

On (k, f, l) -Chordal Polygons

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*Dedicated to Prof. Dr. Constantin UDRIȘTE
on the occasion of his sixtieth birthday*

Abstract

The article generalizes the concept of the k -chordal polygons and the (k, λ, l) -chordal polygons as introduced in papers [3], [4] respectively. Namely, here we describe the positive real function f where $f(a_1), \dots, f(a_n)$ are the side lengths of a l -chordal polygon and in the same time a_1, \dots, a_n are the lengths of the sides of a k -chordal polygon.

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1 Preliminary definitions and results

This article is an addendum to the papers [3], [4], where the concepts of the k -chordal and its generalized variant the (k, λ, l) -chordal polygons are introduced and discussed. The (k, λ, l) property means the following. Let a_1, \dots, a_n be the lengths of the sides of a k -chordal polygon \underline{A} , and $a_1^\lambda, \dots, a_n^\lambda, \lambda \in \mathbb{R}_+$ are the side lengths of a l -chordal polygon, then \underline{A} is called (k, λ, l) . The properties of these general geometrical objects are exposed in detail in [4]. Here we are interested in the following natural generalization of the main results of the paper [4].

Our main goal is now to describe a positive, real function f so that it maps the side lengths a_1, \dots, a_n of a k -chordal polygon into $f(a_1), \dots, f(a_n)$, being these the lengths of the sides of a l -chordal polygon.

For convenience we first repeat briefly some definitions and the results of article [3].

Definition 1.1. Let $\underline{A} = A_1 A_2 \dots A_n$ be a chordal polygon and let $\mathcal{C}_{\underline{A}}^n$ be its circumcircle. By S_{A_i} and \widehat{S}_{A_i} we denote the semicircles so that

$$S_{A_i} \cup \widehat{S}_{A_i} = \mathcal{C}_{\underline{A}}^n, \quad A_i \in S_{A_i} \cap \widehat{S}_{A_i}.$$

The polygon \underline{A} is said to be of the first kind if it is fulfilled: (i) not all vertices A_1, A_2, \dots, A_n lie on the same semicircle; (ii) for every three consecutive vertices A_i, A_{i+1}, A_{i+2} it is valid that $A_i \in S_{A_{i+1}} \implies A_{i+2} \in \widehat{S}_{A_i}$; (iii) any two consecutive vertices A_i, A_{i+1} do not lie on the same diameter.

Definition 1.2. Let $\underline{A} = A_1A_2 \cdots A_n$ be a chordal polygon and let k be a positive integer. The polygon \underline{A} is said to be k -inscribed and called k -chordal polygon if it is of the first kind and if $\sum_{i=1}^n \angle A_i C A_{i+1} = 2k\pi$, where C denotes the centre of the circumscribed circle to the polygon \underline{A} .

In the sequel we will write β_i for $\angle C A_i A_{i+1}$ according to the notations introduced in [3],[4]. It follows that for a k -chordal polygon \underline{A} it results that

$$(1) \quad \sum_{i=1}^n \beta_i = (n - 2k)\pi.$$

The result that we need frequently throughout the paper and concerning the sum of the side lengths in the k -chordal polygons, is: *If a_1, a_2, \dots, a_n are the side lengths of the k -chordal polygon \underline{A} , then*

$$(2) \quad \sum_{i=1}^n a_i > 2ka^*,$$

where $a^* = \max_{1 \leq j \leq n} a_j$, cf. [3], Corrolary 1.2.

Finally we are ready to introduce the concept of the (k, f, l) -chordal polygons.

Definition 1.3. When \underline{A} is k -chordal polygon with the sides of lengths a_1, \dots, a_n and in the same time $f(a_1), \dots, f(a_n)$ are the side lengths of a l -chordal polygon, then \underline{A} is called (k, f, l) -chordal polygon. The polygon that consists of the sides of the lengths $f(a_i)$, we denote as $f(\underline{A})$.

We also need the definition of the *convexity* and the definition of the *star-likeness with respect to the origin*. According to the references [1], [2] we give these definitions.

Definition 1.4. The real function $f : [a, b] \rightarrow R$ is said to be convex on $[a, b]$ when for all $x, y \in [a, b]$ and all $t \in [0, 1]$ we have $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$.

Definition 1.5. The real function $f : [0, b] \rightarrow R$; $f(0) = 0$, that satisfies $f(tx) \leq tf(x)$ for all $x \in [0, b]$ and $t \in [0, 1]$ is called star-like with respect to the origin.

In the following section we give our principal results. Namely, we are focusing to the sufficient conditions for the existence of the (k, f, l) -chordal polygons and we are looking for some sufficient conditions upon the lengths $f(a_i); i = \overline{1, n}$ that ensures the existence of the chordal polygons \underline{A} and $f(\underline{A})$.

2 Main results

Now, according to the already given definitions we are ready to give our principal results concerning the existence of the introduced (k, f, l) -chordal polygons. At first we give the sufficient conditions for the existence of this geometrical object, when we possess the information that \underline{A} is k -chordal.

Theorem 2.1. Let \underline{A} be a k -chordal polygon with the sides of the lengths a_1, \dots, a_n where $k \in \mathbb{N}$; $k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Then $f(a_1), \dots, f(a_n)$ are the side lengths of a l -chordal polygon, i.e. \underline{A} is (k, f, l) -chordal if

1. $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is convex and monotonously nondecreasing,
- 2.

$$(3) \quad \arcsin \frac{f\left(\frac{2k}{n}a^*\right)}{f(a^*)} > \frac{l\pi}{n}.$$

Proof. It is clear that if we prove the existence of some angles β_i , $i = \overline{1, n}$ that satisfies

$$(4) \quad \beta_1 + \beta_2 + \dots + \beta_n = (n - 2l)\frac{\pi}{2}$$

$$(5) \quad \frac{\cos \beta_1}{f(a_1)} = \dots = \frac{\cos \beta_n}{f(a_n)},$$

then $f(\underline{A})$ is obviously l -chordal, i.e. \underline{A} is (k, f, l) -chordal.

For this purpose consider

$$L := \sum_{i=1}^n \arcsin \frac{f(a_i)}{f(a^*)}.$$

As the main part of $\arcsin x$ is convex when $x \in (0, 1)$, and f is convex as well, by the assumptions of the Theorem, it follows that

$$(6) \quad \begin{aligned} L &\geq n \arcsin \left\{ \frac{1}{nf(a^*)} \sum_{i=1}^n f(a_i) \right\} \geq n \arcsin \frac{f\left(\frac{1}{n} \sum_{i=1}^n a_i\right)}{f(a^*)} \\ &> n \arcsin \frac{f\left(\frac{2k}{n}a^*\right)}{f(a^*)} > \frac{l\pi}{n}. \end{aligned}$$

Here we use the fact that \underline{A} is k -chordal, that means it holds (2), and the arcsine functions are always well-defined, since f is nondecreasing. Therefore (6) ensures the existence of the unique $\theta \in (0, 1)$ so that

$$(7) \quad \sum_{i=1}^n \arcsin \left\{ \frac{f(a_i)}{f(a^*)} \theta \right\} = l\pi.$$

But this is equivalent to

$$\sum_{i=1}^n \arccos \left\{ \frac{f(a_i)}{f(a^*)} \theta \right\} = (n - 2l)\frac{\pi}{2}.$$

There is also an unique $\beta^* \in (0, \frac{\pi}{2})$ corresponding to the maximal side length $f(a^*)$ that $\theta = \cos \beta^*$. Consecutively putting

$$\beta_i = \arccos \left\{ \frac{f(a_i)}{f(a^*)} \cos \beta^* \right\}, \quad i = \overline{1, n},$$

we get the angles β_i of a l -chordal polygon the sides of which are of the lengths $f(a_i)$, $i = \overline{1, n}$ so that (4) and (5) are valid. Thus, the proof is complete. **Q.E.D.**

In the continuation we formulate a modest improvement of the previous Theorem, but with more weaker assumptions. Namely, we have not assumed the k -chordality of the initial polygon \underline{A} , but the price of this fact has to be paid by some additional conditions upon f .

Theorem 2.2. *Let a_1, \dots, a_n be any given lengths with the property (2), i.e. so that $a_1 + \dots + a_n > 2ka^*$. Then \underline{A} is (j, f, l) -chordal, $j = \overline{1, k}$ if*

1. $f : R_+ \rightarrow R_+$, is convex, monotonously increasing,
2. $f(0) = 0$,
- 3.

$$(8) \quad \arcsin \frac{f\left(\frac{2k}{n}a^*\right)}{f(a^*)} > \frac{l\pi}{n}.$$

Proof. The first step in proving procedure is completely the same as in the previous Theorem. Therefore $f(\underline{A})$ is clearly l -chordal. Now it remains to prove that \underline{A} is k -chordal under the assumed conditions. For this purpose let us consider (8). This inequality is equivalent to

$$(9) \quad f\left(\frac{2k}{n}a^*\right) > f(a^*) \sin \frac{l\pi}{n}.$$

It is not hard to see that f is star-like with respect to the origin, because it is convex and $f(0) = 0$ ¹. Therefore it is

$$(10) \quad f\left(\frac{2k}{n}a^*\right) > f(a^*) \sin \frac{l\pi}{n} \geq f\left(a^* \sin \frac{l\pi}{n}\right).$$

Furthermore, since f is bijective too, i.e. f is convex on a compact and it is increasing, by (10) we conclude that

$$(11) \quad \arcsin \frac{2k}{n} > \frac{l\pi}{n}.$$

Now, repeating the procedure for finding the exact angles β_i so that (1) holds and

$$\frac{\cos \beta_i}{a_i} = \frac{\cos \beta_j}{a_j}; \quad 1 \leq i < j \leq n,$$

¹Speaking in the convexity hierarchy terminology, we assume that $f \in K_1(1)$. The $K_3(1)$ class consists of all functions being star-like with respect to the origin. So, as $K_1(1) \subset K_3(1)$ consult e.g. [2], the star-likeness of f is obvious.

exposed in the proof of the Theorem 2.1., we finish the proof. **Q.E.D.**

Finally, we give a result in some way reverse to the Theorem 2.1. Namely, we are looking for the broadest possible class of functions so that a condition like (2) e.g.

$$a_1 + \dots + a_n > \gamma(k, n)a^*$$

suffices for the existence of the k -chordal polygon with the side lengths $f(a_i)$, $i = \overline{1, n}$. The existence of such constant $\gamma(k, n)$ is shown in [4], and it has to be $\gamma(k, n) \geq n \sin \frac{k\pi}{n}$, cf. [4], Theorem 3.4.

Theorem 2.3. *Let a_1, a_2, \dots, a_n be any given lengths so that*

$$(12) \quad f(a_1) + f(a_2) + \dots + f(a_n) > \gamma(k, n)f(a^*),$$

where $k \in N$ and $k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Assume $\gamma(k, n) \geq n \sin \frac{k\pi}{n}$. If the positive increasing function f is star-like with respect to the origin, then there is a (k, f, k) -chordal polygon whose sides have lengths a_1, a_2, \dots, a_n .

Proof. By the convexity of the arcsine function we conclude:

$$(13) \quad \sum_{i=1}^n \arcsin \frac{f(a_i)}{f(a^*)} \geq n \arcsin \left\{ \frac{1}{nf(a^*)} \sum_{i=1}^n f(a_i) \right\} \geq n \arcsin \frac{\gamma(k, n)}{n} \geq k\pi.$$

Now, it follows that there is the unique $\theta_1 \in (0, 1)$, so that by (13) we get

$$\sum_{i=1}^n \arcsin \left\{ \frac{f(a_i)}{f(a^*)} \theta_1 \right\} = k\pi;$$

thus $f(\underline{A})$ is k -chordal.

In the same time it is

$$(14) \quad f(a_i) = f\left(\frac{a_i}{a^*} \cdot a^*\right) \leq \frac{a_i}{a^*} f(a^*),$$

henceforth f is star-like with respect to the origin. That means

$$(15) \quad \begin{aligned} \gamma(k, n) &< \frac{f(a_1) + \dots + f(a_n)}{f(a^*)} \\ &\leq \frac{\frac{a_1}{a^*} f(a^*) + \dots + \frac{a_n}{a^*} f(a^*)}{f(a^*)} \\ &= \sum_{i=1}^n \frac{a_i}{a^*} < \sum_{i=1}^n \arcsin \frac{a_i}{a^*}. \end{aligned}$$

Finally the k -chordality of \underline{A} immediately follows from (15), or, what is the same, \underline{A} is (k, f, k) -chordal. **Q.E.D.**

Remark 2.1 Taking $f(x) = x^\lambda$ the Theorems 2.1., 2.2. and 2.3. give the main results of the articles [3], [4] respectively as $\lambda = 1$ and $\lambda \geq 1$.

References

- [1] D.S. Mitrinović, *Analytic Inequalities*, Građevinska knjiga, Belgrade, 1970 (in Serbian)
- [2] D.S. Mitrinović and E.L. Pečarić, *Means in Mathematics*, Naučna knjiga, Belgrade, 1989 (in Serbian)
- [3] M. Radić, *Some inequalities and properties concerning chordal polygons*, Math. Inequal. Appl. **2**, No.1.(1998), pp. 141-150
- [4] M. Radić and T. Pogány, *Some inequalities concerning the existence of (k, λ, l) -chordal polygons*, (submitted).

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