Some Examples of Almost Kähler 4-Manifolds

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Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

In the framework of P. Nurowski and M. Przanowski [3], we construct 4-dimensional examples of Ricci flat almost Kähler manifolds, almost Kähler manifolds of pointwise constant holomorphic sectional curvature, and weakly $*$-Einstein almost Kähler manifolds.

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1 Introduction

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if its Kähler form Ω is closed. One of the interest problems on almost Kähler manifolds is so called Goldberg's conjecture, which asserts the almost complex structure of a compact Einstein almost Kähler manifold is integrable (and the manifold is necessarily Kähler) [2]. In connection with this conjecture, P. Nurowski and M. Przanowski [3] recently constructed a non-compact example of a strictly almost Kähler, Ricci-flat manifold. This shows that the assumption about compactness of the Einstein manifold is essential for the Goldberg conjecture. This example is also a space of pointwise *positive* constant holomorphic sectional curvature and a weakly $*$-Einstein manifold (see also [4]).

By considering a real expression for the Nurowski-Przanowski example, the author [5] has costructed a new example of almost Kähler manifold of pointwise *negative* constant holomorphic sectional curvature. In the present paper, we shall mainly deal with 4-dimensional almost Kähler manifolds $M(f, u, v, \phi)$ which will be defined in §2, and show that there exists a family of Ricci flat almost Kähler manifolds which include the Nurowski-Przanowski example. In [5], we investigated almost Kähler manifolds $M(f, u, v, \phi)$ with $u = v = 0$ and $\phi = 0$. For the case where u and v are any constants and ϕ is arbitrary, we can obtain the examples of almost Kähler manifolds with pointwise constant holomorphic sectional curvature which are generalizations of [5]. These examples are also weakly $*$-Einstein, but not Einstein. We also show that there are other examples of weakly $*$-Einstein almost Kähler manifolds. Our examples

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are 4-dimensional and non-compact. We do not know whether or not there exist arbitrary dimensional, compact almost Kähler manifolds which are of (pointwise) constant holomorphic sectional curvature, or which are (weakly) $*$-Einstein.

In §2, we recall a characterization for a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature by using the expressions A_{ij} introduced by J. T. Cho and K. Sekigawa [1]. We give a real version of the Nurowski-Przanowski's construction, and define an almost Kähler manifold $M(f, u, v, \phi)$. §3 is devoted to the construction of Ricci-flat examples. By calculating Ricci tensor ρ_{ij}, we find functions u, v and f for ρ_{ij} vanishing. In §4, we show that the Ricci flat almost Kähler manifolds in §3 are also of pointwise constant holomorphic sectional curvature, by using the characterization of [1]. Moreover, when u, v are constants α, β, we derive conditions for $M(f, \alpha, \beta, \phi)$ to be of pointwise constant holomorphic sectional curvature. Taking account of these conditions, we shall obtain examples (Theorem 4.4). In the last §5, we give other examples of weakly $*$-Einstein almost Kähler manifolds. Especially, the last example (Theorem 5.3) shows that it depends on the value of ϕ that $M(f, \alpha, \beta, \phi)$ is weakly $*$-Einstein.

2 Preliminaries

Let $M = (M, J, g)$ be a four-dimensional almost Hermitian manifold with an almost Hermitian structure (J, g). We denote by Ω and N the Kähler form and the Nijenhuis tensor of M defined respectively by $\Omega(X, Y) = g(X, JY)$ and $N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$ for $X, Y \in \mathcal{X}(M)$, where $\mathcal{X}(M)$ is the Lie algebra of all smooth vector fields on M. The Nijenhuis tensor N has the properties

$$N(JX, Y) = N(X, JY) = -JN(X, Y), \qquad X, Y \in \mathcal{X}(M).$$

Further we denote by $\nabla, R, \rho, \tau, \rho^*$ and τ^* the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor, the scalar curvature, the Ricci $*$-tensor and the $*$-scalar curvature of M, respectively. The Ricci $*$-tensor ρ^* satisfies

$$\rho^*(JX, JY) = \rho^*(Y, X), \qquad X, Y \in \mathcal{X}(M).$$

An almost Hermitian manifold M is called a *weakly $*$-Einstein manifold* if it satisfies $\rho^* = \lambda^* g$ for some function λ^* on M. In particular, if λ^* is constant on M, then M is called a *$*$-Einstein manifold* .

The holomorphic sectional curvature $H = H(x) = -R(x, Jx, x, Jx)$ $(x \in T_p(M), \|x\| = 1)$ can be regarded as a differentiable function on the unit tangent bundle $U(M)$ of M. If the function H is constant along each fiber, then M is called a *space of pointwise constant holomorphic sectional curvature.* Especially, if H is constant on the whole of $U(M)$, then M is called a *space of constant holomorphic sectional curvature.*

Now we assume that $M = (M, J, g)$ is a four-dimensional almost Kähler manifold. Then we have

$$2g((\nabla_X J)Y, Z) = g(JX, N(Y, Z)), \tag{2.1}$$

$$\tau - \tau^* = -\frac{1}{2}\|\nabla J\|^2 = -\frac{1}{8}\|N\|^2. \tag{2.2}$$

In the sequel, we adopt the following notational convention: for an orthonormal basis $\{e_i\}$ of a tangent space T_pM, we put

$$\begin{array}{ll} J_{ij} = g(Je_i, e_j), & \nabla_i J_{jk} = g((\nabla_{e_i} J)e_j, e_k), \\ N_{ijk} = g(e_i, N(e_j, e_k)), & R_{ijkl} = R(e_i, e_j, e_k, e_l), \\ \nabla_{\bar{i}} J_{jk} = g((\nabla_{Je_i} J)e_j, e_k), & N_{\bar{i}jk} = g(Je_i, N(e_j, e_k)), \\ N_{\bar{i}\bar{j}k} = g(Je_i, N(Je_j, e_k)), & R_{ij\bar{k}\bar{l}} = R(e_i, e_j, Je_k, Je_l), \qquad etc. \end{array}$$

Then it is easy to see that

$$\begin{array}{l} \nabla_i J_{jk} + \nabla_{\bar{i}} J_{\bar{j}k} = 0, \\ \nabla_{\bar{i}} J_{jk} = \nabla_i J_{\bar{j}k} = \nabla_i J_{j\bar{k}}, \\ N_{ijk} = -2\nabla_{\bar{i}} J_{jk}, \qquad 2\nabla_i J_{jk} = N_{\bar{i}jk}. \end{array}$$

We set

$$A_{ij} = g(e_i, (\nabla_{e_j} N)(e_1, e_3)) = \nabla_j N_{i13}, \tag{2.3}$$

for a unitary basis $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ of $T_pM, p \in M$. We note that

$$A_{ij} - A_{ji} = -2(R_{ij13} - R_{ij24}).$$

By using these A_{ij}, J. T. Cho and K. Sekigawa obtained the following characterization of almost Kähler manifolds of pointwise constant holomorphic sectional curvature:

Proposition 2.1 ([1]). *Let M be a 4-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c = c(p)(p \in M)$. Then*

$$\begin{aligned} R_{1212} &= R_{3434} = -c(p), \\ R_{1234} &= -\frac{c(p)}{2} - \frac{1}{16}(\tau^* - \tau), \\ R_{1324} &= -\frac{c(p)}{4} + \frac{1}{32}(\tau^* - \tau) + \frac{1}{8}(A_{13} - A_{31} - A_{24} + A_{42}), \\ R_{1423} &= \frac{c(p)}{4} + \frac{3}{32}(\tau^* - \tau) + \frac{1}{8}(A_{13} - A_{31} - A_{24} + A_{42}), \end{aligned}$$

$$R_{1313}=-\frac{c(p)}{4}+\frac{1}{32}(\tau^*-\tau)-\frac{3}{8}(A_{13}-A_{31})-\frac{1}{8}(A_{24}-A_{42}),$$
$$R_{1414}=-\frac{c(p)}{4}+\frac{5}{32}(\tau^*-\tau)-\frac{1}{8}(A_{13}+A_{42})-\frac{3}{8}(A_{31}+A_{24}),$$
$$R_{2323}=-\frac{c(p)}{4}+\frac{5}{32}(\tau^*-\tau)+\frac{1}{8}(A_{31}+A_{24})+\frac{3}{8}(A_{13}+A_{42}),$$
$$R_{2424}=-\frac{c(p)}{4}+\frac{1}{32}(\tau^*-\tau)+\frac{3}{8}(A_{24}-A_{42})+\frac{1}{8}(A_{13}-A_{31}),$$
$$R_{1334}=-R_{2434}=-\frac{1}{4}(A_{34}-A_{43}),$$
$$R_{1213}=-R_{1224}=-\frac{1}{4}(A_{12}-A_{21}),$$
$$R_{1434}=R_{2334}=-\frac{1}{4}(A_{33}+A_{44}),$$
$$R_{1214}=R_{1223}=-\frac{1}{4}(A_{11}+A_{22}),$$
$$R_{1323}=\frac{1}{8}(A_{14}+A_{41}+A_{32}-3A_{23}),$$
$$R_{2324}=\frac{1}{8}(A_{14}+A_{41}+A_{23}-3A_{32}),$$
$$R_{1314}=-\frac{1}{8}(A_{23}+A_{32}+A_{14}-3A_{41}),$$
$$R_{1424}=-\frac{1}{8}(A_{23}+A_{32}+A_{41}-3A_{14}),$$

for any unitary basis $\{e_i\}$ *of* T_pM *at each point* $p\in M$.

Let M be an open set of $\mathbf{R}^4$, and let (x_1,x_2,x_3,x_4) be the Euclidean coordinates on M. We put

$$z_1=x_1+\sqrt{-1}x_2,\quad z_2=x_3+\sqrt{-1}x_4.$$

Let f be a non-zero real function and h be a complex function on M. Then P. Nurowski and M. Przanowski proved the following

Lemma 2.2 ([3]). *Let* $(z_1,\bar{z_1},z_2,\bar{z_2})$ *be coordinates on* M. *Then for each value of the real constant* $\phi\in[0,2\pi)$, *the metric*

$$g=2f^2(dz_1+hdz_2)(d\bar{z_1}+\bar{h}d\bar{z_2})+\frac{2}{f^2}dz_2d\bar{z_2}$$

and the almost complex structure

$$J^+_{e^{\sqrt{-1}\phi}}=2Re\left[\sqrt{-1}\ e^{\sqrt{-1}\phi}\Big\{f^2(dz_1+hdz_2)\otimes(\frac{\partial}{\partial\bar{z_2}}-\bar{h}\frac{\partial}{\partial\bar{z_1}})-\frac{1}{f^2}dz_2\otimes\frac{\partial}{\partial\bar{z_1}}\Big\}\right]$$

define an almost Kähler structure on M.

The Riemannian metric $g=(g_{ij})$ and the almost complex structure $J^+_{e^{\sqrt{-1}\phi}}=\left(J_i{}^j\right)$ in the above Lemma 2.2 are given with respect to the real coordinates (x_1,x_2,x_3,x_4) by

$$
\begin{aligned}
&g_{11} = g_{22} = 2f^2,\\
&g_{12} = g_{21} = g_{34} = g_{43} = 0,\\
&g_{13} = g_{31} = g_{24} = g_{42} = f^2 u,\\
&-g_{14} = -g_{41} = g_{23} = g_{32} = f^2 v,\\
&g_{33} = g_{44} = 2f^2(u^2 + v^2 + \frac{1}{f^4}),
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
&{J_1}^1 = -{J_2}^2 = -v\cos\phi + u\sin\phi,\\
&{J_1}^2 = {J_2}^1 = u\cos\phi + v\sin\phi,\\
&{J_1}^3 = -{J_2}^4 = -\sin\phi,\\
&{J_1}^4 = {J_2}^3 = -\cos\phi,\\
&{J_3}^1 = -{J_4}^2 = (u^2 + v^2 + \tfrac{1}{f^4})\sin\phi,\\
&{J_3}^2 = {J_4}^1 = (u^2 + v^2 + \tfrac{1}{f^4})\cos\phi,\\
&-{J_3}^3 = {J_4}^4 = v\cos\phi + u\sin\phi,\\
&{J_3}^4 = {J_4}^3 = -u\cos\phi + v\sin\phi,
\end{aligned} \tag{2.5}
$$

where u and v are the real and imaginary part of the complex function h, respectively. It is easy to see that the Kähler form Ω is given by

$$\Omega = 2\sin\phi\, dx_1 \wedge dx_3 + 2\cos\phi\, dx_1 \wedge dx_4 + 2\cos\phi\, dx_2 \wedge dx_3 - 2\sin\phi\, dx_2 \wedge dx_4,$$

and $(J^+_{e^{\sqrt{-1}\phi}}, g)$ is an almost Kähler structure. In the present paper, we shall mainly deal with this almost Kähler manifold $(M, J^+_{e^{\sqrt{-1}\phi}}, g)$. Since the almost Kähler structure $(J^+_{e^{\sqrt{-1}\phi}}, g)$ is determined by functions f, u, v and a real constant ϕ, we denote this almost Kähler manifold by $M(f, u, v, \phi)$.

We define a unitary frame field $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ on $M(f, u, v, \phi)$ by

$$
\begin{aligned}
e_1 &= \frac{1}{\sqrt{2}f}\frac{\partial}{\partial x_1},\\
e_2 &= \frac{f}{\sqrt{2}}\left\{(-v\cos\phi + u\sin\phi)\frac{\partial}{\partial x_1} + (u\cos\phi + v\sin\phi)\frac{\partial}{\partial x_2}\right.\\
&\qquad \left. - \sin\phi\frac{\partial}{\partial x_3} - \cos\phi\frac{\partial}{\partial x_4}\right\},\\
e_3 &= \frac{1}{\sqrt{2}f}\frac{\partial}{\partial x_2},\\
e_4 &= \frac{f}{\sqrt{2}}\left\{(u\cos\phi + v\sin\phi)\frac{\partial}{\partial x_1} + (v\cos\phi - u\sin\phi)\frac{\partial}{\partial x_2}\right.\\
&\qquad \left. - \cos\phi\frac{\partial}{\partial x_3} + \sin\phi\frac{\partial}{\partial x_4}\right\}.
\end{aligned} \tag{2.6}
$$

With respect to this unitary frame $\{e_i\}_{i=1,2,3,4}$ we set

$$\nabla_{e_i} e_j = \sum \Gamma_{ijk} e_k.$$

Then we have

$$
\begin{aligned}
&\Gamma_{ijj} = 0,\\
&\Gamma_{112} = -\Gamma_{121} = \frac{1}{\sqrt{2}}\{\cos\phi(\partial_4 f + v\partial_1 f - u\partial_2 f + f\partial_1 v)\\
&\qquad\qquad + \sin\phi(\partial_3 f - u\partial_1 f - v\partial_2 f - f\partial_1 u)\},\\
&\Gamma_{113} = -\Gamma_{131} = -\frac{\partial_2 f}{\sqrt{2}f^2},\\
&\Gamma_{114} = -\Gamma_{141} = \frac{1}{\sqrt{2}}\{\cos\phi(\partial_3 f - u\partial_1 f - v\partial_2 f - f\partial_1 u)\\
&\qquad\qquad - \sin\phi(\partial_4 f + v\partial_1 f - u\partial_2 f + f\partial_1 v)\},\\
&\Gamma_{123} = -\Gamma_{132} = \frac{f}{2\sqrt{2}}\{\cos\phi(\partial_1 u - \partial_2 v) + \sin\phi(\partial_2 u + \partial_1 v)\},\\
&\Gamma_{124} = -\Gamma_{142} = \frac{f^3}{2\sqrt{2}}\{(\partial_3 v - u\partial_1 v - v\partial_2 v) + (\partial_4 u + v\partial_1 u - u\partial_2 u)\},\\
&\Gamma_{134} = -\Gamma_{143} = -\frac{f}{2\sqrt{2}}\{\cos\phi(\partial_2 u + \partial_1 v) - \sin\phi(\partial_1 u - \partial_2 v)\},\\
&\Gamma_{212} = -\Gamma_{221} = -\frac{\partial_1 f}{\sqrt{2}f^2},\\
&\Gamma_{213} = -\Gamma_{231} = -\frac{f}{2\sqrt{2}}\{\cos\phi(\partial_1 u + \partial_2 v) - \sin\phi(\partial_2 u - \partial_1 v)\},\\
&\Gamma_{214} = -\Gamma_{241} = \frac{f^3}{2\sqrt{2}}\{(\partial_3 v - u\partial_1 v - v\partial_2 v) + (\partial_4 u + v\partial_1 u - u\partial_2 u)\},\\
&\Gamma_{223} = -\Gamma_{232} = \frac{\partial_2 f}{\sqrt{2}f^2},\\
&\Gamma_{224} = -\Gamma_{242} = -\frac{1}{\sqrt{2}}\{\cos\phi(\partial_3 f - u\partial_1 f - v\partial_2 f)\\
&\qquad\qquad - \sin\phi(\partial_4 f + v\partial_1 f - u\partial_2 f)\},
\end{aligned} \tag{2.7}
$$

$$\Gamma_{234} = -\Gamma_{243} = -\frac{f^3}{2\sqrt{2}}\{(\partial_3 u - u\partial_1 u - v\partial_2 u) - (\partial_4 v + v\partial_1 v - u\partial_2 v)\},$$

$$\Gamma_{312} = -\Gamma_{321} = -\frac{f}{2\sqrt{2}}\{\cos\phi(\partial_1 u - \partial_2 v) + \sin\phi(\partial_2 u + \partial_1 v)\},$$

$$\Gamma_{313} = -\Gamma_{331} = \frac{\partial_1 f}{\sqrt{2}f^2},$$

$$\Gamma_{314} = -\Gamma_{341} = -\frac{f}{2\sqrt{2}}\{\cos\phi(\partial_2 u + \partial_1 v) - \sin\phi(\partial_1 u - \partial_2 v)\},$$

$$\Gamma_{323} = -\Gamma_{332} = -\frac{1}{\sqrt{2}}\{\cos\phi(\partial_4 f + v\partial_1 f - u\partial_2 f - f\partial_2 u)$$
$$+ \sin\phi(\partial_3 f - u\partial_1 f - v\partial_2 f - f\partial_2 v)\},$$

$$\Gamma_{324} = -\Gamma_{342} = -\frac{f^3}{2\sqrt{2}}\{(\partial_3 u - u\partial_1 u - v\partial_2 u) - (\partial_4 v + v\partial_1 v - u\partial_2 v)\},$$

$$\Gamma_{334} = -\Gamma_{343} = \frac{1}{\sqrt{2}}\{\cos\phi(\partial_3 f - u\partial_1 f - v\partial_2 f - f\partial_2 v)$$
$$- \sin\phi(\partial_4 f + v\partial_1 f - u\partial_2 f - f\partial_2 u)\},$$

$$\Gamma_{412} = -\Gamma_{421} = -\frac{f^3}{2\sqrt{2}}\{(\partial_3 v - u\partial_1 v - v\partial_2 v) + (\partial_4 u + v\partial_1 u - u\partial_2 u)\},$$

$$\Gamma_{413} = -\Gamma_{431} = \frac{f}{2\sqrt{2}}\{\cos\phi(\partial_2 u - \partial_1 v) + \sin\phi(\partial_1 u + \partial_2 v)\},$$

$$\Gamma_{414} = -\Gamma_{441} = -\frac{\partial_1 f}{\sqrt{2}f^2},$$

$$\Gamma_{423} = -\Gamma_{432} = -\frac{f^3}{2\sqrt{2}}\{(\partial_3 u - u\partial_1 u - v\partial_2 u) - (\partial_4 v + v\partial_1 v - u\partial_2 v)\},$$

$$\Gamma_{424} = -\Gamma_{442} = \frac{1}{\sqrt{2}}\{\cos\phi(\partial_4 f + v\partial_1 f - u\partial_2 f)$$
$$+ \sin\phi(\partial_3 f - u\partial_1 f - v\partial_2 f)\},$$

$$\Gamma_{434} = -\Gamma_{443} = -\frac{\partial_2 f}{\sqrt{2}f^2},$$

where we denote

$$\partial_i f = \frac{\partial f}{\partial x_i}.$$

By (2.7), we find

(2.8)

$$
\begin{aligned}
\nabla_1 J_{13} &= -\nabla_1 J_{31} = -\nabla_1 J_{24} = \nabla_1 J_{42} = -\nabla_2 J_{14} = \nabla_2 J_{41} = -\nabla_2 J_{23} = \nabla_2 J_{32} \\
&= \frac{1}{2\sqrt{2}} \{ 2\cos\phi(\partial_3 f - u\partial_1 f - v\partial_2 f) - 2\sin\phi(\partial_4 f + v\partial_1 f - u\partial_1 f) \\
&- f\cos\phi(\partial_1 u + \partial_2 v) + f\sin\phi(\partial_2 u - \partial_1 v) \} , \\
\nabla_1 J_{14} &= -\nabla_1 J_{41} = \nabla_1 J_{23} = -\nabla_1 J_{32} = \nabla_2 J_{13} = -\nabla_2 J_{31} = -\nabla_2 J_{24} = \nabla_2 J_{42} \\
&= \frac{1}{2\sqrt{2} f^2} \left\{ 2\partial_2 f - f^5 u(\partial_2 u + \partial_1 v) + f^5 v(\partial_1 u - \partial_2 v) + f^5(\partial_4 u + \partial_3 v) \right\} , \\
\nabla_3 J_{13} &= -\nabla_3 J_{31} = -\nabla_3 J_{24} = \nabla_3 J_{42} = -\nabla_4 J_{14} = \nabla_4 J_{41} = -\nabla_4 J_{23} = \nabla_4 J_{32} \\
&= -\frac{1}{2\sqrt{2}} \{ 2\sin\phi(\partial_3 f - u\partial_1 f - v\partial_2 f) + 2\cos\phi(\partial_4 f + v\partial_1 f - u\partial_1 f) \\
&- f\sin\phi(\partial_1 u + \partial_2 v) - f\cos\phi(\partial_2 u - \partial_1 v) \} , \\
\nabla_3 J_{14} &= -\nabla_3 J_{41} = \nabla_3 J_{23} = -\nabla_3 J_{32} = \nabla_4 J_{13} = -\nabla_4 J_{31} = -\nabla_4 J_{24} = \nabla_4 J_{42} \\
&= \frac{1}{2\sqrt{2} f^2} \left\{ -2\partial_1 f + f^5 u(\partial_1 u - \partial_2 v) + f^5 v(\partial_2 u + \partial_1 v) - f^5(\partial_3 u - \partial_4 v) \right\} , \\
\nabla_i J_{jk} &= 0 \qquad \text{(otherwise)}.
\end{aligned}
$$

By (2.1) and (2.8), we then have

(2.9)

$$
\begin{aligned}
N_{113} &= -N_{131} = -N_{214} = N_{241} = -N_{223} = N_{232} = -N_{124} = N_{142} \\
&= -\frac{1}{\sqrt{2} f^2} \left\{ 2\partial_2 f - f^5 u(\partial_2 u + \partial_1 v) + f^5 v(\partial_1 u - \partial_2 v) + f^5(\partial_4 u + \partial_3 v) \right\} , \\
N_{213} &= -N_{231} = N_{114} = -N_{141} = N_{123} = -N_{132} = -N_{224} = N_{242} \\
&= \frac{1}{\sqrt{2}} \{ 2\cos\phi(\partial_3 f - u\partial_1 f - v\partial_2 f) - 2\sin\phi(\partial_4 f + v\partial_1 f - u\partial_1 f) \\
&- f\cos\phi(\partial_1 u + \partial_2 v) + f\sin\phi(\partial_2 u - \partial_1 v) \} , \\
N_{313} &= -N_{331} = -N_{414} = N_{441} = -N_{423} = N_{432} = -N_{324} = N_{342} \\
&= -\tfrac{1}{\sqrt{2} f^2} \left\{ -2\partial_1 f + f^5 u(\partial_1 u - \partial_2 v) + f^5 v(\partial_2 u + \partial_1 v) - f^5(\partial_3 u - \partial_4 v) \right\} , \\
N_{413} &= -N_{431} = N_{314} = -N_{341} = N_{323} = -N_{332} = -N_{424} = N_{442} \\
&= -\frac{1}{\sqrt{2}} \{ 2\sin\phi(\partial_3 f - u\partial_1 f - v\partial_2 f) + 2\cos\phi(\partial_4 f + v\partial_1 f - u\partial_1 f) \\
&- f\sin\phi(\partial_1 u + \partial_2 v) - f\cos\phi(\partial_2 u - \partial_1 v) \} , \\
N_{ijk} &= 0 \quad \text{(otherwise)}.
\end{aligned}
$$

Now, let $u = \alpha, v = \beta$, where α, β are constants. In this case, we introduce a new coordinates system $(\xi_1, \xi_2, \xi_3, \xi_4)$ on $M(f, \alpha, \beta, \phi)$ by

$$\xi_1 = x_1 + \alpha x_3 - \beta x_4, \ \xi_2 = x_2 + \beta x_3 + \alpha x_4, \ \xi_3 = x_3, \ \xi_4 = x_4. \tag{2.10}$$

With respect to this coordinates system $(\xi_1, \xi_2, \xi_3, \xi_4)$, we adopt the notation $\tilde{\partial}_i f = \dfrac{\partial f}{\partial \xi_i}$. Then, (2.9) can be written as

$$
\begin{aligned}
\Gamma_{112} &= -\Gamma_{121} = -\Gamma_{323} = \Gamma_{332} = \Gamma_{424} = -\Gamma_{442} \\
&= \frac{1}{\sqrt{2}}\{\cos\phi\, \tilde{\partial}_4 f + \sin\phi\, \tilde{\partial}_3 f\}, \\
\Gamma_{113} &= -\Gamma_{131} = -\Gamma_{223} = \Gamma_{232} = \Gamma_{434} = -\Gamma_{443} = -\frac{\tilde{\partial}_2 f}{\sqrt{2} f^2}, \\
\Gamma_{114} &= -\Gamma_{141} = -\Gamma_{224} = \Gamma_{242} = \Gamma_{334} = -\Gamma_{343} \\
&= \frac{1}{\sqrt{2}}\{\cos\phi\, \tilde{\partial}_3 f - \sin\phi\, \tilde{\partial}_4 f\}, \\
\Gamma_{212} &= -\Gamma_{221} = -\Gamma_{313} = \Gamma_{331} = \Gamma_{414} = -\Gamma_{441} = -\frac{\tilde{\partial}_1 f}{\sqrt{2} f^2}, \\
\Gamma_{ijk} &= 0 \quad \text{(otherwise)}.
\end{aligned}
\tag{2.11}
$$

3 Ricci flat examples

In this section, we shall find out a family of Ricci flat metrics in the framework of Nurowski and Przanowski [3]. Let $M(f, u, v, \phi)$ be the almost Kähler 4-manifold defined in §2. Since the Nurowski-Przanowski's Ricci flat example is given by

$$
u = -2x_3,\ v = 2x_4,\ f = \frac{1}{\sqrt{2}}\left\{2x_1 - 2(x_3^2 + x_4^2)\right\}^{-\frac{1}{4}},
$$

we suppose that $u = u(x_3, x_4), v = v(x_3, x_4)$. Then, by a straightforward computation, we obtain, with respect to the unitary basis $\{e_i\}$ in (2.6),

$$
\begin{aligned}
\rho_{11} &= \frac{1}{4 f^4} f^{10} (\partial_4 u + \partial_3 v)^2 - 10 (\partial_1 f)^2 + 6 (\partial_2 f)^2 \\
&- 2 f^4 (\partial_4 f + v\, \partial_1 f - u\, \partial_2 f)^2 + (\partial_3 f - u\, \partial_1 f - v\, \partial_2 f)^2 \\
&+ 2 f \left(\partial_1^2 f - \partial_2^2 f\right) \\
&- 2 f^5 \{(\partial_4 + v\partial_1 - u\partial_2)^2 f + (\partial_3 - u\partial_1 - v\partial_2)^2 f)^2\}],
\end{aligned}
\tag{3.1}
$$

(3.2)
$$\begin{aligned}
\rho_{22} &= -\frac{1}{4\,f^4}\Big[6\,\{(\partial_1 f)^2+(\partial_2 f)^2\}-2f\{\partial_1^2 f+\partial_2^2 f\}\\
&+ f^{10}\,\big\{(\partial_3 u-\partial_4 v)^2+(\partial_4 u+\partial_3 v)^2\big\}\\
&+ 2\,f^4\,\{\,(1+2\cos 2\phi)\,(\partial_4 f+v\partial_1 f-u\partial_2 f)^2\\
&+ (1-2\cos 2\phi)\,(\partial_3 f-u\partial_1 f-v\partial_2 f)^2\\
&+ 4\sin 2\phi(\partial_3 f-u\partial_1 f-v\partial_2 f)(\partial_4 f+v\partial_1 f-u\partial_2 f)\,\}\\
&+ 2\,f^5\,\{\,\cos 2\phi((\partial_4+v\partial_1-u\partial_2)^2 f-(\partial_3-u\partial_1-v\partial_2)^2 f)\\
&+ \sin 2\phi((\partial_3-u\partial_1-v\partial_2)(\partial_4+v\partial_1-u\partial_2)f\\
&+ (\partial_4+v\partial_1-u\partial_2)(\partial_3-u\partial_1-v\partial_2)f)\,\}\Big],
\end{aligned}$$

(3.3)
$$\begin{aligned}
\rho_{33} &= \frac{1}{4\,f^4}\Big[f^{10}\,(\partial_3 u-\partial_4 v)^2+6\,(\partial_1 f)^2-10\,(\partial_2 f)^2\\
&- 2\,f^4\,\{\,(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)^2+(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)^2\,\}\\
&- 2\,f\,\big(\partial_1^2 f-\partial_2^2 f\big)\\
&- 2\,f^5\,\{\,(\partial_4+v\partial_1-u\partial_2)^2 f+(\partial_3-u\partial_1-v\partial_2)^2 f\,\}\Big],
\end{aligned}$$

(3.4)
$$\begin{aligned}
\rho_{44} &= \frac{1}{4\,f^4}\Big[-6\,\{(\partial_1 f)^2+(\partial_2 f)^2\}+2f\{\partial_1^2 f+\partial_2^2 f\}\\
&- f^{10}\,\big\{(\partial_3 u-\partial_4 v)^2+(\partial_4 u+\partial_3 v)^2\big\}\\
&- 2\,f^4\,\{\,(1-2\cos 2\phi)\,(\partial_4 f+v\partial_1 f-u\partial_2 f)^2\\
&+ (1+2\cos 2\phi)\,(\partial_3 f-u\partial_1 f-v\partial_2 f)^2\\
&- 4\sin 2\phi(\partial_3 f-u\partial_1 f-v\partial_2 f)(\partial_4 f+v\partial_1 f-u\partial_2 f)\,\}\\
&+ 2\,f^5\,\{\,\cos 2\phi((\partial_4+v\partial_1-u\partial_2)^2 f-(\partial_3-u\partial_1-v\partial_2)^2 f)\\
&+ \sin 2\phi((\partial_3-u\partial_1-v\partial_2)(\partial_4+v\partial_1-u\partial_2)f\\
&+ (\partial_4+v\partial_1-u\partial_2)(\partial_3-u\partial_1-v\partial_2)f)\,\}\Big],
\end{aligned}$$

(3.5)
$$\begin{aligned}
\rho_{12} &= \frac{1}{4\,f^2}\Big[f^6\{\sin\phi\,\partial_4(\partial_4 u+\partial_3 v)-\cos\phi\,\partial_3(\partial_4 u+\partial_3 v)\}\\
&+ 4\,\partial_1 f\,\,\{\cos\phi\,(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)+\sin\phi\,(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)\}\\
&- 6\,f^5\,(\partial_4 u+\partial_3 v)\{\cos\phi\,(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)-\sin\phi\,(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)\}\Big],
\end{aligned}$$

(3.6)
$$\rho_{13}=-\frac{1}{4\,f^4}\Big[f^{10}\,(\partial_3 u-\partial_4 v)(\partial_4 u+\partial_3 v)+16\,\partial_1 f\,\partial_2 f-4f\partial_1\partial_2 f\Big],$$

(3.7)

$$\begin{aligned}\rho_{14} &= \frac{1}{4f^2}\Big[f^6\{\sin\phi\,\partial_3(\partial_4 u+\partial_3 v)+\cos\phi\,\partial_4(\partial_4 u+\partial_3 v)\}\\ &+ 4\,\partial_1 f\ \{\cos\phi\,(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)-\sin\phi\,(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)\}\\ &+ 6\,f^5\,(\partial_4 u+\partial_3 v)\{\cos\phi\,(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)+\sin\phi\,(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)\}\Big],\end{aligned}$$

(3.8)

$$\begin{aligned}\rho_{23} &= \tfrac{1}{4f^2}\Big[f^6\{\cos\phi\,\partial_3(\partial_3 u-\partial_4 v)-\sin\phi\,\partial_4(\partial_3 u-\partial_4 v)\}\\ &+ 4\,\partial_2 f\ \{\cos\phi\,(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)+\sin\phi\,(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)\}\\ &+ 6\,f^5\,(\partial_3 u-\partial_4 v)\{\cos\phi\,(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)-\sin\phi\,(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)\}\Big],\end{aligned}$$

(3.9)

$$\begin{aligned}\rho_{24} &= \frac{1}{2}f\,\partial_2 f(\partial_3 u-\partial_4 v)-\frac{1}{2}f\,\partial_1 f(\partial_4 u+\partial_3 v)\\ &+ \sin 2\phi\,\{(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)^2-(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)^2\}\\ &- 2\cos 2\phi\,(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)\\ &+ f\cos\phi\sin\phi\,\partial_4(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)-f\cos^2\phi\,\partial_4(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)\\ &+ f\sin^2\phi\,\partial_3(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)-f\cos\phi\sin\phi\,\partial_3(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)\\ &+ f(u\cos\phi+v\sin\phi)\{\ \cos\phi\,\partial_2(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)\\ &- \sin\phi\,\partial_2(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)\ \}\\ &+ f(u\sin\phi-v\cos\phi)\{\ \cos\phi\,\partial_1(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)\\ &- \sin\phi\,\partial_1(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)\ \},\end{aligned}$$

(3.10)

$$\begin{aligned}\rho_{34} &= -\frac{1}{4f^2}\Big[f^6\{\sin\phi\,\partial_3(\partial_3 u-\partial_4 v)+\cos\phi\,\partial_4(\partial_3 u-\partial_4 v)\}\\ &+ 4\,\partial_2 f\ \{\sin\phi\,(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)-\cos\phi\,(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)\}\\ &+ 6\,f^5\,(\partial_3 u-\partial_4 v)\{\sin\phi\,(\partial_3 f-u\,\partial_1 f-v\,\partial_2 f)+\cos\phi\,(\partial_4 f+v\,\partial_1 f-u\,\partial_2 f)\}\Big].\end{aligned}$$

In order that our almost Kähler manifold $M(f,u,v,\phi)$ is Einstein, it is necessary that $\rho_{ij}=0$ for $i\neq j$. From (3.6), if

$$\partial_3 u-\partial_4 v=0 \quad \text{or} \quad \partial_4 u+\partial_3 v=0$$

and

$$\partial_1 f=0 \quad \text{or} \quad \partial_2 f=0,$$

then $\rho_{13}=0$. Taking account of the Nurowski-Przanowski example, we assume that

$$\partial_4 u+\partial_3 v=0 \quad \text{and} \quad \partial_2 f=0. \tag{3.11}$$

By (3.5) and (3.7), if

$$\partial_3 f = u\partial_1 f \quad \text{and} \quad \partial_4 f = -v\partial_1 f, \tag{3.12}$$

then $\rho_{12} = \rho_{14} = \rho_{24} = 0$. Further, from (3.8), if

$$\partial_3 u - \partial_4 v = \lambda (= \text{costant}), \tag{3.13}$$

then $\rho_{23} = \rho_{34} = 0$.

In this case, (3.1) $\sim$ (3.4) reduce to

$$\rho_{11} = -\frac{1}{4\,f^4}\Big[10\,(\partial_1 f)^2 - 2\,f\,\partial_1^2 f\Big], \tag{3.14}$$

$$\rho_{22} = -\frac{1}{4\,f^4}\Big[6\,(\partial_1 f)^2 - 2\,f\,\partial_1^2 f + \lambda^2 f^{10}\Big], \tag{3.15}$$

$$\rho_{33} = \frac{1}{4\,f^4}\Big[6\,(\partial_1 f)^2 - 2\,f\,\partial_1^2 f + \lambda^2 f^{10}\Big], \tag{3.16}$$

$$\rho_{44} = -\frac{1}{4\,f^4}\Big[6\,(\partial_1 f)^2 - 2\,f\,\partial_1^2 f + \lambda^2 f^{10}\Big]. \tag{3.17}$$

Since $\rho_{22} = -\rho_{33}$, it must be $\rho = 0$ for (M, J, g) is Einstein. Comparing (3.14) with (3.15), we find

$$4(\partial_1 f)^2 = \lambda^2 f^{10} \quad \text{and} \quad 2\partial_1 f = \pm\lambda f^5.$$

If $2\partial_1 f = -\lambda f^5$, then $N = 0$ by (2.9), and J is integrable. So, we suppose

$$2\partial_1 f = \lambda f^5, \quad \lambda \neq 0. \tag{3.18}$$

From (3.18), we obtain that f is of in the following form:

$$f = \{-2\lambda x_1 - \varphi(x_3, x_4)\}^{-\frac{1}{4}}, \tag{3.19}$$

where φ is an arbitrary function of x_3 and x_4.

By (3.12), we have

$$\partial_3 \varphi = 2\lambda u \quad \text{and} \quad \partial_4 \varphi = -2\lambda v. \tag{3.20}$$

Taking account of (3.13) and (3.20), we deduce

$$\partial_3^2 \varphi + \partial_4^2 \varphi = 2\lambda^2. \tag{3.21}$$

Here, we suppose that φ is a polynomial of degree 2, for simplicity:

$$\varphi = ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k.$$

Then by (3.20) and (3.21), we have

$$u = \frac{1}{\lambda}(ax_3 + bx_4 + d),$$
$$v = -\frac{1}{\lambda}(bx_3 + cx_4 + e),$$
$$a + c = \lambda^2 (> 0).$$

Summing up the above arguements, we obtain the following
Theorem 3.1. *Let*

$$u = \pm\frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d),$$
$$v = \mp\frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e),$$
$$f = \left\{\mp 2\sqrt{a+c}\,x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{1}{4}},$$

where a, b, c, d, e, k are arbitrary constants such that $a + c > 0$. Then Riemannian metric g given by (2.4) is a Ricci flat, i.e., $M(f, u, v, \phi)$ is a Ricci flat strictly almost Kähler manifold.

Note that the Nurowski-Przanowski's Ricci flat example is obtained by putting $a = c = 8, \lambda = -4, b = d = e = k = 0$.

Next, we assume that

$$\partial_3 u - \partial_4 v = 0 \quad \text{and} \quad \partial_1 f = 0. \tag{3.22}$$

By (3.8) and (3.10), if

$$\partial_3 f = v\partial_2 f \quad \text{and} \quad \partial_4 f = u\partial_2 f, \tag{3.23}$$

then $\rho_{23} = \rho_{24} = \rho_{34} = 0$. Further, from (3.5), if

$$\partial_3 u + \partial_4 v = \mu (= \text{costant}), \tag{3.24}$$

then $\rho_{12} = \rho_{14} = 0$.

In this case, (3.1) $\sim$ (3.4) reduce to

$$\rho_{11} = \frac{1}{4\,f^4}\Big[6\,(\partial_2 f)^2 - 2\,f\,\partial_2^2 f + \mu^2 f^{10}\Big], \tag{3.25}$$

$$\rho_{22} = -\frac{1}{4\,f^4}\Big[6\,(\partial_2 f)^2 - 2\,f\,\partial_2^2 f + \mu^2 f^{10}\Big], \tag{3.26}$$

$$\rho_{33} = -\frac{1}{4\,f^4}\Big[10\,(\partial_2 f)^2 - 2\,f\,\partial_2^2 f\Big], \tag{3.27}$$

$$\rho_{44} = -\frac{1}{4\,f^4}\Big[6\,(\partial_2 f)^2 - 2\,f\,\partial_2^2 f + \mu^2 f^{10}\Big]. \tag{3.28}$$

By the same arguement as above, we have

$$f = \{-2\mu x_2 - \varphi(x_3, x_4)\}^{-\frac{1}{4}}, \tag{3.29}$$

and suppose that

$$\varphi = ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k.$$

Then, we find

$$u = \frac{1}{\mu}(bx_3 + cx_4 + e),$$
$$v = \frac{1}{\mu}(ax_3 + bx_4 + d),$$
$$a + c = \mu^2 (> 0).$$

Consequently, we obtain the following

Theorem 3.2. *Let*

$$u = \pm\frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e),$$
$$v = \pm\frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d),$$
$$f = \left\{\mp 2\sqrt{a+c}\,x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{1}{4}},$$

where a, b, c, d, e, k are arbitrary constants such that $a + c > 0$. Then Riemannian metric g given by (2.4) is a Ricci flat, i.e., $M(f, u, v, \phi)$ is a Ricci flat strictly almost Kähler manifold.

In the next section, we shall show that the almost Kähler manifolds $M(f, u, v, \phi)$ in Theorems 3.1 and 3.2 are also of pointwise constant holomorphic sectional curvature and weakly $*$-Einstein.

Remark 3.3. By considering the integrable case: $2\partial_1 f = -\lambda f^5$ or $2\partial_2 f = -\mu f^5$, we get the Ricci flat Kähler manifolds,

Let

$$u = \pm\frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d),$$
$$v = \mp\frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e),$$
$$f = \left\{\pm 2\sqrt{a+c}\,x_1 + (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{1}{4}},$$

or

$$u = \pm\frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e),$$
$$v = \pm\frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d),$$
$$f = \left\{\pm 2\sqrt{a+c}\,x_2 + (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{1}{4}}.$$

Then $M(f, u, v, \phi)$ are Ricci flat Kähler manifolds.

4 Examples of almost Kähler 4-manifolds with pointwise constant holomorphic sectional curvature

In this section, we construct 4-dimensional almost Kähler manifolds of pointwise constant holomorphic sectional curvature.

First, we show that the almost Kähler manifolds $M(f, u, v, \phi)$ in Theorem 3.1 have pointwise constant holomorphic sectional curvature. Indeed, by direct computations, we obtain

$$\begin{aligned} & A_{13} = -A_{24} = \\ & -\frac{1}{2}(a+c)\left\{\mp 2\sqrt{a+c}\,x_1 = -(ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{3}{2}}, \\ (4.1)\quad & A_{31} = -A_{42} \\ & = \frac{3}{2}(a+c)\left\{\mp 2\sqrt{a+c}\,x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{3}{2}}, \\ & A_{ij} = 0 \qquad \text{(otherwise)}, \end{aligned}$$

$$\begin{aligned} & R_{1212} = R_{1234} = R_{1414} = R_{1423} = R_{2323} = R_{3434} \\ & = -\frac{1}{4}(a+c)\left\{\mp 2\sqrt{a+c}\,x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{3}{2}}, \\ (4.2)\quad & R_{1313} = -R_{1324} = R_{2424} \\ & = \frac{1}{2}(a+c)\left\{\mp 2\sqrt{a+c}\,x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{3}{2}}, \\ & R_{ijkl} = 0 \qquad \text{(otherwise} \quad i<j, k<l), \end{aligned}$$

$$\begin{aligned} & \rho^*_{11} = \rho^*_{22} = \rho^*_{33} = \rho^*_{44} \\ (4.3)\quad & = \frac{1}{2}(a+c)\left\{\mp 2\sqrt{a+c}\,x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{3}{2}}, \\ & \rho^*_{ij} = 0 \qquad (i \neq j), \end{aligned}$$

$$(4.4)\quad \tau^* = 2(a+c)\left\{\mp 2\sqrt{a+c}\,x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{3}{2}}.$$

By virtue of Proposition 2.1, (4.1), (4.2), (4.3), and (4.4), we have the following

Theorem 4.1. *Let $M(f, u, v, \phi)$ be the Ricci flat strictly almost Kähler manifold in Theorem 3.1, i.e., functions u, v and f are given by*

$$\begin{aligned} u &= \pm\frac{1}{\sqrt{a+c}}(ax_3 + bx_4 + d), \\ v &= \mp\frac{1}{\sqrt{a+c}}(bx_3 + cx_4 + e), \\ f &= \left\{\mp 2\sqrt{a+c}\,x_1 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{1}{4}}. \end{aligned}$$

Then $M(f,u,v,\phi)$ is of pointwise constant holomorphic sectional curvature

$$c=\frac{1}{4}(a+c)\left\{\mp 2\sqrt{a+c}\,x_1-(ax_3^2+2bx_3x_4+cx_4^2+2dx_3+2ex_4+k)\right\}^{-\frac{3}{2}},$$

and weakly $*$-Einstein.

Next, let $M(f,u,v,\phi)$ be the almost Kähler manifold in Theorem 3.2 Then, similarly we obtain

$$\begin{aligned}
&A_{13}=-A_{24}\\
&=-\frac{3}{2}(a+c)\left\{\mp 2\sqrt{a+c}\,x_2-(ax_3^2+2bx_3x_4+cx_4^2+2dx_3+2ex_4+k)\right\}^{-\frac{3}{2}},\\
(4.5)\quad &A_{31}=-A_{42}\\
&=\frac{1}{2}(a+c)\left\{\mp 2\sqrt{a+c}\,x_2-(ax_3^2+2bx_3x_4+cx_4^2+2dx_3+2ex_4+k)\right\}^{-\frac{3}{2}},\\
&A_{ij}=0\quad\text{(otherwise)},
\end{aligned}$$

$$\begin{aligned}
&R_{1212}=R_{1234}=R_{1414}=R_{1423}=R_{2323}=R_{3434}\\
&=-\frac{1}{4}(a+c)\left\{\mp 2\sqrt{a+c}\,x_2-(ax_3^2+2bx_3x_4+cx_4^2+2dx_3+2ex_4+k)\right\}^{-\frac{3}{2}},\\
(4.6)\quad &R_{1313}=-R_{1324}=R_{2424}\\
&=\frac{1}{2}(a+c)\left\{\mp 2\sqrt{a+c}\,x_2-(ax_3^2+2bx_3x_4+cx_4^2+2dx_3+2ex_4+k)\right\}^{-\frac{3}{2}},\\
&R_{ijkl}=0\quad(\text{otherwise}i<j,k<l),
\end{aligned}$$

$$\begin{aligned}
&\rho^*_{11}=\rho^*_{22}=\rho^*_{33}=\rho^*_{44}\\
(4.7)\quad &=\frac{1}{2}(a+c)\left\{\mp 2\sqrt{a+c}\,x_2-(ax_3^2+2bx_3x_4+cx_4^2+2dx_3+2ex_4+k)\right\}^{-\frac{3}{2}},\\
&\rho^*_{ij}=0\qquad(i\neq j),
\end{aligned}$$

$$(4.8)\quad \tau^*=2(a+c)\left\{\mp 2\sqrt{a+c}\,x_2-(ax_3^2+2bx_3x_4+cx_4^2+2dx_3+2ex_4+k)\right\}^{-\frac{3}{2}}.$$

Therefore we have the following

Theorem 4.2. *Let $M(f,u,v,\phi)$ be the Ricci flat strictly almost Kähler manifold in Theorem 3.2, i.e., u,v and f are given by*

$$\begin{aligned}
u&=\pm\frac{1}{\sqrt{a+c}}(bx_3+cx_4+e),\\
v&=\pm\frac{1}{\sqrt{a+c}}(ax_3+bx_4+d),\\
f&=\left\{\mp 2\sqrt{a+c}\,x_2-(ax_3^2+2bx_3x_4+cx_4^2+2dx_3+2ex_4+k)\right\}^{-\frac{1}{4}}.
\end{aligned}$$

Then $M(f,u,v,\phi)$ is of pointwise constant holomorphic sectional curvature

$$c = \frac{1}{4}(a+c)\left\{\mp 2\sqrt{a+c}\,x_2 - (ax_3^2 + 2bx_3x_4 + cx_4^2 + 2dx_3 + 2ex_4 + k)\right\}^{-\frac{3}{2}},$$

and weakly $*$-Einstein.

Now, we shall construct other examples. In the previous paper [5], we assumed that $u = v = 0$ and $\phi = 0$ for the sake of simplicity. Here, we shall consider the case where $u = \alpha, v = \beta$ (α and β are constants) and arbitrary ϕ. Then, with respect to the unitary frame $\{e_i\}$ in (2.6) and the coordinates system (ξ_i) in (2.10), we have the following expressions:

$$(4.9)\quad \begin{aligned}
A_{11} &= \frac{3\tilde{\partial}_1 f\tilde{\partial}_2 f}{f^4} - \frac{\tilde{\partial}_1\tilde{\partial}_2 f}{f^3} - \cos 2\phi\,\tilde{\partial}_3 f\tilde{\partial}_4 f - \frac{\sin 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\},\\
A_{12} &= \frac{\cos\phi}{f^2}\{2\tilde{\partial}_1 f\tilde{\partial}_3 f - 2\tilde{\partial}_2 f\tilde{\partial}_4 f + f\tilde{\partial}_2\tilde{\partial}_4 f\}\\
&- \frac{\sin\phi}{f^2}\{2\tilde{\partial}_2 f\tilde{\partial}_3 f + 2\tilde{\partial}_1 f\tilde{\partial}_4 f - f\tilde{\partial}_2\tilde{\partial}_3 f\},\\
A_{13} &= -\frac{1}{f^4}\{(\tilde{\partial}_1 f)^2 - 2(\tilde{\partial}_2 f)^2 + f\tilde{\partial}_2^2 f\} - \{\cos\phi\,\tilde{\partial}_3 f - \sin\phi\,\tilde{\partial}_4 f\}^2,\\
A_{14} &= -\frac{\cos\phi}{f^2}\{\tilde{\partial}_2 f\tilde{\partial}_3 f + \tilde{\partial}_1 f\tilde{\partial}_4 f - f\tilde{\partial}_2\tilde{\partial}_3 f\}\\
&- \frac{\sin\phi}{f^2}\{\tilde{\partial}_1 f\tilde{\partial}_3 f - \tilde{\partial}_2 f\tilde{\partial}_4 f + f\tilde{\partial}_2\tilde{\partial}_4 f\},\\
A_{21} &= -\frac{\cos\phi}{f^2}\{2\tilde{\partial}_2 f\tilde{\partial}_4 f - f\tilde{\partial}_1\tilde{\partial}_3 f\} - \frac{\sin\phi}{f^2}\{2\tilde{\partial}_2 f\tilde{\partial}_3 f + f\tilde{\partial}_1\tilde{\partial}_4 f\},\\
A_{22} &= \frac{\tilde{\partial}_1 f\tilde{\partial}_2 f}{f^4} - \cos 2\phi\,\tilde{\partial}_3 f\tilde{\partial}_4 f - \frac{\sin 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\}\\
&- f\cos 2\phi\,\tilde{\partial}_3\tilde{\partial}_4 f - \frac{f\sin 2\phi}{2}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\},\\
A_{23} &= -\frac{\cos\phi}{f^2}\{\tilde{\partial}_2 f\tilde{\partial}_3 f - \tilde{\partial}_1 f\tilde{\partial}_4 f - f\tilde{\partial}_2\tilde{\partial}_3 f\}\\
&+ \frac{\sin\phi}{f^2}\{\tilde{\partial}_1 f\tilde{\partial}_3 f + \tilde{\partial}_2 f\tilde{\partial}_4 f - f\tilde{\partial}_2\tilde{\partial}_4 f\},\\
A_{24} &= \frac{(\tilde{\partial}_2 f)^2}{f^4} + \{\cos\phi\,\tilde{\partial}_4 f + \sin\phi\,\tilde{\partial}_3 f\}^2 - \frac{f}{2}\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\}\\
&- \frac{f\cos 2\phi}{2}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\} + f\sin 2\phi\,\tilde{\partial}_3\tilde{\partial}_4 f,\\
A_{31} &= -\frac{1}{f^4}\{2(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2 - f\tilde{\partial}_1^2 f\} + \{\cos\phi\,\tilde{\partial}_4 f + \sin\phi\,\tilde{\partial}_3 f\}^2,
\end{aligned}$$

$$
\begin{aligned}
A_{32} &= \frac{\cos\phi}{f^2}\{\tilde\partial_2 f\tilde\partial_3 f+\tilde\partial_1 f\tilde\partial_4 f-f\tilde\partial_1\tilde\partial_4 f\}\\
&+ \frac{\sin\phi}{f^2}\{\tilde\partial_1 f\tilde\partial_3 f-\tilde\partial_2 f\tilde\partial_4 f-f\tilde\partial_1\tilde\partial_3 f\},\\
A_{33} &= -\frac{3\tilde\partial_1 f\tilde\partial_2 f}{f^4}+\frac{\tilde\partial_1\tilde\partial_2 f}{f^3}+\cos 2\phi\,\tilde\partial_3 f\tilde\partial_4 f+\frac{\sin 2\phi}{2}\{(\tilde\partial_3 f)^2-(\tilde\partial_4 f)^2\},\\
A_{34} &= \frac{\cos\phi}{f^2}\{2\tilde\partial_1 f\tilde\partial_3 f-2\tilde\partial_2 f\tilde\partial_4 f-f\tilde\partial_1\tilde\partial_3 f\}\\
&- \frac{\sin\phi}{f^2}\{2\tilde\partial_2 f\tilde\partial_3 f+2\tilde\partial_1 f\tilde\partial_4 f-f\tilde\partial_1\tilde\partial_4 f\},\\
A_{41} &= -\frac{\cos\phi}{f^2}\{\tilde\partial_2 f\tilde\partial_3 f-\tilde\partial_1 f\tilde\partial_4 f+f\tilde\partial_1\tilde\partial_4 f\}\\
&+ \frac{\sin\phi}{f^2}\{\tilde\partial_1 f\tilde\partial_3 f+\tilde\partial_2 f\tilde\partial_4 f-f\tilde\partial_1\tilde\partial_3 f\},\\
A_{42} &= -\frac{(\tilde\partial_1 f)^2}{f^4}-\{\cos\phi\tilde\partial_3 f-\sin\phi\tilde\partial_4 f\}^2+\frac{f}{2}\{\tilde\partial_3^2 f+\tilde\partial_4^2 f\}\\
&- \frac{f\cos 2\phi}{2}\{\tilde\partial_3^2 f-\tilde\partial_4^2 f\}+f\sin 2\phi\,\tilde\partial_3\tilde\partial_4 f,\\
A_{43} &= \frac{\cos\phi}{f^2}\{2\tilde\partial_1 f\tilde\partial_3 f-f\tilde\partial_2\tilde\partial_4 f\}-\frac{\sin\phi}{f^2}\{2\tilde\partial_1 f\tilde\partial_4 f+f\tilde\partial_2\tilde\partial_3 f\},\\
A_{44} &= -\frac{\tilde\partial_1 f\tilde\partial_2 f}{f^4}+\cos 2\phi\,\tilde\partial_3 f\tilde\partial_4 f+\frac{\sin 2\phi}{2}\{(\tilde\partial_3 f)^2-(\tilde\partial_4 f)^2\}\\
&+ f\cos 2\phi\,\tilde\partial_3\tilde\partial_4 f+\frac{f\sin 2\phi}{2}\{\tilde\partial_3^2 f-\tilde\partial_4^2 f\}.
\end{aligned}
$$

Furthermore, we find from (2.11)

$$
\begin{aligned}
R_{1212} &= \frac{1}{2f^4}\{3(\tilde\partial_1 f)^2-(\tilde\partial_2 f)^2-f\tilde\partial_1^2 f\}-\frac{\cos 2\phi}{2}\{(\tilde\partial_3 f)^2-(\tilde\partial_4 f)^2\}\\
&+ \sin 2\phi\,\tilde\partial_3 f\tilde\partial_4 f+\frac{f}{4}\{\tilde\partial_3^2 f+\tilde\partial_4^2 f\}-\frac{f\cos 2\phi}{4}\{\tilde\partial_3^2 f-\tilde\partial_4^2 f\}\\
&+ \frac{f\sin 2\phi}{2}\tilde\partial_3\tilde\partial_4 f,\\
R_{3434} &= \frac{1}{2f^4}\{-(\tilde\partial_1 f)^2+3(\tilde\partial_2 f)^2-f\tilde\partial_2^2 f\}+\frac{\cos 2\phi}{2}\{(\tilde\partial_3 f)^2-(\tilde\partial_4 f)^2\}\\
&- \sin 2\phi\,\tilde\partial_3 f\tilde\partial_4 f+\frac{f}{4}\{\tilde\partial_3^2 f+\tilde\partial_4^2 f\}+\frac{f\cos 2\phi}{4}\{\tilde\partial_3^2 f-\tilde\partial_4^2 f\}\\
&- \frac{f\sin 2\phi}{2}\tilde\partial_3\tilde\partial_4 f,
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
R_{1234} &= 0,\\
R_{1324} &= 0,\\
R_{1423} &= 0,\\
R_{1313} &= \frac{1}{2f^4}\{-(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2 + f\tilde{\partial}_1^2 f + f\tilde{\partial}_2^2 f\} + \frac{1}{2}\{(\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2\},\\
R_{1414} &= \frac{1}{2f^4}\{3(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2 - f\tilde{\partial}_1^2 f\} + \frac{\cos 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\}\\
&- \sin 2\phi\, \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{f}{4}\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\} + \frac{f\cos 2\phi}{4}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\}\\
&- \frac{f\sin 2\phi}{2}\tilde{\partial}_3\tilde{\partial}_4 f,\\
R_{2323} &= \frac{1}{2f^4}\{-(\tilde{\partial}_1 f)^2 + 3(\tilde{\partial}_2 f)^2 - f\tilde{\partial}_2^2 f\} - \frac{\cos 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\}\\
&+ \sin 2\phi\, \tilde{\partial}_3 f \tilde{\partial}_4 f + \frac{f}{4}\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\} - \frac{f\cos 2\phi}{4}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\}\\
&+ \frac{f\sin 2\phi}{2}\tilde{\partial}_3\tilde{\partial}_4 f,\\
R_{2424} &= \frac{1}{2f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} + \frac{1}{2}\{(\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2\} - \frac{f}{2}\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\},\\
R_{1334} &= \frac{\cos\phi}{2f}\tilde{\partial}_1\tilde{\partial}_3 f - \frac{\sin\phi}{2f}\tilde{\partial}_1\tilde{\partial}_4 f,\\
R_{2434} &= -\frac{\cos\phi}{f^2}\tilde{\partial}_2 f\tilde{\partial}_4 f - \frac{\sin\phi}{f^2}\tilde{\partial}_2 f\tilde{\partial}_3 f + \frac{\cos\phi}{2f}\tilde{\partial}_2\tilde{\partial}_4 f + \frac{\sin\phi}{2f}\tilde{\partial}_2\tilde{\partial}_3 f,\\
R_{1213} &= -\frac{\cos\phi}{2f}\tilde{\partial}_2\tilde{\partial}_4 f - \frac{\sin\phi}{2f}\tilde{\partial}_2\tilde{\partial}_3 f,\\
R_{1224} &= \frac{\cos\phi}{f^2}\tilde{\partial}_1 f\tilde{\partial}_3 f - \frac{\sin\phi}{f^2}\tilde{\partial}_1 f\tilde{\partial}_4 f - \frac{\cos\phi}{2f}\tilde{\partial}_1\tilde{\partial}_3 f + \frac{\sin\phi}{2f}\tilde{\partial}_1\tilde{\partial}_4 f,\\
R_{1434} &= \frac{1}{2f^4}\{4\tilde{\partial}_1 f\tilde{\partial}_2 f - f\tilde{\partial}_1\tilde{\partial}_2 f\},\\
R_{2334} &= -\cos 2\phi\, \tilde{\partial}_3 f\tilde{\partial}_4 f - \frac{\sin 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\}\\
&- \frac{f\cos 2\phi}{2}\tilde{\partial}_3\tilde{\partial}_4 f - \frac{f\sin 2\phi}{4}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\},\\
R_{1214} &= \cos 2\phi\, \tilde{\partial}_3 f\tilde{\partial}_4 f + \frac{\sin 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\}\\
&+ \frac{f\cos 2\phi}{2}\tilde{\partial}_3\tilde{\partial}_4 f + \frac{f\sin 2\phi}{4}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\},
\end{aligned}
$$

$$
\begin{aligned}
R_{1223} &= -\frac{1}{2f^4}\{4\tilde{\partial}_1 f\tilde{\partial}_2 f - f\tilde{\partial}_1\tilde{\partial}_2 f\},\\
R_{1323} &= -\frac{\cos\phi}{2f}\tilde{\partial}_1\tilde{\partial}_4 f - \frac{\sin\phi}{2f}\tilde{\partial}_1\tilde{\partial}_3 f,\\
R_{2324} &= -\frac{\cos\phi}{f^2}\tilde{\partial}_2 f\tilde{\partial}_3 f + \frac{\sin\phi}{f^2}\tilde{\partial}_2 f\tilde{\partial}_4 f + \frac{\cos\phi}{2f}\tilde{\partial}_2\tilde{\partial}_3 f - \frac{\sin\phi}{2f}\tilde{\partial}_2\tilde{\partial}_4 f,\\
R_{1314} &= -\frac{\cos\phi}{2f}\tilde{\partial}_2\tilde{\partial}_3 f + \frac{\sin\phi}{2f}\tilde{\partial}_2\tilde{\partial}_4 f,\\
R_{1424} &= -\frac{\cos\phi}{f^2}\tilde{\partial}_1 f\tilde{\partial}_4 f - \frac{\sin\phi}{f^2}\tilde{\partial}_1 f\tilde{\partial}_3 f + \frac{\cos\phi}{2f}\tilde{\partial}_1\tilde{\partial}_4 f + \frac{\sin\phi}{2f}\tilde{\partial}_1\tilde{\partial}_3 f.
\end{aligned}
$$

Then, by (4.1), we obtain easily

$$
\begin{aligned}
\tau &= \frac{1}{f^3}\{\tilde{\partial}_1^2 f + \tilde{\partial}_2^2 f\} - \frac{4}{f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\}\\
&- 2\{(\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2\} - f\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\},
\end{aligned}
\tag{4.11}
$$

$$
\tau^* = \frac{1}{f^3}\{\tilde{\partial}_1^2 f + \tilde{\partial}_2^2 f\} - \frac{2}{f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} - f\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\},
\tag{4.12}
$$

$$
\tau^* - \tau = \frac{2}{f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} + 2\{(\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2\}.
\tag{4.13}
$$

By (4.13), we see that $M(f,\alpha,\beta,\phi)$ is non-Kählerian if and only if f is not constant. By Proposition 2.1, (4.9), (4.1) and (4.13), we find the following

Lemma 4.3. *The almost Kähler manifold $M(f,\alpha,\beta,\phi)$ is of pointwise constant holomorphic sectional curvature $c = c(p)$ if and only if*

$$
\begin{aligned}
&\frac{1}{2f^3}\{\tilde{\partial}_1^2 f - \tilde{\partial}_2^2 f\} + \frac{f\cos 2\phi}{2}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\} - f\sin 2\phi\,\tilde{\partial}_3\tilde{\partial}_4 f\\
&= \frac{2}{f^4}\{(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2\} - \cos 2\phi\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\} + 2\sin 2\phi\,\tilde{\partial}_3 f\tilde{\partial}_4 f,
\end{aligned}
\tag{4.14}
$$

$$
c = -\frac{1}{8}(\tau^* - \tau) = -\frac{1}{4f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} - \frac{1}{4}\{(\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2\},
\tag{4.15}
$$

$$
\begin{aligned}
&\frac{1}{f^3}\{\tilde{\partial}_1^2 f + \tilde{\partial}_2^2 f\} - f\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\}\\
&\qquad = \frac{1}{f^4}\{(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} - \{(\tilde{\partial}_3 f)^2 + (\tilde{\partial}_4 f)^2\},
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
&\cos\phi\{\tilde{\partial}_1\tilde{\partial}_3 f + \tilde{\partial}_2\tilde{\partial}_4 f\} - \sin\phi\{\tilde{\partial}_1\tilde{\partial}_4 f - \tilde{\partial}_2\tilde{\partial}_3 f\}\\
&\qquad = \frac{2\cos\phi}{f}\tilde{\partial}_2 f\tilde{\partial}_4 f + \frac{2\sin\phi}{f}\tilde{\partial}_2 f\tilde{\partial}_3 f,
\end{aligned}
\tag{4.17}
$$

$$\begin{aligned}\cos\phi\{\tilde\partial_1\tilde\partial_3 f+\tilde\partial_2\tilde\partial_4 f\}-\sin\phi\{\tilde\partial_1\tilde\partial_4 f-\tilde\partial_2\tilde\partial_3 f\}\\ =\frac{2\cos\phi}{f}\tilde\partial_1 f\tilde\partial_3 f-\frac{2\sin\phi}{f}\tilde\partial_1 f\tilde\partial_4 f,\end{aligned} \tag{4.18}$$

$$\begin{aligned}\tfrac{\tilde\partial_1\tilde\partial_2 f}{f^3}-f\cos 2\phi\tilde\partial_3\tilde\partial_4 f-\frac{f\sin 2\phi}{2}\{\tilde\partial_3^2 f-\tilde\partial_4^2 f\}\\ =\frac{4\tilde\partial_1 f\tilde\partial_2 f}{f^4}+2\cos 2\phi\tilde\partial_3 f\tilde\partial_4 f+\sin 2\phi\{(\tilde\partial_3 f)^2-(\tilde\partial_4 f)^2\},\end{aligned} \tag{4.19}$$

$$\begin{aligned}\cos\phi\{\tilde\partial_1\tilde\partial_4 f-\tilde\partial_2\tilde\partial_3 f\}+\sin\phi\{\tilde\partial_1\tilde\partial_3 f+\tilde\partial_2\tilde\partial_4 f\}\\ =\frac{\cos\phi}{f}\{\tilde\partial_1 f\tilde\partial_4 f-\tilde\partial_2 f\tilde\partial_3 f\}+\frac{\sin\phi}{f}\{\tilde\partial_1 f\tilde\partial_3 f+\tilde\partial_2 f\tilde\partial_4 f\}.\end{aligned} \tag{4.20}$$

To obtain solutions of the above equations, we first assume that $f=f(\xi_1,\xi_2)$ is a function of two variables $\xi_1,\ \xi_2$. Then (4.14) $\sim$ (4.20) reduce to

$$\tilde\partial_1^2 f-\tilde\partial_2^2 f=\frac{4}{f}\{(\tilde\partial_1 f)^2-(\tilde\partial_2 f)^2\}, \tag{4.21}$$

$$c=-\frac{1}{8}(\tau^*-\tau)=-\frac{1}{4f^4}\{(\tilde\partial_1 f)^2+(\tilde\partial_2 f)^2\}, \tag{4.22}$$

$$f(\tilde\partial_1^2 f+\tilde\partial_2^2 f)=(\tilde\partial_1 f)^2+(\tilde\partial_2 f)^2, \tag{4.23}$$

$$f\tilde\partial_1\tilde\partial_2 f=4\tilde\partial_1 f\tilde\partial_2 f. \tag{4.24}$$

By the same way as [5], we see that $f=K(\xi_1^2+\xi_2^2)^{-\frac{1}{3}}$ satisfies (4.21), (4.23) and (4.24), where K is a constant.

Next, we assume that $f=f(\xi_3,\xi_4)$ is a function of $\xi_3,\ \xi_4$. Then (4.14) $\sim$ (4.15) reduce to

$$f(\tilde\partial_3^2 f-\tilde\partial_4^2 f)=-2\{(\tilde\partial_3 f)^2-(\tilde\partial_4 f)^2\}, \tag{4.25}$$

$$c=-\frac{1}{8}(\tau^*-\tau)=-\frac{1}{4}\{(\tilde\partial_3 f)^2+(\tilde\partial_4 f)^2\}, \tag{4.26}$$

$$f(\tilde\partial_3^2 f+\tilde\partial_4^2 f)=(\tilde\partial_3 f)^2+(\tilde\partial_4 f)^2, \tag{4.27}$$

$$f\tilde\partial_3\tilde\partial_4 f=-2\tilde\partial_3 f\tilde\partial_4 f. \tag{4.28}$$

Similarly, we can easily see that $f=L(\xi_3^2+\xi_4^2)^{\frac{1}{3}}$ satisfies (4.25),(4.27) and (4.28), where L is a constant.

Consequently, we obtain the following

Theorem 4.4. *Let $u = \alpha, v = \beta$ (α, β are constants) and*

$$f = K\{(x_1 + \alpha x_3 - \beta x_4)^2 + (x_2 + \beta x_3 + \alpha x_4)^2\}^{-\frac{1}{3}},$$

or

$$f = L(x_3^2 + x_4^2)^{\frac{1}{3}},$$

where K and L are non-zero constants. Then the almost Kähler manifold $M(f, u, v, \phi)$ is of pointwise constant holomorphic sectional curvature with

$$c = -\frac{1}{9K^2}\{(x_1 + \alpha x_3 - \beta x_4)^2 + (x_2 + \beta x_3 + \alpha x_4)^2\}^{-\frac{1}{3}}$$

or

$$c = -\frac{1}{9}L^2(x_3^2 + x_4^2)^{-\frac{1}{3}},$$

respectively.

We note that the pointwise constant c in the above theorem is negative contrary to the ones in Theorems 4.1 and 4.2.

5 Weakly $*$-Einstein examples

In this section, we provide weakly $*$-Einstein almost Kähler 4-manifolds. In §4, we have already shown that the Ricci flat almost Kähler manifolds $M(f, u, v, \phi)$ in Theorems 3.1 and 3.2 are weakly $*$-Einstein.

Here, we consider the almost Kähler manifolds $M(f, \alpha, \beta, \phi)$ for any constants α, β. In this case, by using the coordinates $(\xi_1, \xi_2, \xi_3, \xi_4)$ of (2.10), the components of the Ricci $*$-tensor with respect to the unitary frame $\{e_i\}$ in (2.6) is given by

$$\begin{aligned}
\rho^*_{11} = \rho^*_{22} &= \frac{\tilde{\partial}_1^2 f}{2f^3} - \frac{f}{4}\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\} + \frac{f\cos 2\phi}{4}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\} - \frac{f\sin 2\phi}{2}\tilde{\partial}_3\tilde{\partial}_4 f \\
&+ \frac{1}{2f^4}\{-3(\tilde{\partial}_1 f)^2 + (\tilde{\partial}_2 f)^2\} + \frac{\cos 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\} - \sin 2\phi\,\tilde{\partial}_3 f\tilde{\partial}_4 f, \\
\rho^*_{33} = \rho^*_{44} &= \frac{\tilde{\partial}_2^2 f}{2f^3} - \frac{f}{4}\{\tilde{\partial}_3^2 f + \tilde{\partial}_4^2 f\} - \frac{f\cos 2\phi}{4}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\} + \frac{f\sin 2\phi}{2}\tilde{\partial}_3\tilde{\partial}_4 f \\
&+ \frac{1}{2f^4}\{(\tilde{\partial}_1 f)^2 - 3(\tilde{\partial}_2 f)^2\} - \frac{\cos 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\} + \sin 2\phi\,\tilde{\partial}_3 f\tilde{\partial}_4 f, \\
\rho^*_{12} = \rho^*_{21} &= \rho^*_{34} = \rho^*_{43} = 0,
\end{aligned}$$

$$\rho^*_{13} = \rho^*_{42} = \rho^*_{24} = \rho^*_{31} = \frac{\tilde{\partial}_1\tilde{\partial}_2 f}{2f^3} - 2\frac{\tilde{\partial}_1 f\tilde{\partial}_2 f}{f^4} - \frac{f\cos 2\phi}{2}\tilde{\partial}_3\tilde{\partial}_4 f - \cos 2\phi\,\tilde{\partial}_3 f\tilde{\partial}_4 f$$
$$-\frac{f\sin 2\phi}{4}\{\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f\} - \frac{\sin 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\},$$
$$\rho^*_{14} = -\rho^*_{32} = \frac{\cos\phi}{2f}(\tilde{\partial}_1\tilde{\partial}_3 f - \tilde{\partial}_2\tilde{\partial}_4 f) - \frac{\sin\phi}{2f}(\tilde{\partial}_2\tilde{\partial}_3 f + \tilde{\partial}_1\tilde{\partial}_4 f),$$
$$\rho^*_{23} = -\rho^*_{41} = \frac{\cos\phi}{2f}(\tilde{\partial}_1\tilde{\partial}_3 f - \tilde{\partial}_2\tilde{\partial}_4 f) - \frac{\cos\phi}{f^2}(\tilde{\partial}_1 f\tilde{\partial}_3 f - \tilde{\partial}_2 f\tilde{\partial}_4 f)$$
$$-\frac{\sin\phi}{2f}(\tilde{\partial}_2\tilde{\partial}_3 f + \tilde{\partial}_1\tilde{\partial}_4 f) + \frac{\sin\phi}{f^2}(\tilde{\partial}_2 f\tilde{\partial}_3 f + \tilde{\partial}_1 f\tilde{\partial}_4 f),$$

By (5.1), we have the following

Lemma 5.1. *The almost Kähler* 4*-manifold* $M(f,\alpha,\beta,\phi)$ *is weakly* $*$*-Einstein if and only if*

$$\text{(5.2)}\qquad \begin{aligned}&\frac{1}{2f^3}(\tilde{\partial}_1^2 f - \tilde{\partial}_2^2 f) + \frac{f\cos 2\phi}{2}(\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f) - f\sin 2\phi\,\tilde{\partial}_3\tilde{\partial}_4 f\\ &= \frac{2}{f^4}\{(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2\} - \cos 2\phi\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\} + 2\sin 2\phi\,\tilde{\partial}_3 f\tilde{\partial}_4 f,\end{aligned}$$

$$\text{(5.3)}\qquad \begin{aligned}&\frac{\tilde{\partial}_1\tilde{\partial}_2 f}{2f^3} - \frac{f\cos 2\phi}{2}\tilde{\partial}_3\tilde{\partial}_4 f - \frac{f\sin 2\phi}{4}(\tilde{\partial}_3^2 f - \tilde{\partial}_4^2 f)\\ &= 2\frac{\tilde{\partial}_1 f\tilde{\partial}_2 f}{f^4} + \cos 2\phi\,\tilde{\partial}_3 f\tilde{\partial}_4 f + \frac{\sin 2\phi}{2}\{(\tilde{\partial}_3 f)^2 - (\tilde{\partial}_4 f)^2\},\end{aligned}$$

$$\text{(5.4)}\qquad \cos\phi(\tilde{\partial}_1\tilde{\partial}_3 f - \tilde{\partial}_2\tilde{\partial}_4 f) - \sin\phi(\tilde{\partial}_2\tilde{\partial}_3 f + \tilde{\partial}_1\tilde{\partial}_4 f) = 0,$$

$$\text{(5.5)}\qquad \begin{aligned}&\cos\phi(\tilde{\partial}_1\tilde{\partial}_3 f - \tilde{\partial}_2\tilde{\partial}_4 f) - \sin\phi(\tilde{\partial}_2\tilde{\partial}_3 f + \tilde{\partial}_1\tilde{\partial}_4 f)\\ &= \frac{2}{f}\{\cos\phi(\tilde{\partial}_1 f\tilde{\partial}_3 f - \tilde{\partial}_2 f\tilde{\partial}_4 f) - \sin\phi(\tilde{\partial}_2 f\tilde{\partial}_3 f + \tilde{\partial}_1 f\tilde{\partial}_4 f)\}.\end{aligned}$$

If f is a function of ξ_1, ξ_2, then (5.2) $\sim$ (5.5) reduce to

$$\text{(5.6)}\qquad f(\tilde{\partial}_1^2 f - \tilde{\partial}_2^2 f) = 4\{(\tilde{\partial}_1 f)^2 - (\tilde{\partial}_2 f)^2\},$$

$$\text{(5.7)}\qquad f\tilde{\partial}_1\tilde{\partial}_2 f = 4\tilde{\partial}_1 f\tilde{\partial}_2 f.$$

Since (5.6), (5.7) are same as (4.21), (4.24), we see that

$$\text{(5.8)}\qquad f = K(\xi_1^2 + \xi_2^2)^{-\frac{1}{3}}$$

is a solution.

If f is a function of ξ_3, ξ_4, then (5.2) $\sim$ (5.5) reduce to

$$\begin{aligned}(5.9)\qquad &\frac{f\cos 2\phi}{2}(\tilde{\partial}_3^2 f-\tilde{\partial}_4^2 f)-f\sin 2\phi\,\tilde{\partial}_3\tilde{\partial}_4 f\\ &=-\cos 2\phi\{(\tilde{\partial}_3 f)^2-(\tilde{\partial}_4 f)^2\}+2\sin 2\phi\,\tilde{\partial}_3 f\tilde{\partial}_4 f,\end{aligned}$$

$$\begin{aligned}(5.10)\qquad &-\frac{f\cos 2\phi}{2}\tilde{\partial}_3\tilde{\partial}_4 f-\frac{f\sin 2\phi}{4}(\tilde{\partial}_3^2 f-\tilde{\partial}_4^2 f)\\ &=\cos 2\phi\,\tilde{\partial}_3 f\tilde{\partial}_4 f+\frac{\sin 2\phi}{2}\{(\tilde{\partial}_3 f)^2-(\tilde{\partial}_4 f)^2\}.\end{aligned}$$

Similarly, we see that

$$(5.11)\qquad f=L(\xi_3^2+\xi_4^2)^{\frac{1}{3}}$$

is a solution of (5.9), (5.10). Thus we have the following
Theorem 5.2. *Let*

$$f=K\{(x_1+\alpha x_3-\beta x_4)^2+(x_2+\beta x_3+\alpha x_4)^2\}^{-\frac{1}{3}},$$

or

$$f=L(x_3^2+x_4^2)^{\frac{1}{3}},$$

where K and L are non-zero constants. Then the almost Kähler manifold $M(f,\alpha,\beta,\phi)$ is weakly $*$-Einstein. The $*$-scalar curvature is given respectively by

$$\tau^*=-\frac{4}{9K^2}\{(x_1+\alpha x_3-\beta x_4)^2+(x_2+\beta x_3+\alpha x_4)^2\}^{-\frac{1}{3}},$$

or

$$\tau^*=-\frac{4}{9}L^2(x_3^2+x_4^2)^{-\frac{1}{3}}.$$

Now, we give another example. It is easy to see that

$$(5.12)\qquad f=(a_1\xi_1+a_2\xi_2+b_1)^{-\frac{1}{3}}$$

is also a solution of (5.6), (5.7), and

$$(5.13)\qquad f=(a_3\xi_3+a_4\xi_4+b_2)^{\frac{1}{3}}$$

is a solution of (5.9), (5.10).
From (5.12) and (5.13),

$$(5.14)\qquad f=(a_1\xi_1+a_2\xi_2+b_1)^{-\frac{1}{3}}(a_3\xi_3+a_4\xi_4+b_2)^{\frac{1}{3}}$$

satisfies (5.2), (5.3). Moreover, if

$$(5.15)\qquad (a_1a_3-a_2a_4)\cos\phi-(a_2a_3+a_1a_4)\sin\phi=0,$$

then f in (5.14) also satisfies (5.4) and (5.5). Note that the functions f in (5.12), (5.13) are given as special cases of (5.14) satisfying (5.15).
Consequently, we obtain the following

Theorem 5.3. *Let*

$$f = \left\{ \frac{a_3x_3 + a_4x_4 + b_2}{a_1(x_1 + \alpha x_3 - \beta x_4) + a_2(x_2 + \beta x_3 + \alpha x_4) + b_1} \right\}^{\frac{1}{3}},$$

where a_i $(i = 1, 2, 3, 4)$ *and* b_j $(j = 1, 2)$ *are constants satisfying* (5.15). *Then the almost Kähler manifold* $M(f, \alpha, \beta, \phi)$ *is weakly* $*$*-Einstein, and*

$$\begin{aligned} \tau^* &= \frac{2(a_1^2 + a_2^2)}{9}\{a_1(x_1 + \alpha x_3 - \beta x_4) + a_2(x_2 + \beta x_3 + \alpha x_4) + b_1\}^{-\frac{4}{3}} \\ &\times (a_3x_3 + a_4x_4 + b_2)^{-\frac{2}{3}} \\ &+ \frac{2(a_3^2 + a_4^2)}{9}\{a_1(x_1 + \alpha x_3 - \beta x_4) + a_2(x_2 + \beta x_3 + \alpha x_4) + b_1\}^{-\frac{2}{3}} \\ &\times (a_3x_3 + a_4x_4 + b_2)^{-\frac{4}{3}}. \end{aligned}$$

It should be remarked that the almost Kähler manifold $M(f, \alpha, \beta, \phi)$ in Theorem 5.2 is weakly$*$-Einstein for all ϕ. On the contrary $M(f, \alpha, \beta, \phi)$ in Theorem 5.3 is weakly$*$-Einstein for particular values of ϕ and a_i $(i = 1, 2, 3, 4)$ satisfying (5.15).

Finally, we note that the Ricci tensors of these examples in Theorem 5.3 are J-anti-invariant, i.e., $\rho(JX, JY) = -\rho(X, Y)$, and hence $\tau = 0$.

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