

On Normal Sections of Stiefel Submanifold

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Abstract

In this study we consider Stiefel submanifold $\mathbf{V}_{n,k}$ which is a projective m -space \mathbf{P}^m ($n = mk$) isometrically imbedded in $\mathbf{R}^{m+\frac{1}{2}m(m+1)}$ by its first standard imbedding. We also consider the normal sections of $\mathbf{V}_{n,k}$ and show that $\mathbf{V}_{n,k}$ has $P2 - PNS$ property.

Mathematics Subject Classification: 53C40, 53C42

Key words: Normal section, Stiefel manifold

1 Introduction

Let M be an n -dimensional submanifold in $(m + d)$ - dimensional Euclidean space \mathbf{R}^{n+d} . Let ∇ , $\bar{\nabla}$, and $\tilde{\nabla}$ denote the covariant derivatives in $T(M)$, $N(M)$ and \mathbf{R}^{n+d} respectively. Thus $\tilde{\nabla}_X$ is just the directional derivative in the direction X in \mathbf{R}^{n+d} . Then for tangent vector fields X , Y and Z and normal vector field v over M we have $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and $\tilde{\nabla}_X v = -A_v X + \bar{\nabla}_X v$, where h is the second fundamental form and A_v is the shape operator of M [6]. For tangent vector fields X, Y, Z over M we define $\bar{\nabla}h$ as usual by

$$\bar{\nabla}_X h(Y, Z) = (\bar{\nabla}_X h)(Y, Z) + h(\nabla_X Y, Z) + h(Y, \nabla_X Z).$$

Let M be a smooth n -dimensional submanifold in $(n + d)$ -dimensional Euclidean space \mathbf{R}^{n+d} . For a point x in M and a non-zero tangent vector $X \in T_x M$, we define the $(d+1)$ -dimensional affine subspace $E(x, X)$ of \mathbf{R}^{n+d} by $E(x, X) = x + \text{span}\{X, T_x^\perp M\}$. In a neighborhood of x the intersection $M \cap E(x, X)$ is a regular curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$. We suppose the parameter $t \in (-\varepsilon, \varepsilon)$ is a multiple of the arc-length such that $\gamma(0) = x$ and $\gamma'(0) = X$. Each choice of $X \in T(M)$ yields a different curve which is called the *normal section of M at x in the direction of X* , where $X \in T_x(M)$ [8]. For such a normal section we can write

$$(1) \quad \gamma(t) = x + \lambda(t)X + N(t)$$

where $N(t) \in T_x^\perp M$ and $\lambda(t) \in \mathbf{R}$.

Definition 1.1. The submanifold M is said to have pointwise k -planar normal sections ($Pk - PNS$) if for each normal section γ , the higher order derivatives $\gamma'(0)$, $\gamma''(0), \dots, \gamma^{(k+1)}(0)$ are linearly dependent as vectors in \mathbf{R}^{n+d} .

If $k = 1$ then M is totally geodesic. Taking $k = 2$, we note that submanifolds with pointwise 2-planar normal sections have been classified. They have parallel second fundamental form (i. e. $\bar{\nabla}h = 0$) or hypersurfaces [1], see also [10] and [2].

Theorem 1.1. [1] *Let M be a submanifold of \mathbf{R}^{n+d} then M has pointwise 2-planar normal sections ($P2 - PNS$) if and only if*

$$(2) \quad \|h(X, X)\|^2 (\bar{\nabla}_X h)(X, X) = \langle (\bar{\nabla}_X h)(X, X), h(X, X) \rangle h(X, X).$$

2 Stiefel Submanifolds $\mathbf{V}_{n,k}$

For each n, k ($k \leq n$) the Stiefel manifold $\mathbf{V}_{n,k}$ has as its points all orthonormal frames $x = (e_1, \dots, e_k)$ of k vectors in Euclidean n -space (i.e. ordered sequences of k orthonormal vectors in \mathbf{R}^n). Any orthogonal matrix A of degree n sends any such orthonormal frame x to another, namely $Ax = (Ae_1, \dots, Ae_k)$, this defines an action of $O(n)$ on $\mathbf{V}_{n,k}$ which is transitive (see [9]).

Each Stiefel manifold $\mathbf{V}_{n,k}$ can be realized as a non-singular surface in the Euclidean space \mathbf{R}^{nk} in the following way. Fix on an orthonormal basis for \mathbf{R}^n (e.g. the standard basis), and introduce the following notation for the components with respect to this basis of any k -frame (e_1, \dots, e_k) (i.e. point of $\mathbf{V}_{n,k}$):

$$e_i = (x_{i1}, \dots, x_{in}) \quad , \quad i = 1, \dots, k.$$

The nk quantities x_{ij} , $i = 1, \dots, k$; $j = 1, \dots, n$, (in lexicographic order, say) are now to be regarded as the co-ordinates of a point in nk - dimensional Euclidean space \mathbf{R}^{nk} , related by the following $k(k+1)/2$ equations:

$$\langle e_i, e_j \rangle = \delta_{ij} \iff \sum_{s=1}^n x_{is}x_{js} = \delta_{ij} \quad , \quad i, j = 1, \dots, k, i \leq j$$

We now investigate the isotropy group of this homogenous space. Take any orthonormal k -frame e_1, \dots, e_k and enlarge it to an orthonormal basis e_1, \dots, e_n for the whole of Euclidean n -space. Any orthogonal transformation fixing the vectors e_1, \dots, e_k must (relative to the above basis for \mathbf{R}^n) have the form

$$\begin{bmatrix} 1 & & & 0 \\ & \ddots & & 0 \\ & & 1 & \\ 0 & & & A \end{bmatrix} \quad A \in O(n-k)$$

whence the isotropy group is isomorphic to $O(n, k)$, and $\mathbf{V}_{n,k}$ can be identified with $O(n)/O(n-k)$. In fact $\mathbf{V}_{n,k} \cong O(n)/O(n-k)$.

The Stiefel manifolds $\mathbf{V}_{n,k}$ for $k < n$ are also homogeneous spaces for the group $SO(n)$ (see [9]). From this point of view the isotropy group is clearly (isomorphic to) $SO(n-k)$, and therefore also

$$\mathbf{V}_{n,k} \cong SO(n)/SO(n-k).$$

In particular, we have

$$\mathbf{V}_{n,n} \cong O(n) \quad , \quad \mathbf{V}_{n,n-1} \cong SO(n) \quad , \quad \mathbf{V}_{n,1} \cong S^{n-1}.$$

In [11] Jimenez shown that; i) $\mathbf{V}_{n,k}$ belongs to the class of reduced Riemannian Σ -spaces, ii) if $n = 2k > 30$ and $k = n - 2$, $\mathbf{V}_{n,k}$ belongs to the class of all pointwise Riemannian \mathbf{k} -symmetric spaces ($\mathbf{k} \geq 2$) and furthermore, it is not diffeomorphic to the underlying manifold of any symmetric in the class of all regular Riemannian \mathbf{k} -symmetric spaces ($\mathbf{k} \geq 2$), and iii) if n and k are appropriately chosen $\mathbf{V}_{n,k}$ is not diffeomorphic to the underlying manifold of a space in class of all pointwise Riemannian r -symmetric spaces.

3 The First Standard Imbedding Of The Real Projective Space \mathbf{P}^m

In this section we will consider the Stiefel submanifold as the first standard imbedding of the real projective space \mathbf{P}^m ($m = nk$) in $\mathbf{R}^{m+\frac{1}{2}m(m+1)}$. The notation here is essentially the same as in [7].

Let $M(m+1; \mathbf{R})$ be the space of $(m+1) \times (m+1)$ matrices over \mathbf{R} . It is considered as an $(m+1)^2$ -dimensional Euclidean space with the inner product

$$\langle A, B \rangle = \frac{1}{2} \text{trace} AB^T, \text{ where } B^T \text{ is the transpose of the matrix } B.$$

Someone consider \mathbf{P}^m as the quotient space of the hypersphere

$$\mathbf{S}^m = \{ \zeta \in \mathbf{R}^{n+1} : \zeta^T \zeta = I_k \}; \quad m = nk,$$

obtained by identifying ζ with $\zeta\lambda$, where ζ is a column vector and $\lambda \in \mathbf{R}$ such that $|\lambda| = 1$.

Define a mapping $\tilde{\varphi} : \mathbf{S}^m \rightarrow H(m+1; \mathbf{R}) = \{A \in M(m+1; \mathbf{R}) : A^T = A\}$ as follows

$$\tilde{\varphi}(\zeta) = \zeta\zeta^T = \begin{pmatrix} |\zeta_0|^2 & \zeta_0\zeta_1 & - & \zeta_0\zeta_n \\ - & - & - & - \\ \zeta_n\zeta_0 & \zeta_n\zeta_1 & - & |\zeta_n|^2 \end{pmatrix}$$

for $\zeta = (\zeta_i) \in \mathbf{S}^m \subset \mathbf{R}^{m+1}$, $0 \leq i \leq m$. Then it is easy to verify that $\tilde{\varphi}$ induces a mapping φ of \mathbf{P}^m into $H(m+1; \mathbf{R})$:

$$(3) \quad \varphi((\pi(\zeta))) = \tilde{\varphi}(\zeta) = \zeta\zeta^T.$$

where $\pi : \mathbf{S}^m \rightarrow \mathbf{P}^m$ is a Riemannian submersion [7]. We simply denote $\varphi((\pi(\zeta)))$ by $\varphi(\zeta)$.

From (3) the image of \mathbf{P}^m under φ is given by

$$\varphi(\mathbf{P}^m) = \{A \in H(m+1; \mathbf{R}) : A^2 = A \text{ and } \text{trace } A = 1\}, \quad m = nk, .$$

Let $A = \zeta\zeta^T$ be a point in $\varphi(\mathbf{P}^m)$. Consider the curve

$$(4) \quad A(t) = \zeta\zeta^T; \zeta \in \mathbf{R}^{m+1}$$

in $\varphi(\mathbf{P}^m)$ with $A(0) = A$ and $A'(0) = X \in T_A(\mathbf{P}^m)$. From $A^2(t) = A(t)$ one gets $XA^T + AX^T = X$. So we have

$$T_A(\mathbf{P}^m) = \{X \in H(m+1; \mathbf{R}) : XA^T + AX^T = X\}.$$

Proposition 3.1 [7]. *Let Y be a vector field tangent to \mathbf{P}^m and $X \in T_A(\mathbf{P}^m)$. Consider a curve $A(t)$ in $\varphi(\mathbf{P}^m)$ so that $A(0) = A$ and $A'(0) = X$. Denote by $Y(t)$ the restriction of Y to $A(t)$. Then*

$$A(t)Y(t) + Y(t)A(t) = Y(t).$$

Corollary 3.2. *Let $A(t) = \zeta(t)\zeta(t)^T$; $\zeta(t) \in \mathbf{R}^{m+1}$ be a curve in $\varphi(\mathbf{P}^m)$ then*

$$(5) \quad A(0) = X = \xi a^T + a\xi^T$$

where

$$\zeta(0) = \xi, \zeta'(0) = a.$$

Proof. Let $A(t) = \zeta\zeta^T$ be a curve in $\varphi(\mathbf{P}^m)$ with $A(0) = A$ and $A'(0) = X \in T_A(\mathbf{P}^m)$. Differentiating (5) we get

$$A'(t) = \zeta'\zeta^T + \zeta\zeta'^T.$$

At point $t = 0$ since $\zeta(0) = \xi, \zeta'(0) = a$ we get the result.

A vector ν in $H(m+1; \mathbf{R})$ is normal to \mathbf{P}^m at A if and only if $\langle X, \nu \rangle = 0$ for all X in $T_A(\mathbf{P}^m)$. Thus, ν is in $T_A^\perp(\mathbf{P}^m)$ if and only if $\text{trace}(X\nu) = 0$ for all X in $T_A(\mathbf{P}^m)$. Therefore by (5) we obtain

$$T_A^\perp(\mathbf{P}^m) = \{\nu \in H(m+1; \mathbf{R}) : A\nu = \nu A\}.$$

Theorem 3.3 [7]. *The isometric imbedding $\varphi : \mathbf{S}^m \rightarrow H(m+1; \mathbf{R})$, determined by (3), is the first standard imbedding of \mathbf{P}^m into $H(m+1, \mathbf{R})$ and \mathbf{P}^m lies in a hypersphere $\mathbf{S}(r)$ of $H(m+1; \mathbf{R})$ centered at $\frac{1}{m+1}I$ and with radius $r = \sqrt{\frac{m}{2}(m+1)}$.*

Definition 3.1. A real projective m -space \mathbf{P}^m ($m = nk$) isometrically imbedded in $\mathbf{R}^{m+\frac{1}{2}m(m+1)}$ by its first standard imbedding φ determined by (3) is called the Stiefel submanifold $\mathbf{V}_{n,k}$.

4 Normal Sections Of $\mathbf{V}_{n,k}$

An element $A \in \mathbf{V}_{n,k}$ can be written as $A = \zeta\zeta^T$ where $\zeta^T\zeta = I_{k \times k}$ and $\zeta = \xi m + \eta$, $\xi^T\eta = 0$, $m \in M_{k \times k}(\mathbf{R})$. Since ζ is not unique we can replace ζ by ζu where $u \in O(k)$. Put $m = \xi^T\zeta$ then we can find $u \in O(k)$ so that mu is lower triangular and the diagonal element of mu are ≥ 0 by using the Gram-Schmidt orthonormalization process. Further mu is uniquely defined by m and $\xi^T(\zeta - \xi m) = \xi^T\zeta - m = 0$. Thus we can write $A = \zeta\zeta^T$ where $\zeta = \xi m + \eta$ where m is lower triangular $k \times k$ with diagonal term ≥ 0 and $\xi^T\eta = 0$, $m \in M_{k \times k}(\mathbf{R})$ where m and η are unique. So $A = \xi m m^T \xi^T + \xi m \eta^T + \eta m^T \xi^T + \eta \eta^T$. Interchanging A with the curve $A(t)$ in $\varphi(\mathbf{P}^m)$ we get

$$(6) \quad A(t) = \zeta(t)\zeta(t)^T = \xi m(t)m(t)^T \xi^T + \xi m(t)\eta(t)^T + \eta(t)m(t)^T \xi^T + \eta(t)\eta(t)^T.$$

Lemma 4.1. *Let $A(t)$ be a normal section of $\mathbf{V}_{n,k}$ at point $A(0)$ in the direction of $A'(0) = X$. Then*

$$(7) \quad \lambda(t)X = \xi m(t)\eta(t)^T + \eta(t)m(t)^T \xi^T$$

$$(8) \quad N(t) = \xi m(t)m(t)^T \xi^T + \eta(t)\eta(t)^T$$

where $a\xi^T + \xi a^T = X$, $\xi^T \xi = I_k$ and $a^T \xi + \xi^T a = 0$, $a = \eta'(0) \in M_{n \times k}$.

Proof. Since $\xi^T \eta(t) = 0$ then $m(t)\eta(t)^T \xi + \xi^T \eta(t)m(t)^T = 0$ therefore $\xi m(t)\eta(t)^T + \eta(t)m(t)^T \xi^T \in T_x(\mathbf{V}_{n,k})$. Also $\eta(t)\eta(t)^T \in N_x(\mathbf{V}_{n,k})$ since $\text{trace}(a^T \eta(t)\eta(t)^T \xi) = 0$ (whatever $a \in M_{n \times k}$). Also

$$\begin{aligned} a^T(\xi m(t)m(t)^T \xi^T) \xi &= \text{trace}(a^T \xi m(t)m(t)^T) = \text{trace}(m(t)m(t)^T \xi^T a) \\ &= \frac{1}{2} \text{trace}(a^T \xi + \xi^T a) m(t)m(t)^T = 0 \end{aligned}$$

if $a^T \xi + \xi^T a = 0$. Therefore $\xi m(t)m(t)^T \xi^T \in N_x(\mathbf{V}_{n,k})$. Hence comparing (1) with (6) we get the result.

Lemma 4.2. *Let $A(t) = \zeta(t)\zeta(t)^T$ be a normal section of $\mathbf{V}_{n,k}$ at point $A(0)$ in the direction of $A'(0) = X$. Then*

$$(9) \quad m(t)^T m(t) + \eta(t)^T \eta(t) = I_k.$$

Proof. Since $\zeta(t) = \xi m(t) + \eta(t)$,

$$(10) \quad \zeta(t)^T \zeta(t) = I_k,$$

and

$$(11) \quad \xi^T \eta(t) = 0, \eta \in M_{n \times k}$$

we get the result.

Theorem 4.3. *Stiefel submanifold $\mathbf{V}_{n,k}$ (considered as a projective m -space \mathbf{P}^m ($n = mk$) isometrically imbedded in $\mathbf{R}^{m+\frac{1}{2}m(m+1)}$ by its first standard imbedding defined by (3)) has $P2 - PNS$ property.*

Proof. The normal section $A(t)$ of $\mathbf{V}_{n,k}$ at point $A(0) = \xi \xi^T$ in the direction of $A'(0) = X$ is given by $A(t) = \zeta(t)\zeta(t)^T$, where $\zeta(t) = \xi m(t) + \eta(t)$, $\eta, m \in M_{n \times k}$,

$$(12) \quad m(t) = \xi^T \zeta(t)$$

and

$$(13) \quad \eta(t)m(t)^T = \lambda(t)a.$$

At point $t = 0$ we get

$$(14) \quad \left. \begin{aligned} \zeta(0) &= \xi \\ m(0) &= I_k \\ \eta(0) &= 0 \\ A(0) &= \xi \xi^T \end{aligned} \right\}$$

at least for sufficiently small $A(t)$. Thus we now differentiate (9)-(13) and so determine $m', \eta', \lambda', m'', \eta'', \lambda''$. Then using $N(t) = \xi m(t)m(t)^T \xi^T + \eta(t)\eta(t)^T$ we compute N', N'' , etc.

Differentiating (9), (12) and (11) (surprising the dependence of ζ on t to simplify the notation) we get

$$(15) \quad A'(t) = \zeta'(t)\zeta(t)^T + \zeta(t)\zeta'(t)^T = \lambda'(t)X + N'(t),$$

$$(16) \quad \zeta'(t)^T \zeta(t) + \zeta(t)^T \zeta'(t) = 0,$$

$$(17) \quad m'(t) = \xi^T \zeta'(t)$$

Putting $t = 0$ we get

$$(18) \quad A'(0) = X = \xi a^T + a \xi^T$$

$$(19) \quad \zeta(0) = \xi, \zeta'(0) = \eta'(0) = a,$$

$$(20) \quad m'(0) = \xi^T a = 0.$$

Differentiating (13) two times with respect to t we also get

$$\eta''(t)m(t)^T + 2\eta'(t)m'(t)^T + \eta(t)m''(t)^T = \lambda''(t)a.$$

At $t = 0$ by the use of (14) and (20) the above equation gives

$$(21) \quad \eta''(0) = 0.$$

Differentiating (9) and (8) we get

$$(22) \quad \begin{aligned} 0 &= m''(t)^T m(t) + 2m'(t)^T m'(t) + m(t)m''(t)^T + \\ &+ \eta''(t)^T \eta(t) + 2\eta'(t)^T \eta'(t) + \eta(t)\eta''(t)^T \end{aligned}$$

and

$$(23) \quad \begin{aligned} N''(t) &= \xi m''(t)m(t)^T \xi^T + 2\xi m'(t)m'(t)^T \xi^T + \xi m(t)m''(t)^T \xi^T \\ &+ \eta''(t)\eta(t)^T + 2\eta'(t)\eta'(t)^T + \eta(t)\eta''(t)^T. \end{aligned}$$

At $t = 0$ by (14), (20) and (21) we have

$$m''(0)^T + m''(0) = -2a^T a$$

and

$$(24) \quad N''(0) = h(X, X) = -2\xi a^T a \xi^T + 2aa^T.$$

Differentiating (22) and (23) with respect to t we get

$$(25) \quad \begin{aligned} 0 &= m'''(t)^T m(t) + 3m''(t)^T m'(t) + 3m'(t)^T m''(t) + m^T(t) m'''(t) + \\ &+ \eta'''(t)^T \eta(t) + 3\eta''(t)^T \eta'(t) + 3\eta'(t)^T \eta''(t) + \eta^T(t) \eta'''(t) \end{aligned}$$

and

$$(26) \quad \begin{aligned} N'''(t) &= \xi m'''(t) m(t)^T \xi^T + 3\xi m''(t) m'(t)^T \xi^T + 3\xi m'(t) m''(t)^T \xi^T \\ &+ \xi m(t) m'''(t)^T \xi^T + \eta'''(t) \eta(t)^T + 3\eta''(t) \eta'(t)^T \\ &+ 3\eta'(t) \eta''(t)^T + \eta(t) \eta'''(t)^T. \end{aligned}$$

At $t = 0$, substituting (14), (20) and (21) into (25) and (26) one can get respectively

$$m'''(0)^T + m'''(0) = 0,$$

and

$$N'''(0) = (\overline{\nabla}_X h)(X, X) = \xi m'''(0) \xi^T + \xi m'''(0)^T \xi^T = 0.$$

This means that $\mathbf{V}_{n,k}$ has parallel second fundamental form. Therefore by Theorem 1.1 M has $P2$ - PNS property.

Remark. For $k = 1$ the Stiefel submanifold $\mathbf{V}_{n,1}$ becomes a Veronese submanifold. In [3] we have shown that $\mathbf{V}_{n,1}$ has $P2 - PNS$ property.

Acknowledgements. This research is supported by Uludağ University research fund.

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