

Torsion, Curvature and Deflection d-Tensors on $J^1(T, M)$

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Abstract

The aim of this paper is twofold. On the one hand, to study the local representations of d-connections, d-torsions and d-curvatures with respect to adapted bases produced by a nonlinear connection Γ on the jet fibre bundle of order one $J^1(T, M) \rightarrow T \times M$. On the other hand, to open the problem of prolongations of tensors and connections from a product of two manifolds to 1-jet fibre bundle associated to these manifolds.

Section 2 defines the Γ -linear connection on $J^1(T, M)$, and determines its nine local components. Section 3 studies the main twelve components of torsion d-tensor field, and Section 4 describes the eighteen components of curvature d-tensor field, induced by the Γ -linear connection. Section 5 introduces the deflection d-tensors, and, using the Ricci identities, describes the relations that they must satisfy. Section 6 studies the problem of prolongation of vector fields from $T \times M$ to 1-jet space $J^1(T, M)$.

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Key words: 1-jet fibre bundle, nonlinear connection, Γ -linear connection, torsion and curvature d-tensor fields, deflection d-tensors.

1 Introduction

It is well known the jet fibre bundle of order one $J^1(T, M) \rightarrow T \times M$ is a basic object in the study of classical and quantum fields [6]. For that reason, many researchers were studying the differential geometry of this space, in the sense of connections, torsions and curvatures. In this direction, the book [7] develops the geometry of jet fibre bundles of arbitrary orders, which is characterized by a global approach of geometrical objects that are involved. In other way, using as a pattern the geometrical methods from the theory of Lagrange spaces, the book [2] studies the geometry of the particular jet fibre bundle $J^1(\mathcal{R}, M) \equiv \mathcal{R} \times TM \rightarrow M$, in the sense of d-connections, d-torsions and d-curvatures. Also, some interesting geometrical aspects of the jet fibre bundle $J^1(\mathcal{R}, M) \equiv \mathcal{R} \times TM \rightarrow \mathcal{R} \times M$ are exposed in [9].

Extending geometrical results from [2] and [9], and using Lagrangian geometrical methods, our paper analyses the particular local features of the geometrical objects on 1-jet fibre bundle $J^1(T, M)$, produced by a nonlinear connection. This study was imposed by the construction of the contravariant geometrization of multi-time Lagrangians [5], required by important physical domains: theory of bosonic strings [1] or elasticity [6]. At the same time, the authors believe that differential geometry of 1-jet bundles is a very fruitful domain of mathematics, not only because this geometry allows a clear insight to mathematical physics concepts, but also provides many new ideas, suitable for the geometrical theory of PDEs [8].

2 Components of Γ -linear connections on $J^1(T, M)$

Let us consider T (resp. M) a "temporal" (resp. "spatial") manifold of dimension p (resp. n), coordinated by $(t^\alpha)_{\alpha=\overline{1,p}}$ (resp. $(x^i)_{i=\overline{1,n}}$). Let $J^1(T, M) \rightarrow T \times M$ be the jet fibre bundle of order one associated to these manifolds. The *bundle of configurations* $J^1(T, M)$ has the coordinates $(t^\alpha, x^i, x_\alpha^i)$, where $\alpha = \overline{1,p}$ and $i = \overline{1,n}$. Note that, throughout this paper, the indices $\alpha, \beta, \gamma, \dots$ run from 1 to p , and the indices i, j, k, \dots run from 1 to n .

Let us fix a nonlinear connection Γ on $E = J^1(T, M)$, defined by the *temporal* components $M_{(\alpha)\beta}^{(i)}$ and the *spatial* components $N_{(\alpha)j}^{(i)}$. The transformation rules of the local components of the nonlinear connection Γ are expressed by [4]

$$(2.1) \quad \begin{aligned} \tilde{M}_{(\beta)\mu}^{(j)} \frac{\partial \tilde{t}^\mu}{\partial t^\alpha} &= M_{(\gamma)\alpha}^{(k)} \frac{\partial \tilde{x}^j}{\partial x^k} \frac{\partial t^\gamma}{\partial \tilde{t}^\beta} - \frac{\partial \tilde{x}_\beta^j}{\partial t^\alpha}, \\ \tilde{N}_{(\beta)k}^{(j)} \frac{\partial \tilde{x}^k}{\partial x^i} &= N_{(\gamma)i}^{(k)} \frac{\partial \tilde{x}^j}{\partial x^k} \frac{\partial t^\gamma}{\partial \tilde{t}^\beta} - \frac{\partial \tilde{x}_\beta^j}{\partial x^i}. \end{aligned}$$

Let $\left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x_\alpha^i} \right\} \subset \mathcal{X}(E)$ and $\{dt^\alpha, dx^i, \delta x_\alpha^i\} \subset \mathcal{X}^*(E)$ be the dual adapted bases produced by the nonlinear connection $\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$, via the formulas

$$(2.2) \quad \begin{aligned} \frac{\delta}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha} - M_{(\beta)\alpha}^{(j)} \frac{\partial}{\partial x_\beta^j}, \\ \frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N_{(\beta)i}^{(j)} \frac{\partial}{\partial x_\beta^j}, \\ \delta x_\alpha^i &= dx_\alpha^i + M_{(\alpha)\beta}^{(i)} dt^\beta + N_{(\alpha)j}^{(i)} dx^j. \end{aligned}$$

These bases are suitable for the description of geometrical objects on E , because their transformation laws have the following simple form [4]:

$$(2.3) \quad \begin{aligned} \frac{\delta}{\delta t^\alpha} &= \frac{\partial \tilde{t}^\beta}{\partial t^\alpha} \frac{\delta}{\delta \tilde{t}^\beta}, \quad \frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j}, \quad \frac{\partial}{\partial x_\alpha^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial t^\alpha}{\partial \tilde{t}^\beta} \frac{\partial}{\partial \tilde{x}_\beta^j}, \\ dt^\alpha &= \frac{\partial t^\alpha}{\partial \tilde{t}^\beta} d\tilde{t}^\beta, \quad dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j, \quad \delta x_\alpha^i = \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{t}^\beta}{\partial t^\alpha} \delta \tilde{x}_\beta^j. \end{aligned}$$

In order to develop the theory of Γ -linear connections on $E = J^1(T, M)$, we need the following

Proposition 2.1 *i) The Lie algebra $\mathcal{X}(E)$ of vector fields decomposes as*

$$\mathcal{X}(E) = \mathcal{X}(\mathcal{H}_T) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}),$$

where

$$\mathcal{X}(\mathcal{H}_T) = \text{Span} \left\{ \frac{\delta}{\delta t^\alpha} \right\}, \quad \mathcal{X}(\mathcal{H}_M) = \text{Span} \left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{X}(\mathcal{V}) = \text{Span} \left\{ \frac{\partial}{\partial x_\alpha^i} \right\}.$$

ii) The Lie algebra $\mathcal{X}^(E)$ of covector fields decomposes as*

$$\mathcal{X}^*(E) = \mathcal{X}^*(\mathcal{H}_T) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}),$$

where

$$\mathcal{X}^*(\mathcal{H}_T) = \text{Span}\{dt^\alpha\}, \quad \mathcal{X}^*(\mathcal{H}_M) = \text{Span}\{dx^i\}, \quad \mathcal{X}^*(\mathcal{V}) = \text{Span}\{\delta x_\alpha^i\}.$$

Let us consider h_T , h_M (horizontal) and v (vertical) as the canonical projections of the above decompositions. In this context, we have

Corollary 2.2 *i) Any vector field X can be written in the form*

$$X = h_T X + h_M X + v X.$$

ii) Any covector field ω can be written in the form

$$\omega = h_T \omega + h_M \omega + v \omega.$$

Definition 2.1 A linear connection $\nabla : \mathcal{X}(E) \times \mathcal{X}(E) \rightarrow \mathcal{X}(E)$ is called a Γ -linear connection on E if $\nabla h_T = 0$, $\nabla h_M = 0$ and $\nabla v = 0$.

In order to describe, in local terms, a Γ -linear connection ∇ on E , we need nine unique local components,

$$(2.4) \quad \nabla \Gamma = (\bar{G}_{\beta\gamma}^\alpha, G_{i\gamma}^k, G_{(\alpha)(j)\gamma}^{(i)(\beta)}, \bar{L}_{\beta j}^\alpha, L_{ij}^k, L_{(\alpha)(j)k}^{(i)(\beta)}, \bar{C}_{\beta(k)}^{\alpha(\gamma)}, C_{i(k)}^{j(\gamma)}, C_{(\alpha)(j)(k)}^{(i)(\beta)(\gamma)}),$$

which are defined by the relations

$$\begin{aligned} (h_T) \quad \nabla_{\frac{\delta}{\delta t^\alpha}} \frac{\delta}{\delta t^\beta} &= \bar{G}_{\beta\gamma}^\alpha \frac{\delta}{\delta t^\alpha}, \quad \nabla_{\frac{\delta}{\delta t^\alpha}} \frac{\delta}{\delta x^i} = G_{i\gamma}^k \frac{\delta}{\delta x^k}, \quad \nabla_{\frac{\delta}{\delta t^\alpha}} \frac{\partial}{\partial x_\beta^i} = G_{(\alpha)(i)\gamma}^{(k)(\beta)} \frac{\partial}{\partial x_\alpha^k}, \\ (h_M) \quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta t^\beta} &= \bar{L}_{\beta j}^\alpha \frac{\delta}{\delta t^\alpha}, \quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = L_{ij}^k \frac{\delta}{\delta x^k}, \quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial x_\beta^i} = L_{(\alpha)(i)j}^{(k)(\beta)} \frac{\partial}{\partial x_\alpha^k}, \\ (v) \quad \nabla_{\frac{\partial}{\partial x_\alpha^j}} \frac{\delta}{\delta t^\beta} &= \bar{C}_{\beta(j)}^{\alpha(\gamma)} \frac{\delta}{\delta t^\alpha}, \quad \nabla_{\frac{\partial}{\partial x_\alpha^j}} \frac{\delta}{\delta x^i} = C_{i(j)}^{k(\gamma)} \frac{\delta}{\delta x^k}, \quad \nabla_{\frac{\partial}{\partial x_\alpha^j}} \frac{\partial}{\partial x_\beta^i} = C_{(\alpha)(i)(j)}^{(k)(\beta)(\gamma)} \frac{\partial}{\partial x_\alpha^k}. \end{aligned}$$

The transformation laws of the elements $\left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x_\alpha^i} \right\}$, together with the properties of the Γ -linear connection ∇ , imply

Theorem 2.3 *i) The components of the Γ -linear connection ∇ modify by the rules*

$$\begin{aligned}
(h_T) \quad & \left\{ \begin{aligned} \bar{G}_{\alpha\beta}^{\delta} \frac{\partial \tilde{t}^{\varepsilon}}{\partial t^{\delta}} &= \tilde{G}_{\mu\gamma}^{\varepsilon} \frac{\partial \tilde{t}^{\mu}}{\partial t^{\alpha}} \frac{\partial \tilde{t}^{\gamma}}{\partial t^{\beta}} + \frac{\partial^2 \tilde{t}^{\varepsilon}}{\partial t^{\alpha} \partial t^{\beta}} \\ G_{i\gamma}^k &= \tilde{G}_{j\beta}^m \frac{\partial x^k}{\partial \tilde{x}^m} \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial \tilde{t}^{\beta}}{\partial t^{\gamma}} \\ G_{(\gamma)(i)\alpha}^{(k)(\beta)} &= \tilde{G}_{(\varepsilon)(j)\mu}^{(p)(\eta)} \frac{\partial x^k}{\partial \tilde{x}^p} \frac{\partial \tilde{t}^{\varepsilon}}{\partial t^{\gamma}} \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial t^{\beta}}{\partial \tilde{t}^{\eta}} \frac{\partial \tilde{t}^{\mu}}{\partial t^{\alpha}} + \delta_i^k \frac{\partial \tilde{t}^{\mu}}{\partial t^{\alpha}} \frac{\partial \tilde{t}^{\varepsilon}}{\partial t^{\gamma}} \frac{\partial^2 t^{\beta}}{\partial \tilde{t}^{\mu} \partial \tilde{t}^{\varepsilon}}, \end{aligned} \right. \\
(h_M) \quad & \left\{ \begin{aligned} \bar{L}_{\beta j}^{\gamma} \frac{\partial x^j}{\partial \tilde{x}^l} &= \tilde{L}_{\mu l}^{\eta} \frac{\partial t^{\gamma}}{\partial \tilde{t}^{\eta}} \frac{\partial \tilde{t}^{\mu}}{\partial t^{\beta}} \\ L_{ij}^m \frac{\partial \tilde{x}^r}{\partial x^m} &= \tilde{L}_{pq}^r \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^q}{\partial x^j} + \frac{\partial^2 \tilde{x}^r}{\partial x^i \partial x^j} \\ L_{(\gamma)(i)j}^{(k)(\beta)} &= \tilde{L}_{(\nu)(p)l}^{(r)(\eta)} \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial \tilde{t}^{\nu}}{\partial t^{\gamma}} \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial t^{\beta}}{\partial \tilde{t}^{\eta}} \frac{\partial \tilde{x}^l}{\partial x^j} + \delta_{\gamma}^{\beta} \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial^2 \tilde{x}^r}{\partial x^i \partial x^j}, \end{aligned} \right. \\
(v) \quad & \left\{ \begin{aligned} \bar{C}_{\beta(i)}^{\gamma(\alpha)} &= \tilde{C}_{\varepsilon(j)}^{\mu(\delta)} \frac{\partial t^{\gamma}}{\partial \tilde{t}^{\mu}} \frac{\partial \tilde{t}^{\varepsilon}}{\partial t^{\beta}} \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial t^{\alpha}}{\partial \tilde{t}^{\delta}} \\ C_{i(j)}^{k(\alpha)} &= \tilde{C}_{p(r)}^{s(\beta)} \frac{\partial x^k}{\partial \tilde{x}^s} \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^r}{\partial x^j} \frac{\partial t^{\alpha}}{\partial \tilde{t}^{\beta}} \\ C_{(\gamma)(i)(j)}^{(k)(\beta)(\alpha)} &= \tilde{C}_{(\varepsilon)(p)(q)}^{(r)(\mu)(\nu)} \frac{\partial x^k}{\partial \tilde{x}^r} \frac{\partial \tilde{t}^{\varepsilon}}{\partial t^{\gamma}} \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial t^{\beta}}{\partial \tilde{t}^{\mu}} \frac{\partial \tilde{x}^q}{\partial x^j} \frac{\partial t^{\alpha}}{\partial \tilde{t}^{\nu}}. \end{aligned} \right.
\end{aligned}$$

ii) Conversely, to give a Γ -linear connection ∇ on the 1-jet space E is equivalent to give a set of nine local components (2.4), whose local transformations laws are described in i.

Using Theorem 2.3, we can construct a natural and interesting example of Γ -linear connection on the 1-jet space E .

Example 2.1 Suppose that $h_{\alpha\beta}(t)$ (resp. $\varphi_{ij}(x)$) is a pseudo-Riemannian metric on T (resp. M). We denote $H_{\alpha\beta}^{\gamma}$ (resp. γ_{ij}^k) the Christoffel symbols of the metric $h_{\alpha\beta}$ (resp. φ_{ij}). The *canonical nonlinear connection* Γ_0 associated to these metrics is defined by the local components [4]

$$(2.5) \quad M_{(\alpha)\beta}^{(i)} = -H_{\alpha\beta}^{\gamma} x_{\gamma}^i, \quad N_{(\alpha)j}^{(i)} = \gamma_{jm}^i x_{\alpha}^m.$$

In this context, using the well known local transformation rules of the Christoffel symbols $H_{\alpha\beta}^{\gamma}$ and γ_{jk}^i and setting

$$(2.6) \quad \bar{G}_{\alpha\beta}^{\gamma} = H_{\alpha\beta}^{\gamma}, \quad G_{(\gamma)(i)\alpha}^{(k)(\beta)} = -\delta_i^k H_{\alpha\gamma}^{\beta}, \quad L_{ij}^k = \gamma_{ij}^k, \quad L_{(\gamma)(i)j}^{(k)(\beta)} = \delta_{\gamma}^{\beta} \gamma_{ij}^k,$$

we conclude that the set of local components

$$B\Gamma_0 = (H_{\alpha\beta}^{\gamma}, 0, G_{(\gamma)(i)\alpha}^{(k)(\beta)}, 0, \gamma_{ij}^k, L_{(\gamma)(i)j}^{(k)(\beta)}, 0, 0, 0)$$

defines a Γ_0 -linear connection. This is called the *Berwald connection attached to the pair of metrics* $(h_{\alpha\beta}, \varphi_{ij})$.

Remark 2.1 In the particular case $(T, h) = (\mathbb{R}, \delta)$, the Berwald connection is a natural generalization of the canonical nonlinear connection induced by the spray $2G^i = \gamma_{jk}^i y^j y^k$, from the classical theory of Lagrange spaces. For more details, the reader is invited to consult [2], [4].

Now, let ∇ be a Γ -linear connection on E , locally defined by (2.4). The linear connection ∇ induces a natural linear connection on the d -tensors set of the jet fibre bundle $E = J^1(T, M)$, in the following fashion: starting with a vector field X and a d -tensor field D locally expressed by

$$X = X^\alpha \frac{\delta}{\delta t^\alpha} + X^m \frac{\delta}{\delta x^m} + X_{(\alpha)}^{(m)} \frac{\partial}{\partial x_\alpha^m},$$

$$D = D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots} \frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial x_\beta^j} \otimes dt^\gamma \otimes dx^k \otimes \delta x_\delta^l \dots,$$

we introduce the covariant derivative

$$\nabla_X D = X^\varepsilon \nabla_{\frac{\delta}{\delta t^\varepsilon}} D + X^p \nabla_{\frac{\delta}{\delta x^p}} D + X_{(\varepsilon)}^{(p)} \nabla_{\frac{\partial}{\partial x_\varepsilon^p}} D = \left\{ X^\varepsilon D_{\gamma k(\beta)(l)\dots/\varepsilon}^{\alpha i(j)(\delta)\dots} + X^p \right.$$

$$\left. D_{\gamma k(\beta)(l)\dots|p}^{\alpha i(j)(\delta)\dots} + X_{(\varepsilon)}^{(p)} D_{\gamma k(\beta)(l)\dots|(\varepsilon)}^{\alpha i(j)(\delta)\dots} \right\} \frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{\delta x^i} \otimes \frac{\partial}{\partial x_\beta^j} \otimes dt^\gamma \otimes dx^k \otimes \delta x_\delta^l \dots,$$

where

$$(h_T) \quad \left\{ \begin{array}{l} D_{\gamma k(\beta)(l)\dots/\varepsilon}^{\alpha i(j)(\delta)\dots} = \frac{\delta D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots}}{\delta t^\varepsilon} + D_{\gamma k(\beta)(l)\dots}^{\mu i(j)(\delta)\dots} \bar{G}_{\mu\varepsilon}^\alpha + \\ + D_{\gamma k(\beta)(l)\dots}^{\alpha m(j)(\delta)\dots} G_{m\varepsilon}^i + D_{\gamma k(\mu)(l)\dots}^{\alpha i(m)(\delta)\dots} G_{(\beta)(m)\varepsilon}^{(j)(\mu)} + \dots - \\ - D_{\mu k(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots} \bar{G}_{\gamma\varepsilon}^\mu - D_{\gamma m(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots} G_{k\varepsilon}^m - D_{\gamma k(\beta)(m)\dots}^{\alpha i(j)(\mu)\dots} G_{(\mu)(l)\varepsilon}^{(m)(\delta)} - \dots, \end{array} \right.$$

$$(h_M) \quad \left\{ \begin{array}{l} D_{\gamma k(\beta)(l)\dots|p}^{\alpha i(j)(\delta)\dots} = \frac{\delta D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots}}{\delta x^p} + D_{\gamma k(\beta)(l)\dots}^{\mu i(j)(\delta)\dots} \bar{L}_{\mu p}^\alpha + \\ + D_{\gamma k(\beta)(l)\dots}^{\alpha m(j)(\delta)\dots} L_{m p}^i + D_{\gamma k(\mu)(l)\dots}^{\alpha i(m)(\delta)\dots} L_{(\beta)(m)p}^{(j)(\mu)} + \dots - \\ - D_{\mu k(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots} \bar{L}_{\gamma p}^\mu - D_{\gamma m(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots} L_{k p}^m - D_{\gamma k(\beta)(m)\dots}^{\alpha i(j)(\mu)\dots} L_{(\mu)(l)p}^{(m)(\delta)} - \dots, \end{array} \right.$$

$$(v) \quad \left\{ \begin{array}{l} D_{\gamma k(\beta)(l)\dots|(\varepsilon)}^{\alpha i(j)(\delta)\dots} = \frac{\partial D_{\gamma k(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots}}{\partial x_\varepsilon^p} + D_{\gamma k(\beta)(l)\dots}^{\mu i(j)(\delta)\dots} \bar{C}_{\mu(\varepsilon)}^\alpha + \\ + D_{\gamma k(\beta)(l)\dots}^{\alpha m(j)(\delta)\dots} C_{m(\varepsilon)}^i + D_{\gamma k(\mu)(l)\dots}^{\alpha i(m)(\delta)\dots} C_{(\beta)(m)(\varepsilon)}^{(j)(\mu)} + \dots - \\ - D_{\mu k(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots} \bar{C}_{\gamma(\varepsilon)}^\mu - D_{\gamma m(\beta)(l)\dots}^{\alpha i(j)(\delta)\dots} C_{k(\varepsilon)}^m - D_{\gamma k(\beta)(m)\dots}^{\alpha i(j)(\mu)\dots} C_{(\mu)(l)(\varepsilon)}^{(m)(\delta)} - \dots. \end{array} \right.$$

Definition 2.2 The local operators $_{/\varepsilon}$, $_{|p}$ or $_{|(\varepsilon)}$ are called the T -horizontal covariant derivative, M -horizontal covariant derivative or vertical covariant derivative or h_T -, h_M - and v - covariant derivatives of the Γ -linear connection ∇ .

Remarks 2.2 i) In the particular case of a function $f(t^\gamma, x^k, x_\gamma^k)$ on $J^1(T, M)$, the above covariant derivatives reduce to

$$(2.7) \quad \begin{aligned} f_{/\varepsilon} &= \frac{\delta f}{\delta t^\varepsilon} = \frac{\partial f}{\partial t^\varepsilon} - M_{(\gamma)\varepsilon}^{(k)} \frac{\partial f}{\partial x_\gamma^k}, \\ f_{|p} &= \frac{\delta f}{\delta x^p} = \frac{\partial f}{\partial x^p} - N_{(\gamma)p}^{(k)} \frac{\partial f}{\partial x_\gamma^k}, \\ f^{(\varepsilon)}_{|(p)} &= \frac{\partial f}{\partial x_\varepsilon^p}. \end{aligned}$$

ii) Particularly, starting with a d-vector field $D = Y$ on $J^1(T, M)$, locally expressed by

$$Y = Y^\alpha \frac{\delta}{\delta t^\alpha} + Y^i \frac{\delta}{\delta x^i} + Y_{(\alpha)}^{(i)} \frac{\partial}{\partial x_\alpha^i},$$

the following expressions of above covariant derivatives hold good:

$$(h_T) \quad \begin{cases} Y_{/\varepsilon}^\alpha = \frac{\delta Y^\alpha}{\delta t^\varepsilon} + Y^\mu \bar{G}_{\mu\varepsilon}^\alpha \\ Y_{/\varepsilon}^i = \frac{\delta Y^i}{\delta t^\varepsilon} + Y^m G_{m\varepsilon}^i \\ Y_{(\alpha)/\varepsilon}^{(i)} = \frac{\delta Y_{(\alpha)}^{(i)}}{\delta t^\varepsilon} + Y_{(\mu)}^{(m)} G_{(\alpha)(m)\varepsilon}^{(i)}, \end{cases}$$

$$(h_M) \quad \begin{cases} Y_{|p}^\alpha = \frac{\delta Y^\alpha}{\delta x^p} + Y^\mu \bar{L}_{\mu p}^\alpha \\ Y_{|p}^i = \frac{\delta Y^i}{\delta x^p} + Y^m L_{mp}^i \\ Y_{(\alpha)|p}^{(i)} = \frac{\delta Y_{(\alpha)}^{(i)}}{\delta x^p} + Y_{(\mu)}^{(m)} L_{(\alpha)(m)p}^{(i)}, \end{cases}$$

$$(v) \quad \begin{cases} Y^{\alpha|(\varepsilon)}_{(p)} = \frac{\partial Y^\alpha}{\partial x_\varepsilon^p} + Y^\mu \bar{C}_{\mu(p)}^{\alpha(\varepsilon)} \\ Y^{i|(\varepsilon)}_{(p)} = \frac{\partial Y^i}{\partial x_\varepsilon^p} + Y^m C_{m(p)}^{i(\varepsilon)} \\ Y_{(\alpha)|(\varepsilon)}^{(i)}_{(p)} = \frac{\partial Y_{(\alpha)}^{(i)}}{\partial x_\varepsilon^p} + Y_{(\mu)}^{(m)} C_{(\alpha)(m)(p)}^{(i)(\varepsilon)}. \end{cases}$$

iii) The local covariant derivatives associated to the Berwald Γ_0 -linear connection, will be denoted by " $//_\varepsilon$ ", " $||_p$ " and " $||_{(p)}^{(\varepsilon)}$ ".

Denoting by " $:_A$ " one of local covariant derivatives " $/_\varepsilon$ ", " $|_p$ " or " $|_{(p)}^{(\varepsilon)}$ ", we easily deduce

Proposition 2.4 *If D_{\dots} and F_{\dots} are two d-tensor fields on E , then the following statements hold good:*

- i) $D_{\dots:A}$ are the components of a new d-tensor field;
- ii) $(D_{\dots} + F_{\dots})_{:A} = D_{\dots:A} + F_{\dots:A}$;

- iii) $(D^{\dots} \otimes F^{\dots})_{:A} = D^{\dots}_{:A} \otimes F^{\dots} + D^{\dots} \otimes F^{\dots}_{:A}$;
 iv) The operator " : A " commutes with the contraction operation.

3 Components of d -torsion on $J^1(T, M)$

Let us consider a Γ -linear connection ∇ on $E = J^1(T, M)$, defined by the local components (2.4). Obviously, the torsion d -tensor field associated to ∇ is

$$\mathbf{T} : \mathcal{X}(E) \times \mathcal{X}(E) \rightarrow \mathcal{X}(E), \quad \mathbf{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad \forall X, Y \in \mathcal{X}(E).$$

To characterize locally the torsion d -tensor \mathbf{T} of the connection ∇ , we need the next

Proposition 3.1 *The following bracket identities are true:*

$$\begin{aligned} \left[\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta t^\beta} \right] &= R_{(\mu)\alpha\beta}^{(m)} \frac{\partial}{\partial x_\mu^m}, & \left[\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^j} \right] &= R_{(\mu)\alpha j}^{(m)} \frac{\partial}{\partial x_\mu^m}, \\ \left[\frac{\delta}{\delta t^\alpha}, \frac{\partial}{\partial x_\beta^j} \right] &= \frac{\partial M_{(\mu)\alpha}^{(m)}}{\partial x_\beta^j} \frac{\partial}{\partial x_\mu^m}, & \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] &= R_{(\mu)ij}^{(m)} \frac{\partial}{\partial x_\mu^m}, \\ \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial x_\beta^j} \right] &= \frac{\partial N_{(\mu)i}^{(m)}}{\partial x_\beta^j} \frac{\partial}{\partial x_\mu^m}, & \left[\frac{\partial}{\partial x_\alpha^i}, \frac{\partial}{\partial x_\beta^j} \right] &= 0, \end{aligned}$$

where $M_{(\mu)\alpha}^{(m)}$ and $N_{(\mu)i}^{(m)}$ are the local components of the nonlinear connection Γ , while the components $R_{(\mu)\alpha\beta}^{(m)}$, $R_{(\mu)\alpha j}^{(m)}$, $R_{(\mu)ij}^{(m)}$ are d -tensors expressed by

$$R_{(\mu)\alpha\beta}^{(m)} = \frac{\delta M_{(\mu)\alpha}^{(m)}}{\delta t^\beta} - \frac{\delta M_{(\mu)\beta}^{(m)}}{\delta t^\alpha}, \quad R_{(\mu)\alpha j}^{(m)} = \frac{\delta M_{(\mu)\alpha}^{(m)}}{\delta x^j} - \frac{\delta N_{(\mu)j}^{(m)}}{\delta t^\alpha}, \quad R_{(\mu)ij}^{(m)} = \frac{\delta N_{(\mu)i}^{(m)}}{\delta x^j} - \frac{\delta N_{(\mu)j}^{(m)}}{\delta x^i}.$$

As a consequence, by direct local computations, we deduce that the torsion d -tensor field \mathbf{T} of the Γ -linear connection ∇ may be locally described by

Theorem 3.2 *The torsion d -tensor \mathbf{T} of the Γ -linear connection ∇ is determined by the following local expressions:*

$$\begin{aligned} h_T \mathbf{T} \left(\frac{\delta}{\delta t^\beta}, \frac{\delta}{\delta t^\alpha} \right) &= \bar{T}_{\alpha\beta}^\mu \frac{\delta}{\delta t^\mu}, & h_M \mathbf{T} \left(\frac{\delta}{\delta t^\beta}, \frac{\delta}{\delta t^\alpha} \right) &= 0, \\ v \mathbf{T} \left(\frac{\delta}{\delta t^\beta}, \frac{\delta}{\delta t^\alpha} \right) &= R_{(\mu)\alpha\beta}^{(m)} \frac{\partial}{\partial x_\mu^m}, \\ h_T \mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t^\alpha} \right) &= \bar{T}_{\alpha j}^\mu \frac{\delta}{\delta t^\mu}, & h_M \mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t^\alpha} \right) &= T_{\alpha j}^m \frac{\delta}{\delta x^m}, \\ v \mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta t^\alpha} \right) &= R_{(\mu)\alpha j}^{(m)} \frac{\partial}{\partial x_\mu^m}, \end{aligned}$$

$$\begin{aligned}
h_T \mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) &= 0, & h_M \mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) &= T_{ij}^m \frac{\delta}{\delta x^m}, \\
v \mathbf{T} \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) &= R_{(\mu)ij}^{(m)} \frac{\partial}{\partial x_\mu^m}, \\
h_T \mathbf{T} \left(\frac{\partial}{\partial x_\beta^j}, \frac{\delta}{\delta t^\alpha} \right) &= \bar{P}_{\alpha(j)}^{\mu(\beta)} \frac{\delta}{\delta t^\mu}, & h_M \mathbf{T} \left(\frac{\partial}{\partial x_\beta^j}, \frac{\delta}{\delta t^\alpha} \right) &= 0, \\
v \mathbf{T} \left(\frac{\partial}{\partial x_\beta^j}, \frac{\delta}{\delta t^\alpha} \right) &= P_{(\mu)\alpha(j)}^{(m)(\beta)} \frac{\partial}{\partial x_\mu^m}, \\
h_T \mathbf{T} \left(\frac{\partial}{\partial x_\beta^j}, \frac{\delta}{\delta x^i} \right) &= 0, & h_M \mathbf{T} \left(\frac{\partial}{\partial x_\beta^j}, \frac{\delta}{\delta t^\alpha} \right) &= P_{i(j)}^{m(\beta)} \frac{\delta}{\delta x^m}, \\
v \mathbf{T} \left(\frac{\partial}{\partial x_\beta^j}, \frac{\delta}{\delta x^i} \right) &= P_{(\mu)i(j)}^{(m)(\beta)} \frac{\partial}{\partial x_\mu^m}, \\
h_T \mathbf{T} \left(\frac{\partial}{\partial x_\beta^j}, \frac{\partial}{\partial x_\alpha^i} \right) &= 0, & h_M \mathbf{T} \left(\frac{\partial}{\partial x_\beta^j}, \frac{\partial}{\partial x_\alpha^i} \right) &= 0, \\
v \mathbf{T} \left(\frac{\partial}{\partial x_\beta^j}, \frac{\partial}{\partial x_\alpha^i} \right) &= S_{(\mu)(i)(j)}^{(m)(\alpha)(\beta)} \frac{\partial}{\partial x_\mu^m},
\end{aligned}$$

where $R_{(\mu)\alpha\beta}^{(m)}$, $R_{(\mu)\alpha j}^{(m)}$, $R_{(\mu)ij}^{(m)}$ are the d -tensors from Proposition 3.1, and $\bar{T}_{\alpha\beta}^\mu = \bar{G}_{\alpha\beta}^\mu - \bar{G}_{\beta\alpha}^\mu$, $\bar{T}_{\alpha j}^\mu = \bar{L}_{\alpha j}^\mu$, $\bar{P}_{\alpha(j)}^{\mu(\beta)} = \bar{C}_{\alpha(j)}^{\mu(\beta)}$, $T_{\alpha j}^m = -G_{j\alpha}^m$, $T_{ij}^m = L_{ij}^m - L_{ji}^m$, $P_{i(j)}^{m(\beta)} = C_{i(j)}^{m(\beta)}$, $S_{(\mu)(i)(j)}^{(m)(\alpha)(\beta)} = C_{(\mu)(i)(j)}^{(m)(\alpha)(\beta)} - C_{(\mu)(j)(i)}^{(m)(\beta)(\alpha)}$, $P_{(\mu)\alpha(j)}^{(m)(\beta)} = \frac{\partial M_{(\mu)\alpha}^{(m)}}{\partial x_\beta^j} - G_{(\mu)(j)\alpha}^{(m)(\beta)}$, $P_{(\mu)i(j)}^{(m)(\beta)} = \frac{\partial N_{(\mu)i}^{(m)}}{\partial x_\beta^j} - L_{(\mu)(j)i}^{(m)(\beta)}$.

In other words, the torsion \mathbf{T} of the Γ -linear connection ∇ is determined by twelve effective adapted d -tensor fields that we arrange in the following table:

	h_T	h_M	v
$h_T h_T$	$\bar{T}_{\alpha\beta}^\mu$	0	$R_{(\mu)\alpha\beta}^{(m)}$
$h_M h_T$	$\bar{T}_{\alpha j}^\mu$	$T_{\alpha j}^m$	$R_{(\mu)\alpha j}^{(m)}$
$h_M h_M$	0	T_{ij}^m	$R_{(\mu)ij}^{(m)}$
$v h_T$	$\bar{P}_{\alpha(j)}^{\mu(\beta)}$	0	$P_{(\mu)\alpha(j)}^{(m)(\beta)}$
$v h_M$	0	$P_{i(j)}^{m(\beta)}$	$P_{(\mu)i(j)}^{(m)(\beta)}$
vv	0	0	$S_{(\mu)(i)(j)}^{(m)(\alpha)(\beta)}$

Remark 3.1 In the particular case of the Berwald Γ_0 -linear connection associated to the metrics $h_{\alpha\beta}$ and φ_{ij} , all torsion d -tensors vanish, except $R_{(\mu)\alpha\beta}^{(m)} = -H_{\mu\alpha\beta}^\gamma x_\gamma^m$, $R_{(\mu)ij}^{(m)} = r_{ijl}^m x_\mu^l$, where $H_{\mu\beta\gamma}^\gamma$ (resp. r_{ijl}^m) are the curvature tensors of the metric $h_{\alpha\beta}$ (resp. φ_{ij}).

4 Components of d -curvature on $J^1(T, M)$

From the general theory of linear connections, we recall that the curvature d -tensor field associated to the Γ -linear connection ∇ is $\mathbf{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$, $\forall X, Y, Z \in \mathcal{X}(E)$.

Using an adapted basis, the properties of the Γ -linear connection ∇ and of the d -tensor \mathbf{R} , together with the expressions of the local T -, M - horizontal and vertical covariant derivatives attached to the Γ -linear connection ∇ , by complicated local computations, we find

Theorem 4.1 *The curvature d -tensor \mathbf{R} of the Γ -linear connection ∇ is determined by the following local expressions:*

$$\begin{aligned}
\mathbf{R}\left(\frac{\delta}{\delta t^\gamma}, \frac{\delta}{\delta t^\beta}\right) \frac{\delta}{\delta t^\alpha} &= \bar{R}_{\alpha\beta\gamma}^\delta \frac{\delta}{\delta t^\delta}, & \mathbf{R}\left(\frac{\delta}{\delta t^\gamma}, \frac{\delta}{\delta t^\beta}\right) \frac{\delta}{\delta x^i} &= R_{i\beta\gamma}^l \frac{\delta}{\delta x^l}, \\
\mathbf{R}\left(\frac{\delta}{\delta t^\gamma}, \frac{\delta}{\delta t^\beta}\right) \frac{\partial}{\partial x_\alpha^i} &= R_{(\delta)(i)\beta\gamma}^{(l)(\alpha)} \frac{\partial}{\partial x_\delta^l}, \\
\mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta t^\beta}\right) \frac{\delta}{\delta t^\alpha} &= \bar{R}_{\alpha\beta k}^\delta \frac{\delta}{\delta t^\delta}, & \mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta t^\beta}\right) \frac{\delta}{\delta x^i} &= R_{i\beta k}^l \frac{\delta}{\delta x^l}, \\
\mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta t^\beta}\right) \frac{\partial}{\partial x_\alpha^i} &= R_{(\delta)(i)\beta k}^{(l)(\alpha)} \frac{\partial}{\partial x_\delta^l}, \\
\mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta t^\alpha} &= \bar{R}_{\alpha j k}^\delta \frac{\delta}{\delta t^\delta}, & \mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta x^i} &= R_{i j k}^l \frac{\delta}{\delta x^l}, \\
\mathbf{R}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right) \frac{\partial}{\partial x_\alpha^i} &= R_{(\delta)(i) j k}^{(l)(\alpha)} \frac{\partial}{\partial x_\delta^l}, \\
\mathbf{R}\left(\frac{\partial}{\partial x_\gamma^k}, \frac{\delta}{\delta t^\beta}\right) \frac{\delta}{\delta t^\alpha} &= \bar{P}_{\alpha\beta(k)}^{\delta(\gamma)} \frac{\delta}{\delta t^\delta}, & \mathbf{R}\left(\frac{\partial}{\partial x_\gamma^k}, \frac{\delta}{\delta t^\beta}\right) \frac{\delta}{\delta x^i} &= P_{i\beta(k)}^l \frac{\delta}{\delta x^l}, \\
\mathbf{R}\left(\frac{\partial}{\partial x_\gamma^k}, \frac{\delta}{\delta t^\beta}\right) \frac{\partial}{\partial x_\alpha^i} &= P_{(\delta)(i)\beta(k)}^{(l)(\alpha)(\gamma)} \frac{\partial}{\partial x_\delta^l}, \\
\mathbf{R}\left(\frac{\partial}{\partial x_\gamma^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta t^\alpha} &= \bar{P}_{\alpha j(k)}^{\delta(\gamma)} \frac{\delta}{\delta t^\delta}, & \mathbf{R}\left(\frac{\partial}{\partial x_\gamma^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta x^i} &= P_{i j(k)}^l \frac{\delta}{\delta x^l}, \\
\mathbf{R}\left(\frac{\partial}{\partial x_\gamma^k}, \frac{\delta}{\delta x^j}\right) \frac{\partial}{\partial x_\alpha^i} &= P_{(\delta)(i) j(k)}^{(l)(\alpha)(\gamma)} \frac{\partial}{\partial x_\delta^l}, \\
\mathbf{R}\left(\frac{\partial}{\partial x_\gamma^k}, \frac{\partial}{\partial x_\beta^j}\right) \frac{\delta}{\delta t^\alpha} &= \bar{S}_{\alpha(j)(k)}^{\delta(\beta)(\gamma)} \frac{\delta}{\delta t^\delta}, & \mathbf{R}\left(\frac{\partial}{\partial x_\gamma^k}, \frac{\partial}{\partial x_\beta^j}\right) \frac{\delta}{\delta x^i} &= S_{i(j)(k)}^{l(\beta)(\gamma)} \frac{\delta}{\delta x^l}, \\
\mathbf{R}\left(\frac{\partial}{\partial x_\gamma^k}, \frac{\partial}{\partial x_\beta^j}\right) \frac{\partial}{\partial x_\alpha^i} &= S_{(\delta)(i)(j)(k)}^{(l)(\alpha)(\beta)(\gamma)} \frac{\partial}{\partial x_\delta^l},
\end{aligned}$$

whose eighteen adapted components we arrange in the table

	h_T	h_M	v
$h_T h_T$	$\bar{R}_{\alpha\beta\gamma}^\delta$	$R_{i\beta\gamma}^l$	$R_{(\delta)(i)\beta\gamma}^{(l)(\alpha)}$
$h_M h_T$	$\bar{R}_{\alpha\beta k}^\delta$	$R_{i\beta k}^l$	$R_{(\delta)(i)\beta k}^{(l)(\alpha)}$
$h_M h_M$	$\bar{R}_{\alpha j k}^\delta$	$R_{ij k}^l$	$R_{(\delta)(i)j k}^{(l)(\alpha)}$
$v h_T$	$\bar{P}_{\alpha\beta(k)}^\delta(\gamma)$	$P_{i\beta(k)}^l(\gamma)$	$P_{(\delta)(i)\beta(k)}^{(l)(\alpha)}(\gamma)$
$v h_M$	$\bar{P}_{\alpha j(k)}^\delta(\gamma)$	$P_{ij(k)}^l(\gamma)$	$P_{(\delta)(i)j(k)}^{(l)(\alpha)}(\gamma)$
vv	$\bar{S}_{\alpha(j)(k)}^{\delta(\beta)(\gamma)}$	$S_{i(j)(k)}^{l(\beta)(\gamma)}$	$S_{(\delta)(i)(j)(k)}^{(l)(\alpha)(\beta)(\gamma)}$

Moreover, the following expressions of the local adapted components of the curvature d -tensor \mathbf{R} hold good:

$$\begin{aligned}
 (h_T) \quad & \left\{ \begin{aligned}
 1. \quad \bar{R}_{\alpha\beta\gamma}^\delta &= \frac{\delta \bar{G}_{\alpha\beta}^\delta}{\delta t^\gamma} - \frac{\delta \bar{G}_{\alpha\gamma}^\delta}{\delta t^\beta} + \bar{G}_{\alpha\beta}^\mu \bar{G}_{\mu\gamma}^\delta - \bar{G}_{\alpha\gamma}^\mu \bar{G}_{\mu\beta}^\delta + \bar{C}_{\alpha(m)}^{\delta(\mu)} R_{(\mu)\beta\gamma}^{(m)} \\
 2. \quad \bar{R}_{\alpha\beta k}^\delta &= \frac{\delta \bar{G}_{\alpha\beta}^\delta}{\delta x^k} - \frac{\delta \bar{L}_{\alpha k}^\delta}{\delta t^\beta} + \bar{G}_{\alpha\beta}^\mu \bar{L}_{\mu k}^\delta - \bar{L}_{\alpha k}^\mu \bar{G}_{\mu\beta}^\delta + \bar{C}_{\alpha(m)}^{\delta(\mu)} R_{(\mu)\beta k}^{(m)} \\
 3. \quad \bar{R}_{\alpha j k}^\delta &= \frac{\delta \bar{L}_{\alpha j}^\delta}{\delta x^k} - \frac{\delta \bar{L}_{\alpha k}^\delta}{\delta x^j} + \bar{L}_{\alpha j}^\mu \bar{L}_{\mu k}^\delta - \bar{L}_{\alpha k}^\mu \bar{L}_{\mu j}^\delta + \bar{C}_{\alpha(m)}^{\delta(\mu)} R_{(\mu)j k}^{(m)} \\
 4. \quad \bar{P}_{\alpha\beta(k)}^\delta(\gamma) &= \frac{\partial \bar{G}_{\alpha\beta}^\delta}{\partial x_\gamma^k} - \bar{C}_{\alpha(k)/\beta}^{\delta(\gamma)} + \bar{C}_{\alpha(m)}^{\delta(\mu)} P_{(\mu)\beta(k)}^{(m)}(\gamma) \\
 5. \quad \bar{P}_{\alpha j(k)}^\delta(\gamma) &= \frac{\partial \bar{L}_{\alpha j}^\delta}{\partial x_\gamma^k} - \bar{C}_{\alpha(k)|j}^{\delta(\gamma)} + \bar{C}_{\alpha(m)}^{\delta(\mu)} P_{(\mu)j(k)}^{(m)}(\gamma) \\
 6. \quad \bar{S}_{\alpha(j)(k)}^{\delta(\beta)(\gamma)} &= \frac{\partial \bar{C}_{\alpha(j)}^{\delta(\beta)}}{\partial x_\gamma^k} - \frac{\partial \bar{C}_{\alpha(k)}^{\delta(\gamma)}}{\partial x_\beta^j} + \bar{C}_{\alpha(j)}^{\mu(\beta)} \bar{C}_{\mu(k)}^{\delta(\gamma)} - \bar{C}_{\alpha(k)}^{\mu(\gamma)} \bar{C}_{\mu(j)}^{\delta(\beta)},
 \end{aligned} \right. \\
 (h_M) \quad & \left\{ \begin{aligned}
 7. \quad R_{i\beta\gamma}^l &= \frac{\delta G_{i\beta}^l}{\delta t^\gamma} - \frac{\delta G_{i\gamma}^l}{\delta t^\beta} + G_{i\beta}^m G_{m\gamma}^l - G_{i\gamma}^m G_{m\beta}^l + C_{i(m)}^{l(\mu)} R_{(\mu)\beta\gamma}^{(m)} \\
 8. \quad R_{i\beta k}^l &= \frac{\delta G_{i\beta}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta t^\beta} + G_{i\beta}^m L_{mk}^l - L_{ik}^m G_{m\beta}^l + C_{i(m)}^{l(\mu)} R_{(\mu)\beta k}^{(m)} \\
 9. \quad R_{ij k}^l &= \frac{\delta L_{ij}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta x^j} + L_{ij}^m L_{mk}^l - L_{ik}^m L_{mj}^l + C_{i(m)}^{l(\mu)} R_{(\mu)j k}^{(m)} \\
 10. \quad P_{i\beta(k)}^l(\gamma) &= \frac{\partial G_{i\beta}^l}{\partial x_\gamma^k} - C_{i(k)/\beta}^{l(\gamma)} + C_{i(m)}^{l(\mu)} P_{(\mu)\beta(k)}^{(m)}(\gamma) \\
 11. \quad P_{ij(k)}^l(\gamma) &= \frac{\partial L_{ij}^l}{\partial x_\gamma^k} - C_{i(k)|j}^{l(\gamma)} + C_{i(m)}^{l(\mu)} P_{(\mu)j(k)}^{(m)}(\gamma) \\
 12. \quad S_{i(j)(k)}^{l(\beta)(\gamma)} &= \frac{\partial C_{i(j)}^{l(\beta)}}{\partial x_\gamma^k} - \frac{\partial C_{i(k)}^{l(\gamma)}}{\partial x_\beta^j} + C_{i(j)}^{m(\beta)} C_{m(k)}^{l(\gamma)} - C_{i(k)}^{m(\gamma)} C_{m(j)}^{l(\beta)},
 \end{aligned} \right.
 \end{aligned}$$

$$(v) \left\{ \begin{array}{l} 13. \quad R_{(\delta)(i)\beta\gamma}^{(l)(\alpha)} = \frac{\delta G_{(\delta)(i)\beta}^{(l)(\alpha)}}{\delta t^\gamma} - \frac{\delta G_{(\delta)(i)\gamma}^{(l)(\alpha)}}{\delta t^\beta} + G_{(\mu)(i)\beta}^{(m)(\alpha)} G_{(\delta)(m)\gamma}^{(l)(\mu)} - \\ \quad - G_{(\mu)(i)\gamma}^{(m)(\alpha)} G_{(\delta)(m)\beta}^{(l)(\mu)} + C_{(\delta)(i)(m)}^{(l)(\alpha)(\mu)} R_{(\mu)\beta\gamma}^{(m)} \\ 14. \quad R_{(\delta)(i)\beta k}^{(l)(\alpha)} = \frac{\delta G_{(\delta)(i)\beta}^{(l)(\alpha)}}{\delta x^k} - \frac{\delta L_{(\delta)(i)k}^{(l)(\alpha)}}{\delta t^\beta} + G_{(\mu)(i)\beta}^{(m)(\alpha)} L_{(\delta)(m)k}^{(l)(\mu)} - \\ \quad - L_{(\mu)(i)k}^{(m)(\alpha)} G_{(\delta)(m)\beta}^{(l)(\mu)} + C_{(\delta)(i)(m)}^{(l)(\alpha)(\mu)} R_{(\mu)\beta k}^{(m)} \\ 15. \quad R_{(\delta)(i)jk}^{(l)(\alpha)} = \frac{\delta L_{(\delta)(i)j}^{(l)(\alpha)}}{\delta x^k} - \frac{\delta L_{(\delta)(i)k}^{(l)(\alpha)}}{\delta x^j} + L_{(\mu)(i)j}^{(m)(\alpha)} L_{(\delta)(m)k}^{(l)(\mu)} - \\ \quad - L_{(\mu)(i)k}^{(m)(\alpha)} L_{(\delta)(m)j}^{(l)(\mu)} + C_{(\delta)(i)(m)}^{(l)(\alpha)(\mu)} R_{(\mu)jk}^{(m)} \\ 16. \quad P_{(\delta)(i)\beta(k)}^{(l)(\alpha)(\gamma)} = \frac{\partial G_{(\delta)(i)\beta}^{(l)(\alpha)}}{\partial x_\gamma^k} - C_{(\delta)(i)(k)/\beta}^{(l)(\alpha)(\gamma)} + C_{(\delta)(i)(m)}^{(l)(\alpha)(\mu)} P_{(\mu)\beta(k)}^{(m)(\gamma)} \\ 17. \quad P_{(\delta)(i)j(k)}^{(l)(\alpha)(\gamma)} = \frac{\partial L_{(\delta)(i)j}^{(l)(\alpha)}}{\partial x_\gamma^k} - C_{(\delta)(i)(k)j}^{(l)(\alpha)(\gamma)} + C_{(\delta)(i)(m)}^{(l)(\alpha)(\mu)} P_{(\mu)j(k)}^{(m)(\gamma)} \\ 18. \quad S_{(\delta)(i)(j)(k)}^{(l)(\alpha)(\beta)(\gamma)} = \frac{\partial C_{(\delta)(i)(j)}^{(l)(\alpha)(\beta)}}{\partial x_\gamma^k} - \frac{\partial C_{(\delta)(i)(k)}^{(l)(\alpha)(\gamma)}}{\partial x_\beta^j} + C_{(\mu)(i)(j)}^{(m)(\alpha)(\beta)} C_{(\delta)(m)(k)}^{(l)(\mu)(\gamma)} - \\ \quad - C_{(\mu)(i)(k)}^{(m)(\alpha)(\gamma)} C_{(\delta)(m)(j)}^{(l)(\mu)(\beta)}. \end{array} \right.$$

Remark 4.1 In the case of the Berwald Γ_0 -linear connection associated to the pair of metrics $(h_{\alpha\beta}, \varphi_{ij})$, all curvature d -tensors vanish, except

$$R_{\alpha\beta\gamma}^\delta = H_{\alpha\beta\gamma}^\delta, \quad R_{ijk}^l = r_{ijk}^l,$$

where $H_{\alpha\beta\gamma}^\delta$ (resp. r_{ijk}^l) are the curvature tensors of the metric $h_{\alpha\beta}$ (resp. φ_{ij}).

5 Ricci and Bianchi identities on $J^1(T, M)$

Let us consider an arbitrary d -vector field X on the 1-jet space E , which is locally expressed by

$$X = X^\alpha \frac{\delta}{\delta t^\alpha} + X^i \frac{\delta}{\delta x^i} + X_{(\alpha)}^{(i)} \frac{\partial}{\partial x_\alpha^i}.$$

Taking again into account the local form of the T -, M - horizontal and vertical covariant derivatives of the Γ -linear connection ∇ , by a direct calculation, we obtain

Theorem 5.1 *The following Ricci identities associated to the Γ -linear connection ∇ and to the distinguished vector field X hold good:*

$$\begin{aligned}
(h_T) \quad & \left\{ \begin{aligned}
X_{/\beta/\gamma}^\alpha - X_{/\gamma/\beta}^\alpha &= X^\mu \bar{R}_{\mu\beta\gamma}^\alpha - X_{/\mu}^\alpha \bar{T}_{\beta\gamma}^\mu - X^\alpha |_{(m)}^{(\mu)} R_{(\mu)\beta\gamma}^{(m)} \\
X_{/\beta|k}^\alpha - X_{|k/\beta}^\alpha &= X^\mu \bar{R}_{\mu\beta k}^\alpha - X_{/\mu}^\alpha \bar{T}_{\beta k}^\mu - X_{|m}^\alpha T_{\beta k}^m - X^\alpha |_{(m)}^{(\mu)} R_{(\mu)\beta k}^{(m)} \\
X_{|j|k}^\alpha - X_{|k|j}^\alpha &= X^\mu \bar{R}_{\mu j k}^\alpha - X_{|m}^\alpha T_{j k}^m - X^\alpha |_{(m)}^{(\mu)} R_{(\mu)j k}^{(m)} \\
X_{/\beta|k}^\alpha |_{(\gamma)} - X^\alpha |_{(k)/\beta} |_{(\gamma)} &= X^\mu \bar{P}_{\mu\beta(k)}^\alpha |_{(\gamma)} - X_{/\mu}^\alpha \bar{C}_{\beta(k)}^{\mu(\gamma)} - X^\alpha |_{(m)}^{(\mu)} P_{(\mu)\beta(k)}^{(m)} |_{(\gamma)} \\
X_{|j|k}^\alpha |_{(\gamma)} - X^\alpha |_{(k)|j} |_{(\gamma)} &= X^\mu \bar{P}_{\mu j(k)}^\alpha |_{(\gamma)} - X_{|m}^\alpha C_{j(k)}^{m(\gamma)} - X^\alpha |_{(m)}^{(\mu)} P_{(\mu)j(k)}^{(m)} |_{(\gamma)} \\
X^\alpha |_{(j)|k} |_{(\gamma)} - X^\alpha |_{(k)|j} |_{(\beta)} &= X^\mu \bar{S}_{\mu(j)(k)}^{\alpha(\beta)(\gamma)} - X^\alpha |_{(m)}^{(\mu)} S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)},
\end{aligned} \right. \\
(h_M) \quad & \left\{ \begin{aligned}
X_{/\beta/\gamma}^i - X_{/\gamma/\beta}^i &= X^m R_{m\beta\gamma}^i - X_{/\mu}^i \bar{T}_{\beta\gamma}^\mu - X^i |_{(m)}^{(\mu)} R_{(\mu)\beta\gamma}^{(m)} \\
X_{/\beta|k}^i - X_{|k/\beta}^i &= X^m R_{m\beta k}^i - X_{/\mu}^i \bar{T}_{\beta k}^\mu - X_{|m}^i T_{\beta k}^m - X^i |_{(m)}^{(\mu)} R_{(\mu)\beta k}^{(m)} \\
X_{|j|k}^i - X_{|k|j}^i &= X^m R_{m j k}^i - X_{|m}^i T_{j k}^m - X^i |_{(m)}^{(\mu)} R_{(\mu)j k}^{(m)} \\
X_{/\beta|k}^i |_{(\gamma)} - X^i |_{(k)/\beta} |_{(\gamma)} &= X^m P_{m\beta(k)}^i |_{(\gamma)} - X_{/\mu}^i \bar{C}_{\beta(k)}^{\mu(\gamma)} - X^i |_{(m)}^{(\mu)} P_{(\mu)\beta(k)}^{(m)} |_{(\gamma)} \\
X_{|j|k}^i |_{(\gamma)} - X^i |_{(k)|j} |_{(\gamma)} &= X^m P_{m j(k)}^i |_{(\gamma)} - X_{|m}^i C_{j(k)}^{m(\gamma)} - X^i |_{(m)}^{(\mu)} P_{(\mu)j(k)}^{(m)} |_{(\gamma)} \\
X^i |_{(j)|k} |_{(\gamma)} - X^i |_{(k)|j} |_{(\beta)} &= X^m S_{m(j)(k)}^{i(\beta)(\gamma)} - X^i |_{(m)}^{(\mu)} S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)},
\end{aligned} \right. \\
(v) \quad & \left\{ \begin{aligned}
X_{(\alpha)/\beta/\gamma}^{(i)} - X_{(\alpha)/\gamma/\beta}^{(i)} &= X_{(\mu)}^{(m)} R_{(\alpha)(m)\beta\gamma}^{(i)(\mu)} - X_{(\alpha)/\mu}^{(i)} \bar{T}_{\beta\gamma}^\mu - X_{(\alpha)}^{(i)} |_{(m)}^{(\mu)} R_{(\mu)\beta\gamma}^{(m)} \\
X_{(\alpha)/\beta|k}^{(i)} - X_{(\alpha)/k/\beta}^{(i)} &= X_{(\mu)}^{(m)} R_{(\alpha)(m)\beta k}^{(i)(\mu)} - X_{(\alpha)/\mu}^{(i)} \bar{T}_{\beta k}^\mu - X_{(\alpha)|m}^{(i)} T_{\beta k}^m - \\
&\quad - X_{(\alpha)}^{(i)} |_{(m)}^{(\mu)} R_{(\mu)\beta k}^{(m)} \\
X_{(\alpha)|j|k}^{(i)} - X_{(\alpha)|k|j}^{(i)} &= X_{(\mu)}^{(m)} R_{(\alpha)(m)j k}^{(i)(\mu)} - X_{(\alpha)|m}^{(i)} T_{j k}^m - X_{(\alpha)}^{(i)} |_{(m)}^{(\mu)} R_{(\mu)j k}^{(m)} \\
X_{(\alpha)/\beta|k}^{(i)} |_{(\gamma)} - X_{(\alpha)/k/\beta}^{(i)} |_{(\gamma)} &= X_{(\mu)}^{(m)} P_{(\alpha)(m)\beta(k)}^{(i)(\mu)} |_{(\gamma)} - X_{(\alpha)/\mu}^{(i)} \bar{C}_{\beta(k)}^{\mu(\gamma)} - \\
&\quad - X_{(\alpha)}^{(i)} |_{(m)}^{(\mu)} P_{(\mu)\beta(k)}^{(m)} |_{(\gamma)} \\
X_{(\alpha)/j|k}^{(i)} |_{(\gamma)} - X_{(\alpha)/k|j}^{(i)} |_{(\gamma)} &= X_{(\mu)}^{(m)} P_{(\alpha)(m)j(k)}^{(i)(\mu)} |_{(\gamma)} - X_{(\alpha)|m}^{(i)} C_{j(k)}^{m(\gamma)} - \\
&\quad - X_{(\alpha)}^{(i)} |_{(m)}^{(\mu)} P_{(\mu)j(k)}^{(m)} |_{(\gamma)} \\
X_{(\alpha)}^{(i)} |_{(j)|k} |_{(\beta)} - X_{(\alpha)}^{(i)} |_{(k)|j} |_{(\beta)} &= X_{(\mu)}^{(m)} S_{(\alpha)(m)(j)(k)}^{(i)(\mu)(\beta)(\gamma)} - X_{(\alpha)}^{(i)} |_{(m)}^{(\mu)} S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)}.
\end{aligned} \right.
\end{aligned}$$

Proof. Let (Y_A) and (ω^A) , $A \in \{\alpha, i, \binom{\alpha}{i}\}$, be the adapted bases of the nonlinear connection Γ , and let $X = X^M Y_M$ be an arbitrary d-vector field on $J^1(T, M)$. In this context, using the equalities

$$\begin{aligned}
[\mathbf{R}(Y_C, Y_B)X] \cdot \omega^B \cdot \omega^C &= \{\nabla_{Y_C} \nabla_{Y_B} X - \nabla_{Y_B} \nabla_{Y_C} X - \nabla_{[Y_B, Y_C]} X\} \cdot \omega^B \cdot \omega^C, \\
\mathbf{R}(Y_C, Y_B)Y_A &= \mathbf{R}_{ABC}^M Y_M, \quad \mathbf{T}(Y_C, Y_B) = \mathbf{T}_{BC}^M = \{\Gamma_{BC}^M - \Gamma_{CB}^M - R_{CB}^M\} Y_M, \\
\nabla_{Y_C} \omega^B &= -\Gamma_{MC}^B \omega^M, \quad [Y_B, Y_C] = R_{BC}^M Y_M, \quad \nabla_{Y_C} Y_B = \Gamma_{BC}^M Y_M,
\end{aligned}$$

where " \cdot " represents the tensorial product " \otimes ", by a direct calcul, we obtain the relations

$$(5.1) \quad X_{:B:C}^A - X_{:C:B}^A = X^M \mathbf{R}_{MBC}^A - X_{:M}^A \mathbf{T}_{BC}^M,$$

where " $:D$ " represents one from the local covariant derivatives " $/\beta$ ", " $|_j$ " or " $|_{(j)}^{(\beta)}$ " associated to the Γ -linear connection ∇ .

Taking into account the indices A, B, C, \dots belong to set $\{\alpha, i, \binom{\alpha}{i}\}$, after enough complicated computations, the identities (5.1) imply the required Ricci identities. ■

Remark 5.1 For the arbitrary vector fields $X, Y, Z \in \mathcal{X}(E)$ and the arbitrary 1-form $\omega \in \mathcal{X}^*(E)$ on $J^1(T, M)$, the relations

$$(5.2) \quad \begin{aligned} \mathbf{R}(X, Y)\omega &= -\omega \circ \mathbf{R}(X, Y), \\ \mathbf{R}(X, Y)(Z \otimes \omega) &= \mathbf{R}(X, Y)Z \otimes \omega + Z \otimes \mathbf{R}(X, Y)\omega, \end{aligned}$$

are true. These relations allow us to generalize the Ricci identities to the d -tensors set of the 1-jet fibre bundle E . The generalization is a natural one, but the expressions of Ricci identities become extremely complicated. For that reason, we exemplify this generalization writing just one Ricci identity. For example, if $D = (D_{\alpha j(\eta)(l)\dots}^{\delta i(k)(\varepsilon)\dots})$ is an arbitrary d -tensor field on E , then the following Ricci identity

$$\begin{aligned} D_{\alpha j(\eta)(l)\dots/\beta/\gamma}^{\delta i(k)(\varepsilon)\dots} - D_{\alpha j(\eta)(l)\dots/\gamma/\beta}^{\delta i(k)(\varepsilon)\dots} &= D_{\alpha j(\eta)(l)\dots}^{\mu i(k)(\varepsilon)\dots} \bar{R}_{\mu\beta\gamma}^{\alpha} + D_{\alpha j(\eta)(l)\dots}^{\delta m(k)(\varepsilon)\dots} R_{m\beta\gamma}^i + \\ + D_{\alpha j(\mu)(l)\dots}^{\delta i(m)(\varepsilon)\dots} R_{(\eta)(m)\beta\gamma}^{(k)(\mu)} + \dots - D_{\mu j(\eta)(l)\dots}^{\delta i(k)(\varepsilon)\dots} \bar{R}_{\alpha\beta\gamma}^{\mu} - D_{\alpha m(\eta)(l)\dots}^{\delta i(k)(\varepsilon)\dots} R_{i\beta\gamma}^m - \\ - D_{\alpha j(\eta)(m)\dots}^{\delta i(k)(\mu)\dots} R_{(\mu)(l)\beta\gamma}^{(m)(\varepsilon)} - \dots - D_{\alpha j(\eta)(l)\dots/\mu}^{\delta i(k)(\varepsilon)\dots} \bar{T}_{\beta\gamma}^{\mu} - D_{\alpha j(\eta)(l)\dots}^{\delta i(k)(\varepsilon)\dots} |_{(m)}^{(\mu)} R_{(\mu)\beta\gamma}^{(m)} \end{aligned}$$

holds good.

Now, let us consider the Liouville canonical vector field $\mathbf{C} = x_{\alpha}^i \frac{\partial}{\partial x_{\alpha}^i}$ on $J^1(T, M)$, together with the *deflection d -tensors associated to the Γ -linear connection ∇* , which are defined by the local components

$$\bar{D}_{(\alpha)\beta}^{(i)} = x_{\alpha/\beta}^i, \quad D_{(\alpha)j}^{(i)} = x_{\alpha|j}^i, \quad d_{(\alpha)(j)}^{(i)(\beta)} = x_{\alpha}^i |_{(j)}^{(\beta)}.$$

By a direct calculation, we find the the following local expressions of the deflection d -tensors:

$$(5.3) \quad \begin{aligned} \bar{D}_{(\alpha)\beta}^{(i)} &= -M_{(\alpha)\beta}^{(i)} + G_{(\alpha)(m)\beta}^{(i)(\mu)} x_{\mu}^m \\ D_{(\alpha)j}^{(i)} &= -N_{(\alpha)j}^{(i)} + L_{(\alpha)(m)j}^{(i)(\mu)} x_{\mu}^m \\ d_{(\alpha)(j)}^{(i)(\beta)} &= \delta_j^i \delta_{\alpha}^{\beta} + C_{(\alpha)(m)(j)}^{(i)(\mu)(\beta)} x_{\mu}^m. \end{aligned}$$

Applying the v -set of the Ricci identities to the components of the Liouville vector field \mathbf{C} , we obtain

Theorem 5.2 *The deflection d -tensors of the Γ -linear connection ∇ satisfy the identities:*

$$\begin{aligned}
\bar{D}_{(\alpha)\beta/\gamma}^{(i)} - \bar{D}_{(\alpha)\gamma/\beta}^{(i)} &= x_\mu^m R_{(\alpha)(m)\beta\gamma}^{(i)(\mu)} - \bar{D}_{(\alpha)\mu}^{(i)} \bar{T}_{\beta\gamma}^\mu - d_{(\alpha)(m)}^{(i)(\mu)} R_{(\mu)\beta\gamma}^{(m)}, \\
\bar{D}_{(\alpha)\beta|k}^{(i)} - D_{(\alpha)k/\beta}^{(i)} &= x_\mu^m R_{(\alpha)(m)\beta k}^{(i)(\mu)} - \bar{D}_{(\alpha)\mu}^{(i)} \bar{T}_{\beta k}^\mu - D_{(\alpha)m}^{(i)} T_{\beta k}^m - d_{(\alpha)(m)}^{(i)(\mu)} R_{(\mu)\beta k}^{(m)}, \\
D_{(\alpha)j|k}^{(i)} - D_{(\alpha)k|j}^{(i)} &= x_\mu^m R_{(\alpha)(m)jk}^{(i)(\mu)} - D_{(\alpha)m}^{(i)} T_{jk}^m - d_{(\alpha)(m)}^{(i)(\mu)} R_{(\mu)jk}^{(m)}, \\
\bar{D}_{(\alpha)\beta|k}^{(i)(\gamma)} - d_{(\alpha)(k)/\beta}^{(i)(\gamma)} &= x_\mu^m P_{(\alpha)(m)\beta(k)}^{(i)(\mu)(\gamma)} - \bar{D}_{(\alpha)\mu}^{(i)} \bar{C}_{\beta(k)}^{\mu(\gamma)} - d_{(\alpha)(m)}^{(i)(\mu)} P_{(\mu)\beta(k)}^{(m)(\gamma)}, \\
D_{(\alpha)j|k}^{(i)(\gamma)} - d_{(\alpha)(k)|j}^{(i)(\gamma)} &= x_\mu^m P_{(\alpha)(m)j(k)}^{(i)(\mu)(\gamma)} - D_{(\alpha)m}^{(i)} C_{j(k)}^{m(\gamma)} - d_{(\alpha)(m)}^{(i)(\mu)} P_{(\mu)j(k)}^{(m)(\gamma)}, \\
d_{(\alpha)(j)|k}^{(i)(\beta)(\gamma)} - d_{(\alpha)(k)|j}^{(i)(\beta)(\gamma)} &= x_\mu^m S_{(\alpha)(m)(j)(k)}^{(i)(\mu)(\beta)(\gamma)} - d_{(\alpha)(m)}^{(i)(\mu)} S_{(\mu)(j)(k)}^{(m)(\beta)(\gamma)}.
\end{aligned}$$

At the end of this section, we point out the torsion \mathbf{T} and the curvature \mathbf{R} of the Γ -linear connection ∇ are not independent. They verify the general Bianchi identities

$$\begin{cases} \sum_{\{X,Y,Z\}} \{(\nabla_X \mathbf{T})(Y, Z) - \mathbf{R}(X, Y)Z + \mathbf{T}(\mathbf{T}(X, Y), Z)\} = 0, & \forall X, Y, Z \in \mathcal{X}(E) \\ \sum_{\{X,Y,Z\}} \{(\nabla_X \mathbf{R})(U, Y, Z) + \mathbf{R}(\mathbf{T}(X, Y), Z)U\} = 0, & \forall X, Y, Z, U \in \mathcal{X}(E), \end{cases}$$

where $\{X, Y, Z\}$ means cyclic sum.

In the adapted basis $(X_A) = \left(\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x_\alpha^i} \right)$, we have sixty-four effective local Bianchi identities attached to a Γ -linear connection, that can be obtained by the relations

$$(5.4) \quad \begin{cases} \sum_{\{A,B,C\}} \{R_{ABC}^F - T_{AB:C}^F - T_{AB}^G T_{CG}^F\} = 0 \\ \sum_{\{A,B,C\}} \{R_{DAB:C}^F + T_{AB}^G R_{DAG}^F\} = 0, \end{cases}$$

where $\mathbf{R}(X_A, X_B)X_C = R_{CBA}^D X_D$, $\mathbf{T}(X_A, X_B) = T_{BA}^D X_D$ and " :A " represents one of the covariant derivatives " / α ", " | i " or " | i ^(α)". The large number and the complicated form of the above Bianchi identities determined us to describe them effectively only for *h-normal Γ -linear connections*, which are studied in [3]. In that case, the number of Bianchi identities reduces to thirty.

6 Jet prolongation of vector fields

A general vector field X^* on $J^1(T, M)$ can be written under the form

$$X^* = X^\alpha \frac{\partial}{\partial t^\alpha} + X^i \frac{\partial}{\partial x^i} + X_{(\alpha)}^{(i)} \frac{\partial}{\partial x_\alpha^i},$$

where the components X^α , X^i , $X_{(\alpha)}^{(i)}$ are functions of $(t^\alpha, x^i, x_\alpha^i)$.

The prolongation of a vector field X on $T \times M$ to a vector field on the 1-jet bundle $J^1(T, M)$ was solved by Olver, using [6]

Definition 6.1 Let X be a vector field on $T \times M$ with corresponding (local) one-parameter group $\exp(\varepsilon X)$. The *1-th prolongation* of X , denoted by $pr^{(1)}X$, is a vector field on the 1-jet space $J^1(T, M)$, which is defined as the infinitesimal generator of the corresponding prolonged one-parameter group $pr^{(1)}[\exp(\varepsilon X)]$. In other words, we have

$$(6.1) \quad [pr^{(1)}X](t^\alpha, x^i, x_\alpha^i) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} pr^{(1)}[\exp(\varepsilon X)](t^\alpha, x^i, x_\alpha^i).$$

In order to write the components of the 1-th prolongation of a vector field X on $T \times M$, Olver use the α -th total derivative D_α of an arbitrary function $f(t^\alpha, x^i)$ on $T \times M$, which is defined by the relation

$$(6.2) \quad D_\alpha f = \frac{\partial f}{\partial t^\alpha} + \frac{\partial f}{\partial x^i} x_\alpha^i.$$

In this direction, starting with $X = X^\alpha(t, x) \frac{\partial}{\partial t^\alpha} + X^i(t, x) \frac{\partial}{\partial x^i}$ like a vector field on $T \times M$, Olver conclude the 1-th prolongation of X is the vector field

$$(6.3) \quad pr^{(1)}X = X + X_{(\alpha)}^{(i)}(t^\beta, x^j, x_\beta^j) \frac{\partial}{\partial x_\alpha^i},$$

where

$$X_{(\alpha)}^{(i)} = D_\alpha X^i - (D_\alpha X^\beta) x_\beta^i = \frac{\partial X^i}{\partial t^\alpha} + \frac{\partial X^i}{\partial x^j} x_\alpha^j - \left(\frac{\partial X^\beta}{\partial t^\alpha} + \frac{\partial X^\beta}{\partial x^j} x_\alpha^j \right) x_\beta^i.$$

Now, let us study certain geometrical aspects for obtaining the jet prolongation of a given vector field X on $T \times M$.

If we assume that is given a nonlinear connection $\Gamma = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$ on $J^1(T, M)$, we deduce the α -th total derivative used by Olver can be written as

$$(6.4) \quad D_\alpha f = \frac{\delta f}{\delta t^\alpha} + \frac{\delta f}{\delta x^i} x_\alpha^i,$$

and, consequently, $D_\alpha f$ represent the local components of the distinguished 1-form $Df = (D_\alpha f) dt^\alpha$ on $J^1(T, M)$.

In this geometrical context, supposing that $J^1(T, M)$ is also endowed with a Γ -linear connection (2.4), we may define the geometrical 1-th jet prolongation of X as the vector field

$$(6.5) \quad pr^{(1)}X = X^\alpha \frac{\delta}{\delta t^\alpha} + X^i \frac{\delta}{\delta x^i} + Y_{(\alpha)}^{(i)}(t^\beta, x^j, x_\beta^j) \frac{\partial}{\partial x_\alpha^i},$$

where

$$Y_{(\alpha)}^{(i)} = X_{/\alpha}^i + X_{|j}^i x_\alpha^j - X_{/\alpha}^\beta x_\beta^i - X_{|j}^\beta x_\alpha^j x_\beta^i + \\ + X^\mu \left(M_{(\alpha)\mu}^{(i)} + \bar{G}_{\mu\alpha}^{i\beta} x_\beta^i + \bar{L}_{\mu j}^\beta x_\alpha^j x_\beta^i \right) - X^m \left(N_{(\alpha)m}^{(i)} + G_{m\alpha}^i + L_{mj}^i x_\alpha^j \right).$$

Remarks 6.1 i) Taking into account the expressions of local covariant derivatives applied to a vector field (see Remarks 1.2 ii), we conclude our geometrical 1-th jet prolongation coincides with that constructed by Olver. Moreover, the following relations of connection are true:

$$(6.6) \quad Y_{(\alpha)}^{(i)} = X_{(\alpha)}^{(i)} + M_{(\alpha)\mu}^{(i)} X^\mu + N_{(\alpha)m}^{(i)} X^m,$$

where $M_{(\alpha)\mu}^{(i)}$ and $N_{(\alpha)m}^{(i)}$ are the components of the nonlinear connection Γ .

ii) In the particular case of the Berwald Γ_0 -linear connection associated to the metrics $h_{\alpha\beta}$ and φ_{ij} , the expressions of $Y_{(\alpha)}^{(i)}$ reduce to

$$(6.7) \quad Y_{(\alpha)}^{(i)} = X_{//\alpha}^i + X_{\parallel j}^i x_{\alpha}^j - X_{//\alpha}^{\beta} x_{\beta}^i - X_{\parallel j}^{\beta} x_{\alpha}^j x_{\beta}^i - 2\gamma_{jm}^i x_{\alpha}^j X^m,$$

where γ_{jk}^i represent the Christoffel symbols of the metric φ_{ij} .

Open problem. – Study the prolongations of vectors, 1-forms, tensors, G -structures from $T \times M$ to $J^1(T, M)$.

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References

- [1] M. J. Gotay, J. Isenberg, J. E. Marsden, *Momentum Maps and the Hamiltonian Structure of Classical Relativistic Fields*, <http://xxx.lanl.gov/abs/hep/9801019>, (1998).
- [2] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, (1994).
- [3] M. Neagu, *Upon h -normal Γ -linear connections on $J^1(T, M)$* , (2000), <http://xxx.lanl.gov/abs/math.DG/0009070>.
- [4] M. Neagu, C. Udriște, *Multi-Time Dependent Sprays and Harmonic Maps on $J^1(T, M)$* , Third Conference of Balkan Society of Geometers, Politehnica University of Bucharest, Romania, July 31-August 3, (2000); <http://xxx.lanl.gov/abs/math.DG/0009049>, (2000).
- [5] M. Neagu, C. Udriște, *The Geometry of Metrical Multi-Time Lagrange Spaces*, <http://xxx.lanl.gov/abs/math.DG/0009071>, (2000).
- [6] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, (1986).
- [7] D. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press, New York, London, (1989).
- [8] C. Udriște, M. Neagu, *From PDE Systems and Metrics to Generalized Field Theories*, <http://xxx.lanl.gov/abs/math.DG/0101207>, (2001).
- [9] A. Vondra, *Symmetries of Connections on Fibered Manifolds*, Archivum Mathematicum, Tomus 30, 97-115, (1994).

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