

# On Lagrange Epimorphisms and Lagrange Submersions

M. Popescu and P. Popescu

*Dedicated to Prof.Dr. Constantin UDRIȘTE  
on the occasion of his sixtieth birthday*

## Abstract

The aim of the paper is to define and study some aspects of Lagrange epimorphisms, which are generalizations for vector bundles of Lagrange submersions. We construct a Lagrangian on  $f^*\xi''$  canonically associated with a Lagrange vector bundle  $\xi$  and an  $f_0$ -epimorphism of vector bundles  $f : \xi \rightarrow \xi''$ . The main result is that a Lagrangian  $L$  on  $\xi$  has a locally  $f$ -projectable metric on  $\xi''$  iff it is a local Lagrange submersion. All the definitions and results in the paper can be stated in particular for Lagrange submersions.

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A *Lagrangian* on the vector bundle  $\xi = (E, \pi, M)$  is a real and differentiable function  $L : E \rightarrow \mathbb{R}$ . A *regular Lagrangian* on  $\xi$  is a Lagrangian  $L$  which has  $Hess(L)$  non-degenerate.. All the Lagrangians considered in the sequel are regular. Notice that every Lagrangian on  $\xi$  define a (pseudo)metric on the fibres of the vertical bundle  $V\xi$ . In local coordinates, if  $(x^i)$  and  $(x^i, y^a)$  are coordinates on  $M$  and  $E$  respectively, then  $Hess(L)$  is a bilinear form on the vertical bundle  $V\xi$ , which has the matrix:

$$g_{bc}(x^i, y^a) = \frac{\partial^2 L}{\partial y^b \partial y^c}.$$

The Legendre transform associated with the Lagrangian  $L$  on  $\xi$  is:

$$\mathcal{L} : E \rightarrow E^*, \mathcal{L}(x^i, y^a) = \left( x^i, \frac{\partial L}{\partial y^a}(x^i, y^a) \right).$$

If  $L$  is regular, then  $\mathcal{L}$  is a local diffeomorphism. Since most of our constructions are local, we can suppose, without loss of generality, that  $\mathcal{L}$  is a global diffeomorphism.

In the sequel  $\xi = (E, \pi, M)$  and  $\xi'' = (E'', \pi'', M'')$  are two vector bundles,  $f_0 : M \rightarrow M''$  is a submersion and  $f : \xi \rightarrow \xi''$  is an  $f_0$ -epimorphism (i.e.  $f$  is surjective on fibers). The restriction of the differential  $f_*$  to the vertical bundle  $V\xi \subset \tau E$  defines an  $f$ -(epi)morphism of vector bundle

$$(1) \quad F : V\xi \cong \pi^* \xi \rightarrow V\xi'' \cong (\pi'')^* \xi''$$

If  $(\xi, L)$  is a Lagrange space, the *vertical distribution* and *horizontal distribution* of  $f$  are  $Vf = \ker F$  and  $Hf = (\ker F)^\perp$  respectively, where the orthogonal is taken according the Lagrange metric, assuming that it exists. Notice that there is a canonical Whitney sum decomposition  $V\xi = Vf \oplus Hf$ .

We say that the Lagrangian  $L$  on  $\xi$  has an *f-projectable* metric if there exists a metric on  $V\xi''$  such that the  $f$ -epimorphism  $F : V\xi \rightarrow V\xi''$  is Riemannian, i.e. the fibers of  $Hf$  are isometric with the fibers of  $V\xi''$ .

We say that  $f : \xi \rightarrow \xi'$  is a *Lagrange epimorphism* if there is a Lagrangian on  $\xi''$  such that the induced  $f$ -epimorphism  $F : V\xi \rightarrow V\xi''$  is Riemannian on fibres.

We use in the sequel local coordinates adapted to the submersion  $f_0$  and the epimorphism  $f : (x^i) = (x^u, x^{\bar{u}})$  on  $M$ ,  $(x^{\bar{u}})$  on  $M''$ ,  $(x^u, x^{\bar{u}}, y^a, y^{\bar{a}})$  on  $E$  and  $(x^{\bar{u}}, y^{\bar{a}})$  on  $E''$ , such that  $f_0$  and  $f$  has the local forms  $(x^u, x^{\bar{u}}) \rightarrow (x^{\bar{u}})$  and  $(x^u, x^{\bar{u}}, y^a, y^{\bar{a}}) \rightarrow (x^{\bar{u}}, y^{\bar{a}})$  respectively. We denote as  $\{s_u, s_{\bar{u}}\}$  and  $\{s''_{\bar{u}}\}$  the local bases of sections in the vector bundles  $V\xi$  and  $V\xi''$  respectively. If  $g$  is a (pseudo)metric tensor on  $V\xi$ , then we denote as  $\{g_{ij}\} = \{g_{uv}, g_{\bar{u}\bar{v}} = g_{v\bar{u}}, g_{\bar{u}\bar{v}}\}$  its components using the above base. Notice that  $\{s_u\}$  is a local base of sections in  $Vf$  and  $\{\bar{s}_{\bar{u}} = s_{\bar{u}} - \tilde{g}^{uv} g_{v\bar{u}} s_u\}$  is a local base of sections in  $Hf$ , where  $(\tilde{g}^{uv}) = (g_{uv})^{-1}$  as matrices. It is easy to see that we have  $g(\bar{s}_{\bar{u}}, \bar{s}_{\bar{v}}) = g_{\bar{u}\bar{v}} - g_{\bar{u}u} \tilde{g}^{uv} g_{v\bar{v}}$ .

The annihilator of the vector subbundle  $Vf \subset \xi$  is a vector subbundle  $Vf^* \subset \xi^*$ , defined by the linear forms in  $\xi^*$  which are null on vectors in  $V\xi$ . If  $(x^u, x^{\bar{u}}, p_a, p_{\bar{a}})$  and  $(x^{\bar{u}}, p_{\bar{a}})$  are local coordinates on  $E^*$  and  $E''^*$  respectively, then the local coordinates on  $Vf^*$  are  $(x^u, x^{\bar{u}}, p_{\bar{a}})$  and the inclusion  $Vf^* \subset \xi^*$  has the local form  $(x^u, x^{\bar{u}}, p_{\bar{a}}) \rightarrow (x^u, x^{\bar{u}}, p_{\bar{a}}, 0)$ . Since  $\mathcal{L}$  is a diffeomorphism, then  $WE = \mathcal{L}^{-1}(Vf^*) \subset E$  is a submanifold. Notice the manifold  $WE$  has as coordinates  $(x^u, x^{\bar{u}}, Q^u(x^u, x^{\bar{u}}, y^{\bar{a}}), y^{\bar{a}})$ , such that

$$(2) \quad \frac{\partial L}{\partial y^a}(x^u, x^{\bar{u}}, Q^a(x^u, x^{\bar{u}}, y^{\bar{a}}), y^{\bar{a}}) = 0.$$

Let us observe that  $f|_{WE} : WE \rightarrow f_0^* E''$  is a diffeomorphism, where  $f_0^* E''$  is the total space of the induced bundle:

$$\begin{array}{ccc} M & \xrightarrow{f_0} & M'' \\ & & \uparrow \pi'' \\ & & E'' \end{array}$$

Indeed, the local form of  $f|_{WE}$  is

$$(3) \quad (x^u, x^{\bar{u}}, Q^a(x^u, x^{\bar{u}}, y^{\bar{a}}), y^{\bar{a}}) \rightarrow (x^u, x^{\bar{u}}, y^{\bar{a}}).$$

Let us define

$$S = f|_{WE}^{-1} : f_0^* E'' \rightarrow WE$$

and

$$\bar{L}'' = L \circ S : \pi^* E'' \rightarrow \mathbb{R}.$$

In local coordinates

$$\bar{L}''(x^u, x^{\bar{u}}, y^{\bar{a}}) = L(x^u, x^{\bar{u}}, Q^u(x^u, x^{\bar{u}}, y^{\bar{a}}), y^{\bar{a}}).$$

Notice that

The map  $S$  is a section of the fibered manifold defined by the epimorphism  $E \xrightarrow{f_0^*} f_0^* E''$  of vector bundles over  $M$ .

The couple  $(f_0^* \xi'', \bar{L}'')$  is a Lagrange vector bundle and we call  $\bar{L}''$  the *canonical Lagrangian* on  $f_0^* \xi''$  (induced by  $L$  and  $f$ ).

Consider again the  $f$ -epimorphism (1), denote as  $I : WE \rightarrow E$  the inclusion and consider the vector bundle  $I^* Hf$  (i.e. the fibres of  $Hf$  along the submanifold  $WE$ ).

**Lemma 1** *The restriction  $f|_{WE} : I^* Hf \rightarrow f_0^* \xi''$  is an  $f|_{WE}$ -isomorphism which is an isometry on fibres according to the Lagrangians  $L|_{I^* Hf}$  and  $\bar{L}''$  respectively.*

**Proof.** We use local coordinates. The local form of  $f|_{WE}$  is given by (3). The local correspondence of bases of sections by mean of  $f|_{WE}$  are  $\bar{s}_a = s_a - \tilde{g}^{ab} g_{ba} s_a \rightarrow s''_a$ . We have

$$(4) \quad g(\bar{s}_a, \bar{s}_b) = g_{ab} - g_{aa} \tilde{g}^{ab} g_{bb}$$

and

$$\bar{g}''(s''_a, s''_b) = \frac{\partial^2 \bar{L}''}{\partial y^a \partial y^b},$$

where

$$\frac{\partial^2 \bar{L}''}{\partial y^a \partial y^b}(x^u, x^{\bar{u}}, y^{\bar{a}}) = \frac{\partial^2}{\partial y^a \partial y^b} L(x^u, x^{\bar{u}}, Q^a(x^u, x^{\bar{u}}, y^{\bar{a}}), y^{\bar{a}}).$$

But

$$\frac{\partial}{\partial y^b} L(x^u, x^{\bar{u}}, Q^a(x^u, x^{\bar{u}}, y^{\bar{b}}), y^{\bar{b}}) = \frac{\partial L}{\partial y^b}(x^u, x^{\bar{u}}, Q^b(x^u, x^{\bar{u}}, y^{\bar{b}}), y^{\bar{v}})$$

thus

$$\begin{aligned} \frac{\partial^2}{\partial y^a \partial y^b} L(x^u, x^{\bar{u}}, Q^a(x^u, x^{\bar{u}}, y^{\bar{a}}), y^{\bar{a}}) &= \frac{\partial^2 L}{\partial y^a \partial y^b}(x^u, x^{\bar{u}}, Q^v(x^u, x^{\bar{u}}, y^{\bar{v}}), y^{\bar{v}}) + \\ &+ \frac{\partial Q^c}{\partial y^a}(x^u, x^{\bar{u}}, y^{\bar{a}}) \frac{\partial^2 L}{\partial y^{\bar{u}} \partial y^c}(x^u, x^{\bar{u}}, Q^a(x^u, x^{\bar{u}}, y^{\bar{a}}), y^{\bar{a}}) = g_{ab} + \frac{\partial Q^c}{\partial y^a} g_{bc}. \end{aligned}$$

Differentiating partially the relation (2) with respect to  $y^{\bar{a}}$  we obtain that

$$\frac{\partial Q^a}{\partial y^{\bar{a}}} = -\tilde{g}^{ab} g_{b\bar{a}},$$

where

$$(\tilde{g}^{ab}) = (g_{ab})^{-1}.$$

Thus

$$\bar{g}''(s''_a, s''_b) = g_{ab} - g_{aa} \tilde{g}^{ab} g_{bb}.$$

Comparing with relations (4), the conclusion follows.  $\square$

Notice that using a linear algebra computation we obtain the following equality of matrices:

$$(g_{ab} - g_{aa} \tilde{g}^{ab} g_{bb}) = (g^{\bar{a}\bar{b}})^{-1}.$$

Using the Lemma 1 we obtain:

**Proposition 1** *If  $f$  is a Lagrange epimorphism then  $\xi \xrightarrow{f_0^* f} f_0^* \xi''$  is a Lagrange epimorphism, where the canonical Lagrangian  $\bar{L}''$  is considered on  $f_0^* \xi''$ .*

Using also Lemma 1 we obtain:

**Proposition 2** *Let  $f : \xi \rightarrow \xi''$  be an epimorphism of vector bundles over the same base and  $L$  be a Lagrangian on  $\xi$  which is non-degenerate on  $Vf$  and has an  $f$ -projectable metric.*

*Then  $f$  is a Lagrange submersion.*

We say that a Lagrangian  $\bar{L}''$  on  $f_0^* \xi''$  projects on  $\xi''$  if there is a Lagrangian  $L''$  on  $\xi''$  such that  $\bar{L}'' = L'' \circ h$ , where  $h : f_0^* \xi'' \rightarrow \xi''$  is the canonical  $f_0$ -morphism which is the identity on fibers.

We are going to give sufficient conditions in order to induce a Lagrange submersion.

**Proposition 3** *Let  $f : \xi \rightarrow \xi''$  be an  $f_0$ -epimorphism and  $L$  a Lagrangian on  $\xi$  which is non-degenerate on  $Vf$ . Consider on  $f_0^* \xi''$  the canonical Lagrangian.*

*If  $\xi \xrightarrow{f_0^* f} f_0^* \xi''$  is a Lagrange epimorphism and the canonical Lagrangian on  $f_0^* \xi''$  is projectable on  $\xi''$  then there is a Lagrangian  $L''$  on  $\xi''$  such that  $f$  is a Lagrange submersion.*

**Proof.** Since  $h \circ f_0^* f = f$ , the conclusion follows using the composition of the Lagrange epimorphisms

$$\xi \xrightarrow{f_0^* f} f_0^* \xi'' \xrightarrow{h} \xi'',$$

which is a Lagrange epimorphism.  $\square$

**Lemma 2** *Let  $f_0 : M \rightarrow M''$  be a submersion,  $\xi'' = (E'', \pi'', M'')$  be a vector bundle and  $g : f_0^* \xi'' \rightarrow \xi''$  be the canonical  $f_0$ -morphism of vector bundles.*

*Assuming that a Lagrangian  $\bar{L}''$  on  $f_0^* \xi''$  has a  $g$ -projectable metric on  $\xi''$ , then  $\bar{L}''$  projects locally on  $\xi''$ .*

**Proof.** Consider local coordinates  $(x^u, x^{\bar{u}})$  on  $M$ ,  $(x^{\bar{u}})$  on  $M''$  and  $(x^{\bar{u}}, y^{\bar{a}})$  on  $\xi''$ , thus  $g$  has the local form  $(x^u, x^{\bar{u}}, y^{\bar{a}}) \rightarrow (x^{\bar{u}}, y^{\bar{a}})$ . The condition that  $\bar{L}''$  has a  $g$ -projectable metric on  $\xi''$  reads that the local functions  $\frac{\partial^2 \bar{L}''}{\partial y^{\bar{a}} \partial y^{\bar{b}}}$  do not depend on  $x^u$ .

It follows that there are local and real functions  $L'' : \xi''_{U''} \rightarrow \mathbb{R}$ , where  $U'' \subset M''$  and  $\xi''_{U''}$  is the restriction of the vector bundle  $\xi''$  to  $U''$ , such that

$$\frac{\partial^2 \bar{L}''}{\partial y^{\bar{a}} \partial y^{\bar{b}}}(x^u, x^{\bar{u}}, y^{\bar{a}}) = \frac{\partial^2 L''}{\partial y^{\bar{a}} \partial y^{\bar{b}}}(x^{\bar{u}}, y^{\bar{a}}).$$

$\square$

Using Proposition 3 and Lemma 2 we obtain the main result:

**Theorem 1** *Let  $f : \xi \rightarrow \xi''$  be an  $f_0$ -epimorphism and  $L$  a Lagrangian on  $\xi$  which is non-degenerate on  $Vf$ .*

*Then  $L$  has a locally  $f$ -projectable metric on  $\xi''$  iff it is a local Lagrange submersion.*

Let  $L : TM \rightarrow \mathbb{R}$  be a regular Lagrangian on  $M$  and  $f : M \rightarrow M''$  be a surjective submersion. Notice that all the definitions and the results stated above for vector bundles apply in this case, for example:

$f$  is a *Lagrange submersion* if  $f_*$  is a Lagrange submersion;

A Lagrangian  $L$  on  $M$  is  $f$ -projectable if it is  $f_*$ -projectable.

The case when  $f$  is *flat* (i.e. the horizontal distribution  $Hf_*$  is integrable) is need to Lagrange foliations (see [4]).

A theory of Lagrange submersion, in an analogous way of Riemannian submersion, will be done elsewhere.

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University of Craiova  
Department of Mathematics  
11, Al.I. Cuza St., Craiova, 1100, Romania  
e-mail:paul@udjmatj2.sfos.ro