline{U}),$$

$$'W_2(T,\overline{Y},Z,U) +' W_2(T,Y,Z,\overline{U}) = 0.$$

Proof. Using (1.2), (1.3)b, (1.4)a, (1.7) and (1.8), we get the results. □

Theorem 2.2. *In an A-Einstein Sasakian manifold, we have*

$$(a)\quad 'E(T,Y,Z,U) = \frac{1}{2}\left[1-\frac{\alpha}{n-1}\right]\big\{A(U)g(Y,Z)-A(Z)g(Y,U)\big\},$$

$$(2.2)\qquad (b)\quad 'E(X,Y,Z,T) = \frac{1}{2}\left[1-\frac{\alpha}{n-1}\right]\big\{A(X)g(Y,Z)-A(Y)g(X,Z)\big\},$$

$$(c)\quad 'E(T,Y,Z,T) = \frac{1}{2}\left[g(Y,Z)(1-\frac{\alpha}{n-1})-\frac{\beta}{n-1}A(Y)A(Z)\right].$$

Proof. Using (1.9), (1.3)b, (1.5), (1.6) and (1.7), we get the results. □

Corollary 2.2. *For the A-Sasakian manifold, we have*

$$'E(T,Y,Z,\overline{U}) = -\frac{1}{2}\left[1-\frac{\alpha}{n-1}\right]A(Z)g(Y,\overline{U}),$$
$$'E(\overline{Y},Z,U,T) = -\frac{1}{2}\left[1-\frac{\alpha}{n-1}\right]A(Z)g(\overline{Y},U),$$
$$'E(\overline{Y},Z,U,T) + \ 'E(T,Y,Z,\overline{U}) = 0,$$
$$'E(T,\overline{Y},\overline{Z},T) = \frac{1}{2}\left[1-\frac{\alpha}{n-1}\right]\{g(Y,Z) - A(Y)A(Z)\}.$$

Proof. Using (1.2), (1.3)b, (1.4)a, (1.7) and (1.9), we get the results. □

Theorem 2.3. *For an A-Einstein Sasakian manifold, we have*

(2.3)
$$(a)\quad 'F(X,Y,Z,T) = \frac{\beta}{2(n-1)}\left[g(X,Z)A(Y) - g(Y,Z)A(X)\right]$$
$$(b)\quad 'F(T,Y,Z,U) = \frac{\beta}{2(n-1)}\left[2A(Y)A(Z)A(U) - \right.$$
$$\left. -A(U)g(Y,Z) - A(Z)g(Y,U)\right]$$
$$(c)\quad 'F(T,Y,Z,T) = \frac{\beta}{2(n-1)}\left[A(Y)A(Z) - g(Y,Z)\right].$$

Proof. Using (1.10), (1.3)b, (1.6) and (1.7), we get the result. □

Corollary 2.3. *In an A-Einstein Sasakian manifold, we have*

$$'F(\overline{X},Y,Z,T) = \frac{\beta}{2(n-1)}\left[g(\overline{X},Z)A(Y)\right]$$
$$'F(\overline{X},\overline{Y},Z,T) =' F(T,Y,\overline{Z},\overline{W}) = 0$$
$$'F(T,\overline{Y},\overline{Z},T) = \frac{-\beta}{2(n-1)}\left[g(Y,Z) - A(Y)A(Z)\right]$$
$$'F(T,Y,Z,\overline{U}) = \frac{-\beta}{2(n-1)}\left[g(Y,\overline{U}) - A(Z)\right]$$
$$'F(\overline{Y},Z,U,T) +' F(T,Y,Z,\overline{U}) = 0.$$

Proof. Using (1.10), (1.2), (1.3) and (1.7), we get the results. □

3 Symmetric A-Einstein Sasakian manifolds

We consider an A-Einstein Sasakian manifold M_n and define the following

Definition 3.1. The manifold M_n is called *E-Symmetric* and *F-symmetric* provided

(3.1) $$(a)\quad (D_Y E)(Z,U,V) = 0, \quad \text{and} \quad (b)\quad (D_Y F)(Z,U,V) = 0$$

are satisfied, where D_Y denotes the covariant differentiation. From (3.1), we have

$$R(X,Y,E(Z,U,V)) - E((R(X,Y,Z),U,V) - E(Z,R(X,Y,U),V) - E(Z,U,R(X,Y,V)) = 0.$$

This equation implies

$$'R(T,Y,E(Z,U,V),T) -' E((R(T,Y,Z),U,V,T)-$$
$$-'E(Z,R(T,Y,U),V,T) - \ 'E(Z,U,R(T,Y,V),T) = 0.$$

Using (1.3), (1.5), (1.6) and (1.9), we get

$$'E(Z,U,V,Y) + \frac{1}{2}\Big[g(Y,U)\Big\{g(Z,V) - \frac{Ric(Z,V)}{n-1}\Big\} - g(Y,Z)\Big\{g(U,V) - \frac{Ric(U,V)}{n-1}\Big\}+$$
$$+A(Z)A(V)\Big\{g(Y,U) - \frac{Ric(Y,U)}{n-1}\Big\} - A(U)A(V)\Big\{g(Y,Z) - \frac{Ric(Y,Z)}{n-1}\Big\}\Big] = 0.$$

Using (1.7), we find

$$'E(Z,U,V,Y) + \frac{1}{2}(1 - \frac{\alpha}{n-1})\{g(Y,U)g(Z,V)-$$
$$-g(Y,Z)g(U,V) + A(Z)A(V)g(Y,U) - A(U)A(V)g(Y,Z)\}+$$
$$+\frac{\beta}{2(n-1)}\{g(Y,Z)A(U)A(V) - g(Y,U)A(Z)A(V)\} = 0.$$

Thus, we have

Theorem 3.1. *If the A-Einstein Sasakian manifold M_n is E-Symmetric, then E is given by*

(3.2)
$$'E(Z,U,V,Y) = \frac{1}{2}\left(1 - \frac{\alpha}{n-1}\right)\{g(Y,Z)g(U,V)-$$
$$-g(Y,U)g(Z,V) - A(Z)A(V)g(Y,U) + A(U)A(V)g(Y,Z)\}+$$
$$+\frac{\beta}{2(n-1)}[g(Y,U)A(Z)A(V) - g(Y,Z)A(U)A(V)\}.$$

On similar lines, we have the following theorem.

Theorem 3.2. *If an A-Einstein Sasakian manifold M_n is F-Symmetric, then F is given by*

(3.3)
$$'F(Z,U,V,Y) = \frac{\beta}{2(n-1)}[g(Y,U)g(Z,V) - g(Y,Z)g(U,V)].$$

Proof. The proof follows the pattern of Theorem 3.1. □

4 Discussion

The W_2-curvature tensor was introduced on the line of Weyl curvature tensor and by breaking W_2 into skew-symmetric parts the tensor E has been defined. Rainich conditions for the existence of the non-null electrovariance can be obtained by W_2 and E, if we replace the matter tensor by the contracted part of these tensors. The tensor E enables to extend Pirani formulation of gravitational waves to Einstein space [4]. In Sasakian manifold and other manifolds, tensors W_2 and E satisfy properties, some of which are similar to that of Weyl's projective tensor and conformal curvature tensor respectively. Thus, these tensors can alternatively be used to study physical and geometrical properties of manifolds.

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