On the Symmetries of the Differential Equation Governing Damped Harmonic Vibration

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Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

The application of group theoretic method of Lie to ODE is discussed as it provides a unified view of the diverse and ad-hoc integration procedures of such equations. Following consequences of one-parameter point transformations, the symmetries of the differential equation describing damped harmonic vibration are determined. A particular case is considered to illustrate the generality of the approach.

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1 Introduction

The symmetries of differential equations are often found useful in finding their solutions. The knowledge about the underlying transformations and their generators are essential for this purpose. Let us consider the invertible point transformation of the x-y plane

$$\bar{x} = \phi(x, y, \gamma), \ \ \bar{y} = \psi(x, y, \gamma) \tag{1.1}$$

with the conditions

$$\phi|_{\gamma=0} = x \ \text{ and } \ \psi|_{\gamma=0} = y \tag{1.2}$$

The point transformations are said to form a one-parameter group if successive action of two transformations is equivalent to the action of another transformation of the form (1.1) (Ibragimov [1]). Taylor series expansions with respect to γ in the neighbourhood of $\gamma = 0$ of (1.1) yield the infinitesimal transformations

$$\bar{x} = x + \xi(x, y)\gamma, \quad \bar{y} = y + \eta(x, y)\gamma \tag{1.3}$$

where

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$$\xi(x,y) \;=\; \frac{\partial\phi}{\partial\gamma}\Big|_{\gamma=0}, \quad \eta(x,y) \;=\; \frac{\partial\psi}{\partial\gamma}\Big|_{\gamma=0} \tag{1.4}$$

A tangent vector field in terms of the first-order differential operator, is written as

$$X \;=\; \xi(x,y)\,\frac{\partial}{\partial x} + \eta(x,y)\,\frac{\partial}{\partial y} \tag{1.5}$$

Lie [2] termed X as the infinitesimal generator of the transformation. The unknown functions $\xi(x,y)$ and $\eta(x,y)$ may be found out from the determining equation for Lie point symmetries.

For the second-order differential equation

$$y'' = f(x,y,y'); \;\; y' = \frac{dy}{dx}, \;\; y'' = \frac{d^2y}{dx^2} \tag{1.6}$$

the determining equation for the Lie point symmetries derived as (Ibragimov [1])

$$\begin{aligned} &\eta_{xx} + (2\eta_{xy} - \xi_{xx})\,y' + (\eta_{yy} - 2\xi_{xy})\,y'^2 - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)\,f \\ &- \left[\eta_x + (\eta_y - \xi_x)\,y' - y'^2\xi_y\right]\, f_{y'} - \xi f_x - \eta f_y \;=\; 0 \end{aligned} \tag{1.7}$$

where $f(x,y,y')$ is a known function and
(1.8)

$$\xi_x = \frac{\partial\xi}{\partial x},\; \xi_y = \frac{\partial\xi}{\partial y},\; \eta_x = \frac{\partial\eta}{\partial x},\; \eta_y = \frac{\partial\eta}{\partial y},\; \eta_{xx} = \frac{\partial^2\eta}{\partial x^2},\; \xi_{xx} = \frac{\partial^2\xi}{\partial x^2},\; \xi_{yy} = \frac{\partial^2\xi}{\partial y^2},$$

$$\xi_{xy} = \frac{\partial^2\xi}{\partial x\partial y},\; \eta_{yy} = \frac{\partial^2\eta}{\partial y^2},\; \eta_{xy} = \frac{\partial^2\eta}{\partial x\partial y},\; f_y = \frac{\partial f}{\partial y},\; f_x = \frac{\partial f}{\partial x},\; f_{y'} = \frac{\partial f}{\partial y'}$$

Treating the left-hand side of equation (1.7) as a polynomial of third degree in y', we can split it into an overdetermined system of equations. The symmetries of the differential equation (1.6) are determined generally, by integrating the system. The principle discussed above is to be applied here to the dynamical equation governing the damped harmonic vibration.

2 Symmetries of the differential equation for damped harmonic vibration

If a particle be oscillating in a fluid (air or liquid) the equation of motion for its small displacement can be written as

$$\frac{d^2y}{dt^2} + k\frac{dy}{dt} + \omega^2 y \;=\; 0 \tag{2.1}$$

where k is defined as the fluid resistance or the damping effect to the motion of a particle of unit mass moving with unit velocity, y is the co-ordinate and ω is the angular velocity.

Equation (2.1) is re-written in the form

$$y'' = f(t, y, y') = -ky' - \omega^2 y; \quad y' = \frac{dy}{dx}, \ y'' = \frac{d^2y}{dx^2} \tag{2.2}$$

Let the infinitesimal generator in this case is

$$X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y} \tag{2.3}$$

$\xi = \xi(t, y)$, $\eta = \eta(t, y)$, and the determining equation for the Lie symmetry, is given by

$$\begin{aligned} &\eta_{tt} + (2\eta_{ty} - \xi_{tt})\, y' + (\eta_{yy} - 2\xi_{ty})\, y'^2 - y'^3 \xi_{yy} + (\eta_y - 2\xi_t - 3y'\xi_y)\, f \\ &- \left[\eta_t + (\eta_y - \xi_t)\, y' - y'^2 \xi_y\right] f_{y'} - \xi f_t - \eta f_y = 0 \end{aligned} \tag{2.4}$$

where

$$\xi_t = \frac{\partial \xi}{\partial t}, \ \xi_y = \frac{\partial \xi}{\partial y}, \ \eta_t = \frac{\partial \eta}{\partial t}, \ \eta_y = \frac{\partial \eta}{\partial y}, \ \eta_{tt} = \frac{\partial^2 \eta}{\partial t^2}, \ \xi_{tt} = \frac{\partial^2 \xi}{\partial t^2}, \ \xi_{yy} = \frac{\partial^2 \xi}{\partial y^2},$$

$$\xi_{ty} = \frac{\partial^2 \xi}{\partial t \partial y}, \ \eta_{yy} = \frac{\partial^2 \eta}{\partial y^2}, \ \eta_{ty} = \frac{\partial^2 \eta}{\partial t \partial y}, \ f_y = \frac{\partial f}{\partial y}, \ f_t = \frac{\partial f}{\partial t}, \ f_{y'} = \frac{\partial f}{\partial y'}$$

Treating the left-hand side of equation (2.4) as a polynomial in y', we can split it up into the following system of equations :

$$y'^3 \ : \ \xi_{yy} = 0 \tag{2.5}$$

$$y'^2 \ : \ \eta_{yy} - 2\xi_{ty} + 2k\xi_y = 0 \tag{2.6}$$

$$y' \ : \ 2\eta_{ty} - \xi_{tt} + k\xi_t + 3\omega^2 y \xi_y = 0 \tag{2.7}$$

$$y'^0 \ : \ \eta_{tt} - \omega^2 \left[y(\eta_y - 2\xi_t) - \eta\right] + k\eta_t = 0 \tag{2.8}$$

Solution of (2.5) can be written as

$$\xi = p(t)y + a(t) \tag{2.9}$$

In view of (2.9), equation (2.6) is transformed to

$$\eta_{yy} - p_t + 2kp = 0; \qquad p_t = \frac{dp}{dt} \tag{2.10}$$

Solving (2.10), we obtain η as

$$\eta = (p_t - kp)y^2 + b(t)y + c(t) \tag{2.11}$$

Taking (2.9) and (2.11) into account in equation (2.7), we obtain

(2.12) $$3y\left(p_{tt} - kp_t + \omega^2 p\right) - \left(a_{tt} - ka_t - 2b_t\right) = 0$$

Since left-hand side of (2.12) can be treated as a simple polynomial in y, we can write

(2.13) $$p_{tt} - kp_t + \omega^2 p = 0$$

(2.14) $$a_{tt} - ka_t - 2b_t = 0$$

where

$$a_t = \frac{\partial a}{\partial t},\ p_{tt} = \frac{\partial^2 p}{\partial t^2},\ a_{tt} = \frac{\partial^2 a}{\partial t^2},\ b_t = \frac{\partial b}{\partial t}$$

Two independent solutions of equation (2.13), are given by

(2.15) $$p_1 = A_1 e^{\frac{k+\sqrt{k^2-4\omega^2}}{2}t}$$

(2.16) $$p_2 = A_2 e^{\frac{k-\sqrt{k^2-4\omega^2}}{2}t}$$

where A_1, A_2 are constants, and the general solution can be written as

(2.17) $$p = A_1 e^{\frac{k+\sqrt{k^2-4\omega^2}}{2}t} + A_2 e^{\frac{k-\sqrt{k^2-4\omega^2}}{2}t}$$

Integrating (2.14), we obtain

(2.18) $$a(t) = e^{kt} \int e^{-kt}\ \left[2b(t) + a_1\right] dt + a_2$$

where a_1 and a_2 are constants of integration.

Lastly, taking the relations (2.9) and (2.11) into account in equation (2.8), we obtain after some calculations

(2.19)
$$y^2\left[p_{ttt} + p_t\left(\omega^2 - k^2\right) + kp\omega^2\right] + y\left(b_{tt} + 2\omega^2 a_t + kb_t\right) + \left(c_{tt} + c\omega^2 + kc_t\right) = 0$$

Considering (2.19) as a polynomial in y, we can write

(2.20) $$p_{ttt} + p_t\left(\omega^2 - k^2\right) + kp\omega^2 = 0$$

(2.21) $$b_{tt} + 2\omega^2 a_t + kb_t = 0$$

(2.22) $$c_{tt} + c\omega^2 + kc_t = 0$$

where

$$p_{ttt} = \frac{d^3 p}{dt^3}$$

Equation (2.20) is automatically satisfied since it can be rearranged as

$$\frac{d}{dt}\left(p_{tt} - kp_t + p\omega^2\right) + k\left(p_{tt} - kp_t + p\omega^2\right) = 0 \tag{2.23}$$

and $p_{tt} - kp_t + p\omega^2 = 0$ by virtue of equation (2.13).

Solution of (2.21), is given by

$$b(t) = e^{-kt} \int \left[-2\omega^2 a(t) + b_1\right] e^{kt} dt + b_2 \tag{2.24}$$

where b_1 and b_2 are constants of integration.

Two independent solutions of equation (2.22), are given

$$c_1 = C_1 e^{\frac{-k+\sqrt{k^2-4\omega^2}}{2}t} \tag{2.25}$$

$$c_2 = C_2 e^{\frac{-k-\sqrt{k^2-4\omega^2}}{2}t} \tag{2.26}$$

where C_1, C_2 are constants. The general solution of equation (2.22) can therefore be written as

$$c = C_1 e^{\frac{-k+\sqrt{k^2-4\omega^2}}{2}t} + C_2 e^{\frac{-k-\sqrt{k^2-4\omega^2}}{2}t} \tag{2.27}$$

Thus with the symmetry conditions prescribed through the relations (2.5) $-$ (2.8), the generator X is given by

$$X = [p(t)y + a(t)] \frac{\partial}{\partial t} + \left[(p_t - kp)y^2 + b(t)y + c(t)\right] \frac{\partial}{\partial y} \tag{2.28}$$

where $p(t)$ and $c(t)$ are respectively, given by the relations (2.17) and (2.27). $a(t)$ and $b(t)$ are determined by the coupled relations (2.18) and (2.24).

It is to be noticed that the generator (2.28) exhibits a case of eight-parameter symmetry of the differential equation (2.1) as it involves exactly eight parametric constants A_1, A_2, C_1, C_2, a_1, a_2, b_1 and b_2. Stephani [3] has shown that the maximum number of symmetries admitted by a second-order ODE is also eight.

3 A particular case

We attempt now for some simple solutions of the problem by taking $a = b = 0$ and considering the independent solutions for p and c as given, respectively by (2.15), (2.16) and (2.25), (2.26). Equations (2.9) and (2.11) reduce simply to

$$\xi = p(t)y \quad and \quad \eta = (p_t - kp)y^2 + c(t) \tag{3.1}$$

For the sake of briefness, we proceed by writing $p = Ae^{\alpha t}$ and $c = Ce^{\beta t}$. Thus, ξ and η can be written as

$$\xi = Ae^{\alpha t} \quad and \quad \eta = Ae^{\alpha t}(\alpha - k)y^2 + Ce^{\beta t} \tag{3.2}$$

Substituting (3.2) for ξ and η in the characteristic equation (Olver [4]) b_1

$$\frac{dt}{\xi} = \frac{dy}{\eta} \tag{3.3}$$

we obtain

$$\frac{dt}{y} = \frac{dy}{(\alpha - k)y^2 + Be^{(\beta-\alpha)t}}; \; B = \frac{C}{A} \tag{3.4}$$

Solution of (3.4), is given by

$$y^2 = \frac{2B}{(\beta - 3\alpha + 2k)} e^{(\beta-\alpha)t} + \; constant \; .e^{-2(k-\alpha)t} \tag{3.5}$$

Employing (3.5), independent solutions for y corresponding to paired values of α and β possible are written as

(i)

$$\left[\begin{array}{l} \alpha = e^{\frac{k+\sqrt{k^2-4\omega^2}}{2}}, \; \beta = e^{\frac{-k+\sqrt{k^2-4\omega^2}}{2}}, \; B = \dfrac{C_1}{A_1}; \\ y = \left\{\dfrac{2B}{-\sqrt{k^2-4\omega^2}} e^{-kt} + \; constant \; .e^{-kt} \; .e^{\sqrt{k^2-4\omega^2}\; t}\right\}^{\frac{1}{2}} \end{array}\right. \tag{3.6}$$

(ii)

(3.7)

$$\left[\begin{array}{l} \alpha = e^{\frac{k+\sqrt{k^2-4\omega^2}}{2}}, \; \beta = e^{\frac{-k-\sqrt{k^2-4\omega^2}}{2}}, \; B = \dfrac{C_2}{A_1}; \\ y = \left\{\dfrac{2B}{-2\sqrt{k^2-4\omega^2}} e^{(-k-\sqrt{k^2-4\omega^2})\; t} + \; constant \; .e^{-kt} \; .e^{\sqrt{k^2-4\omega^2}\; t}\right\}^{\frac{1}{2}} \end{array}\right.$$

(iii)

(3.8)

$$\left[\begin{array}{l} \alpha = e^{\frac{k-\sqrt{k^2-4\omega^2}}{2}}, \; \beta = e^{\frac{-k+\sqrt{k^2-4\omega^2}}{2}}, \; B = \dfrac{C_1}{A_2}; \\ y = \left\{\dfrac{2B}{2\sqrt{k^2-4\omega^2}} e^{(-k+\sqrt{k^2-4\omega^2})\; t} + \; constant \; .e^{-kt} \; .e^{-\sqrt{k^2-4\omega^2}\; t}\right\}^{\frac{1}{2}} \end{array}\right.$$

(iv)

$$\left[\begin{array}{l} \alpha = e^{\frac{k-\sqrt{k^2-4\omega^2}}{2}}, \; \beta = e^{\frac{-k-\sqrt{k^2-4\omega^2}}{2}}, \; B = \dfrac{C_2}{A_2}; \\ y = \left\{\dfrac{2B}{\sqrt{k^2-4\omega^2}} e^{-kt} + \; constant \; .e^{-kt} \; .e^{-\sqrt{k^2-4\omega^2}\; t}\right\}^{\frac{1}{2}} \end{array}\right. \tag{3.9}$$

By the process of linear combination more solutions can be constructed from (3.6) − (3.9). Now, in addition to $a = b = 0$ if we take $c = 0$ then all the first terms from the right hand sides of equations (3.6) − (3.9) will vanish. This is obvious since $C_1 = C_2 = 0$ will make $B = 0$ in each cases.

From the reduced versions (i.e., $C_1 = C_2 = 0$) we can combine (3.6) linearly with (3.8) or (3.9), and (3.7) linearly with (3.8) or (3.9) to obtain an exact solution of the form

$$y = e^{-\frac{k}{2}\, t} \left[R_1 e^{\frac{\sqrt{k^2-4\omega^2}}{2} t} + R_2 e^{\frac{-\sqrt{k^2-4\omega^2}}{2} t} \right] \tag{3.10}$$

where R_1 and R_2 are constants.

(3.10) is well referred solution of equation (2.1) for the simple harmonic vibration (Ahsan [5]).

Remarks: From the above analysis, we may conclude the following :

I. Lie's group theoretic approach is very useful in finding the symmetries, if they exist of the differential equations and integrate them in turn.

II. It may not always be simple to find the Lie point symmetry, even of the ordinary second-order linear differential equation.

III. Whenever an exact solution of a differential equation can be obtained, the underlying property is the symmetry of that equation.

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