Example of Extrinsically Homogeneous Real Hypersurface in $H_3(\mathbf{C})$

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Abstract

The purpose of this paper is to give an example of extrinsically homogeneous real hypersurface in a complex hyperbolic space $H_3(\mathbf{C})$, which is an orbit under a solvable Lie subgroup of the isometry group of $H_3(\mathbf{C})$ and not a Hopf hypersurface.

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Introduction

Let $H_n(\mathbf{C})$ be the hyperbolic complex space form of complex dimension $n(\geq 2)$ endowed with the metric of constant holomorphic sectional curvature c. A submanifold in $H_n(\mathbf{C})$ is said to be *extrinsically homogeneous* if it is an orbit under a closed subgroup of the group of isometries on $H_n(\mathbf{C})$. If the structure vector field of a real hypersurface M in $H_n(\mathbf{C})$ is principal, then M is called a *Hopf hypersurface.*

As proposed also in R. Niebergall and P. J. Ryan ([3]), the following is an open problem : *Classify all extrinsically homogeneous real hypersurfaces in* $H_n(\mathbf{C})$.

As a partial answer of this problem, there is a theorem of J. Berndt ([1]) to the effect that if M is an extrinsically homogeneous real hypersurface in $H_n(\mathbf{C})$ and M is a Hopf hypersurface, then M is congruent to one of well-known homogeneous model spaces of A_0, A_1, A_2 and B type.

In this paper, we shall give an example of extrinsically homogeneous real hypersurface in $H_3(\mathbf{C})$ which is not a Hopf hypersurface.

1 A constuction of an example

At first we construct an example of extrinsically homogeneous real hypersurface in $H_3(\mathbf{C})$. Basically we shall adopt the notations in S. Helgason ([2]).

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Let $GL(4, \mathbf{C})$ be the general linear group of degree 4 over $\mathbf{C}$, and E_{jk} the element $(\delta_{ja}\delta_{kb})_{1\leq a,\, b\leq 4}$ of $GL(4, \mathbf{C})$, where $1 \leq j,\, k \leq 4$. For $I = E_{11} - E_{22} - E_{33} - E_{44}$, we put $G = \{\sigma \in GL(4, \mathbf{C}) |{}^t\sigma\ I\ \bar{\sigma} = I,\ \det \sigma = 1\}$ and

$$K = \left\{ \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix} \middle| \sigma \in U(1), \quad \tau \in U(3),\ \det \sigma \det \tau = 1 \right\}.$$

Then K is a closed subgroup of G, and the homogeneous space G/K is just the hyperbolic complex space form of complex dimension 3, which is denoted by H_3. The Riemannian metric and the complex structure on H_3 will be stated later.

In the following, given a Lie group (e.g. G), we denote the associated Lie algebra of G by the corresponding bold character(e.g. $\mathbf{g}$). Conversely, given a subalgebra (e.g. $\mathbf{l}$) of $\mathbf{g}$, we denote by the corresponding Roman character(e.g. L) the connected Lie subgroup of G whose Lie algebra is $\mathbf{l}$.

We put

$$\begin{array}{lll} A_1 = iE_{11} - iE_{33}, & A2 = iE_{11} - iE_{22}, & A_3 = iE_{11} - iE_{44}, \\ Y_1 = iE_{23} + iE_{32}, & Y_2 = E_{23} - E_{32}, & Y_3 = iE_{24} + iE_{42}, \\ Y_4 = E_{24} - E_{42}, & Y_5 = iE_{34} + iE_{43}, & Y_6 = E_{34} - E_{43}, \\ X_1 = iE_{12} - iE_{21}, & X_2 = iE_{13} - iE_{31}, & X_3 = E_{13} + E_{31}, \\ X_4 = E_{12} + E_{21}, & X_5 = iE_{14} - iE_{41}, & X_6 = E_{14} + E_{41}. \end{array}$$

Then the set of the above eight vectors (resp. the set $\{A_1, A_2, Y_1, Y_2\}$) forms basis for $\mathbf{g}$ (resp. $\mathbf{k}$). By a simple computation of bracket product operation in $\mathbf{g}$, we have the following table :

$$\begin{array}{lll} [A_1, A_2] = 0, & [A_1, A_3] = 0, & [A_1, Y_1] = -Y_2, \\ [A_1, Y_2] = Y_1, & [A_1, Y_3] = 0, & [A_1, Y_4] = 0, \\ [A_1, Y_5] = Y_6, & [A_1, Y_6] = -Y_5, & [A_1, X_1] = -X_4, \\ [A_1, X_2] = -2X_3, & [A_1, X_3] = 2X_2, & [A_1, X_4] = X_1, \\ [A_1, X_5] = -X_6, & [A_1, X_6] = X_5, & [A_2, A_3] = 0, \\ [A_2, Y_1] = Y_2, & [A_2, Y_2] = -Y_1, & [A_2, Y_3] = Y_4, \\ [A_2, Y_4] = -Y_3, & [A_2, Y_5] = 0, & [A_2, Y_6] = 0, \\ [A_2, X_1] = -2X_4, & [A_2, X_2] = -X_3, & [A_2, X_3] = X_2, \\ [A_2, X_4] = 2X_1, & [A_2, X_5] = -X_6, & [A_2, X_6] = X_5, \\ [A_3, Y_1] = 0, & [A_3, Y_2] = 0, & [A_3, Y_3] = -Y_4, \\ [A_3, Y_4] = Y_3, & [A_3, Y_5] = -Y_6, & [A_3, Y_6] = Y_5, \\ [A_3, X_1] = -X_4, & [A_3, X_2] = -X_3, & [A_3, X_3] = X_2, \\ [A_3, X_4] = X_1, & [A_3, X_5] = -2X_6, & [A_3, X_6] = 2X_5, \\ [Y_1, Y_2] = 2A_2 - 2A_1, & [Y_1, Y_3] = -Y_6, & [Y_1, Y_4] = Y_5, \\ [Y_1, Y_5] = -Y_4, & [Y_1, Y_6] = Y_3, & [Y_1, X_1] = X_3, \\ [Y_1, X_2] = X_4, & [Y_1, X_3] = -X_1, & [Y_1, X_4] = -X_2, \\ [Y_1, X_5] = 0, & [Y_1, X_6] = 0, & [Y_2, Y_3] = -Y_5, \\ [Y_2, Y_4] = -Y_6, & [Y_2, Y_5] = Y_3, & [Y_2, Y_6] = Y_4, \\ [Y_2, X_1] = -X_2, & [Y_2, X_2] = X_1, & [Y_2, X_3] = X_4, \\ [Y_2, X_4] = -X_3, & [Y_2, X_5] = 0, & [Y_2, X_6] = 0, \\ [Y_3, Y_4] = 2A_2 - 2A_3, & [Y_3, Y_5] = -Y_2, & [Y_3, Y_6] = -Y_1, \\ [Y_3, X_1] = X_6, & [Y_3, X_2] = 0, & [Y_3, X_3] = 0, \\ [Y_3, X_4] = -X_5, & [Y_3, X_5] = X_4, & [Y_3, X_6] = -X_1, \\ [Y_4, Y_5] = Y_1, & [Y_4, Y_6] = -Y_2, & [Y_4, X_1] = -X_5, \end{array} \tag{1.1}$$

$$\begin{array}{lll}
[Y_4, X_2] = 0, & [Y_4, X_3] = 0, & [Y_4, X_4] = -X_6, \\
[Y_4, X_5] = X_1, & [Y_4, X_6] = X_4, & [Y_5, Y_6] = 2A_1 - 2A_3, \\
[Y_5, X_1] = 0, & [Y_5, X_2] = X_6, & [Y_5, X_3] = -X_5, \\
[Y_5, X_4] = 0, & [Y_5, X_5] = X_3, & [Y_5, X_6] = -X_2, \\
[Y_6, X_1] = 0, & [Y_6, X_2] = -X_5, & [Y_6, X_3] = -X_6, \\
[Y_6, X_4] = 0, & [Y_6, X_5] = X_2, & [Y_6, X_6] = X_3, \\
[X_1, X_2] = Y_2, & [X_1, X_3] = -Y_1, & [X_1, X_4] = 2A_2, \\
[X_1, X_5] = Y_4, & [X_1, X_6] = -Y_3, & [X_2, X_3] = 2A_1, \\
[X_2, X_4] = -Y_1, & [X_2, X_5] = Y_6, & [X_2, X_6] = -Y_5, \\
[X_3, X_4] = -Y_2, & [X_3, X_5] = Y_5, & [X_3, X_6] = Y_6, \\
[X_4, X_5] = Y_3, & [X_4, X_6] = Y_4, & [X_5, X_6] = 2A_3.
\end{array}$$

We put $\mathbf{p} = \mathbf{R}X_1 + \mathbf{R}X_2 + \mathbf{R}X_3 + \mathbf{R}X_4 + \mathbf{R}X_5 + \mathbf{R}X_6$. Then we have a Cartan decomposition of $\mathbf{g}$

$$\mathbf{g} = \mathbf{k} + \mathbf{p}.$$

For any element X of $\mathbf{g}$, we denote the $\mathbf{k}$ (resp. $\mathbf{p}$)-component of X by $X_{\mathbf{k}}$ (resp. $X_{\mathbf{p}}$).

We shall identify $\mathbf{p}$ with the tangent space $T_o(H_3)$ of H_3 at the origin o. For a constant $c(<0)$, we give on H_3, regarded as a symmetric space, a Riemannian metric $\langle\ ,\ \rangle$ in such a way that

$$\langle \sqrt{-c}\ X_i,\ \sqrt{-c}\ X_j \rangle = \delta_{ij} \qquad (1 \leq i, j \leq 6)$$

at o. Such an H_3 is the hyperbolic complex space form of constant holomorphic sectional curvature $4c$ of complex dimension 3, which is denoted by $H_3(\mathbf{C})$. Then G acts on $H_3(\mathbf{C})$ as a group of isometries. The complex structure J on $H_3(\mathbf{C})$ is given by (cf. Helgason [2], p. 393)

$$(1.2) \qquad J(X_1) = X_4, \qquad J(X_2) = X_3 \qquad \text{and} \qquad J(X_5) = X_6,$$

where we have $J = ad\left(-\frac{1}{4}(A_1 + A_2 + A_3)\right)$.

For any element σ of G and for any 4×4 matrix X over $\mathbf{C}$, we put

$$Ad(\sigma)\ X = \sigma\ X\ \sigma^{-1}.$$

Then $Ad(\sigma)$ $(\sigma \in G)$ is an isomorphism of G as well as an inner automorphism of $\mathbf{g}$. The exponential map exp of $\mathbf{g}$ into G is given by

$$\exp X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \qquad \text{for} \qquad X \in \mathbf{g}.$$

Then the followings are well-known, or can be easily checked :

$$(1.3) \qquad Ad(\sigma)\ \mathbf{p} \subset \mathbf{p} \quad \text{for} \quad \sigma \in K,$$

$$(1.4) \qquad \left.\frac{d}{dt}\right|_o Ad(\exp\ tX)\ Y = [X,\ Y] \quad \text{for} \quad X,\ Y \in \mathbf{g},$$

$$\sigma\ (\exp\ X)\ \sigma^{-1} = \exp(\sigma\ X\ \sigma^{-1}) \quad \text{for} \quad \sigma \in G,\ X \in \mathbf{g}, \tag{1.5}$$

$$\begin{aligned} &\exp(sA_1 + tA_2 + uA_3) = e^{i(s+t+u)}E_{11} + e^{-it}E_{22} + e^{-is}E_{33} + e^{-iu}E_{44} \\ &\text{for} \quad s,\ t,\ u \in \mathbf{R}, \end{aligned} \tag{1.6}$$

(1.7) The group $Ad(K)$ acts on any hypersphere of $\mathbf{p}$ centered at the origin transitively.

Remark 1.1. The statement (1.7) is a property of a symmetric space of rank 1.

For two subalgebras $\mathbf{l}_1$ and $\mathbf{l}_2$ of $\mathbf{g}$, if there is an element σ of K such that

$$Ad(\sigma)\ \mathbf{l}_1 = \mathbf{l}_2,$$

then the corresponding two orbits $L_1(o)$ and $L_2(o)$ are congruent in $H_3(\mathbf{C})$ since $\sigma(L_1(o)) = L_2(o)$ by (1.5).

Put $Z_1 = X_1 + Y_3,\ Z_2 = X_2 + Y_5,\ Z_3 = X_5 - A_3,\ Z_4 = X_6$ and $Z_5 = X_3 - Y_6$. Then it follows from (1.1) that

$$\begin{aligned} &[Z_1,\ Z_2] = [Z_1,\ Z_3] = [Z_1,\ Z_5] = [Z_2,\ Z_3] = [Z_3,\ Z_5] = 0, \\ &[Z_1,\ Z_4] = -Z_1,\ [Z_2,\ Z_4] = -Z_2,\ [Z_4,\ Z_5] = Z_5, \\ &[Z_3,\ Z_4] = [Z_2,\ Z_5] = -2Z_3. \end{aligned} \tag{1.8}$$

If we define a subspace $\mathbf{l}$ of $\mathbf{g}$ by

$$\mathbf{l} = \mathbf{R}Z_1 + \mathbf{R}Z_2 + \mathbf{R}Z_3 + \mathbf{R}Z_4 + \mathbf{R}Z_5, \tag{1.9}$$

then we see from (1.8) that $\mathbf{l}$ is a solvable Lie subalgebra of $\mathbf{g}$.

Now we can state our example as follows.

Theorem 1.1. *Let L be the connected Lie subgroup of G whose associated Lie algebra is $\mathbf{l}$ given in (1.9), and denote by σ_t the 1-parameter subgroup $\exp tX_4$ of G. Then, for any $t \in \mathbf{R}$, the orbit $L(\sigma_t(o))$ of the point $\sigma_t(o)$ under L is an extrinsically homogeneous real hypersurface in $H_3(\mathbf{C})$ whose structure vector is not principal.*

In order to prove Theorem 1.1, we must make some preperations. Let ∇ be the Riemannian connection of $H_3(\mathbf{C})$ with respect to the Riemannian metric $\langle\ ,\ \rangle$. Each element X of $\mathbf{g}$ induces a differentiable vector field X^* on $H_3(\mathbf{C})$ such as

$$X^*_p = \frac{d}{dt}\Big|_o (\exp\ tX)(p), \qquad p \in H_3(\mathbf{C}).$$

Then the following results are well-known in the theory of a symmetric spaces

(1.10) For any $X \in \mathbf{g}$, X^* is a Killing vector field on $H_3(\mathbf{C})$,

$$[X^*,\ Y^*] = -[X,\ Y]^* \ \text{ for any } X,\ Y \in \mathbf{g}, \tag{1.11}$$

$$\nabla_{X^*_o} Y^* = 0 \quad \text{for any} \quad X,\ Y \in \mathbf{p}. \tag{1.12}$$

It is clear that, for any $X \in \mathbf{g}$, we have $X_o^* = X_{\mathbf{p}} \in \mathbf{p} \equiv T_o(H_3(\mathbf{C}))$. In particular, we see that

$$X_o^* = \begin{cases} 0 & \text{if} \quad X \in \mathbf{k} \\ X & \text{if} \quad X \in \mathbf{p}. \end{cases}$$

To find out the shape operator of an orbit under G, we shall prove

Proposition 1.2. *Let $\boldsymbol{m}$ be any Lie subalgbra of $\mathbf{g}$, and M be the corresponding analytic Lie subgroup of G such that* $\dim\ M(o)\ \leq 5$. *Let $\nu \in \mathbf{p}$ be a normal vector of the orbit $M(o)$ at o. Then the shape operator T_ν of $M(o)$ in the direction ν is given by $T_\nu(X_{\mathbf{p}}) = [X_{\mathbf{k}},\ \nu]_M$ for $X \in \boldsymbol{m}$, where $[X_{\mathbf{k}},\ \nu]_M$ indicates the $T_o(M)$-component of $[X_{\mathbf{k}},\ \nu]$.*

Proof. First we assert that

$$\nabla_{X_o^*} Y^* = -[X,\ Y] \quad \text{for any } X \in \mathbf{p} \text{ and } Y \in \mathbf{k}. \tag{1.13}$$

In fact, we have

$$\nabla_{X^*} Y^* = \nabla_{Y^*} X^* + [X^*,\ Y^*] = \nabla_{Y^*} X^* - [X,\ Y]^*$$

by (1.11). Evaluating this equation at o, we obtain (1.13).

Next, by use of the result in R.Takagi and T.Takahashi ([4], 471p) and the equations (1.12) and (1.13), we get

$$T_\nu(X_{\mathbf{p}}) = -\nabla_\nu X^* = -\nabla_{\nu_o^*}(X_{\mathbf{k}}^* + X_{\mathbf{p}}^*) = [X_{\mathbf{k}},\ \nu]_M,$$

and this completes the proof. □

Remark 1.2. As seen from the above proof, Proposition 1.2 holds for any symmetric space and the isometry group of it.

Proof of Theorem 1.1. It is clear by definition that the orbit $L(\sigma_t(o))$ is an extrinsically homogeneous real hypersurface in $H_3(\mathbf{C})$.

Since the orbit $L(\sigma_t(o))$ is congruent to the orbit $(Ad(\sigma_t^{-1})L)(o)$ under $Ad(\sigma_t^{-1})L$ in $H_3(\mathbf{C})$, we shall investigate the shape operator and the structure vector on the latter. For simplicity, we put $c_t = \cosh t$ and $s_t = \sinh t$. Then we see that $\sigma_t = c_t E_{11} + c_t E_{22} + E_{33} + E_{44} + s_t E_{12} + s_t E_{21}$. By a simple calculation, we have

$$\begin{aligned} Ad(\sigma_t)Z_1 &= -2s_t c_t A_2 + c_t Y_3 + (c_t^2 + s_t^2)X_1 + s_t X_5, \\ Ad(\sigma_t)Z_2 &= s_t Y_1 + Y_5 + c_t X_2, \\ Ad(\sigma_t)Z_3 &= -s_t^2 A_2 - A_3 + s_t Y_3 + s_t c_t X_1 + c_t X_5, \\ Ad(\sigma_t)Z_4 &= s_t Y_4 + c_t X_6, \\ Ad(\sigma_t)Z_5 &= s_t Y_2 - Y_6 + c_t X_3. \end{aligned} \tag{1.14}$$

Since $\nu = X_4$ is the normal vector of $(Ad(\sigma_t^{-1})L)(o)$, it follows from (1.1), Proposition 1.2 and (1.14) that

$$\begin{aligned} T_\nu((c_t^2 + s_t^2)X_1 + s_t X_5) &= -4s_t c_t X_1 - c_t X_5, \\ T_\nu(s_t c_t X_1 + c_t X_5) &= -(2s_t^2 + 1)X_1 - s_t X_5, \\ T_\nu(c_t X_2) &= -s_t X_2, \\ T_\nu(c_t X_3) &= -s_t X_3, \\ T_\nu(c_t X_6) &= -s_t X_6. \end{aligned}$$

Therefore, with respect to a basis $\{X_1,\ X_2,\ X_3\}$ for $T_o((Ad(\sigma_t^{-1})L)(o))$, the shape operator $T := T_{\sqrt{-c}X_4}$ of $(Ad(\sigma_t^{-1})L)(o)$ is expressed by

$$
\begin{aligned}
T(X_1) &= \sqrt{-c}(\tanh^3 t - 3\tanh t)X_1 - \sqrt{-c}\,\mathrm{sech}^3 t\ X_5,\\
T(X_2) &= -\sqrt{-c}\tanh t\ X_2,\\
T(X_3) &= -\sqrt{-c}\tanh t\ X_3, \qquad (1.15)\\
T(X_5) &= -\sqrt{-c}\,\mathrm{sech}^3 t\ X_1 - \sqrt{-c}\tanh^3 t\ X_5,\\
T(X_6) &= -\sqrt{-c}\tanh t\ X_6.
\end{aligned}
$$

Since the structure vector of $(Ad(\sigma_t^{-1})L)(o)$ is X_1 by (1.2) and sech $t \neq 0$ for any $t \in \mathbf{R}$, (1.15) shows that X_1 is not principal. This completes the proof. □

Remark 1.3. By the above computation we see that the vector $X_2,\ X_3$ and X_6 are always principal, and for any $t \in \mathbf{R}$ the orbit $L(\sigma_t(o))$ has the following three distinct principal curvatures $-\sqrt{-c}\ \tanh t$ (of multiplicity 3), $-\sqrt{-c}\left(\frac{3}{2}\tanh t \pm \sqrt{1-\frac{3}{4}\tanh^2 t}\right)$. In particular, the orbit $L(o)$ is minimal.

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