Dynamical Systems and Lagrangian Spaces

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

For the general case of the dynamical systems which are not necessarily Lagrangian, a geometrical method for the determination of a second order Lagrangian (linear with respect to accelerations) is described.

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1 Introduction

Several physical theories use second order Lagrangians. Generally speaking any geometric formulation of the higher order Lagrangian systems is developed in the framework of higher order tangent bundles by using their underlying canonical structures.

In this Note we discuss the dynamical systems on manifolds locally expressed by second order differential equations of the form

$$\frac{d^2x^i}{dt^2} = S^i\left(x, \frac{dx}{dt}\right), \quad i = \overline{1,n}. \tag{1}$$

It is known that for such a dynamical system there exists a Lagrangian, which is linear with respect to accelerations [6], [7] that is

$$L(x, \dot{x}, \ddot{x}) = A_i(x, \dot{x})\ddot{x}^i + B(x, \dot{x}). \tag{2}$$

Our aim is to present a geometrical manner to obtain a Lagrangian for (1). In order to do this we use the following arguments: 1) The existence of an infinite Riemannian structures on the tangent manifold of a Lagrange space (in sense of [3]); 2) For a first order dynamical system,

$$\frac{dx^i}{dt} = f^i(x), \quad i = \overline{1,n}, \tag{3}$$

on a Riemannian space (M, g), there exists a nondegenerate Lagrangian of the form [5], [8]

$$L(x, \dot{x}) = \frac{1}{2} g_{ij}(x)(\dot{x}^i - f^i)(\dot{x}^j - f^j),. \tag{4}$$

Let us remark that the result is still valid if g is an indefinite Riemannian metric on M.

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2 Geometrical structures associated to sprays

Let M be an n-dimensional differentiable manifold. A spray on M is a vector field S on the tangent manifold TM such that

$$JS = C,$$

where $C \in \mathcal{X}(TM)$ is the canonical vector field on TM and $J \in \mathcal{X}_1^1(TM)$ is the natural almost tangent structure on TM. Locally,

$$J = dx^i \otimes \frac{\partial}{\partial y^j}, \quad C = y^i \frac{\partial}{\partial y^i}$$

and

$$S = y^i \frac{\partial}{\partial x^i} + S^i \frac{\partial}{\partial y^i}, \tag{5}$$

where (x^i, y^i) are the local coordinates on TM and $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \right\}$ is the natural local basis.

To give a second order system (1) on M is equivalent to give a spray S on M, and the trajectories of S are the solutions of (1).

The geometry induced by S follows from the existence of the nonlinear connection N associated to S. Locally N is determined by the coefficients

$$N_j^i = -\frac{1}{2} \frac{\partial S^i}{\partial y^j}. \tag{6}$$

The local basis adapted to the horizontal distribution HTM of N and to the vertical distribution VTM on TM is given by

$$\left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i} \right\} \tag{7}$$

and the dual basis of (7) is

$$\{dx^i, \delta y^i = dy^i + N_j^i dx^j\}. \tag{8}$$

A pair (M, L) is called a *Lagrange space* if $L \in C^\infty(TM)$ is a nondegenerate or regular Lagrangian, that is $\det \left\| \frac{\partial^2 L}{\partial y^i \partial y^j} \right\| \neq 0$. ([4]). On a Lagrange space (M, L) there is a canonical spray (the Lagrangian spray associated to L) locally given by (5) with

$$S^i = g^{ij} \left(\frac{\partial L}{\partial x^j} - \frac{\partial^2 L}{\partial x^k \partial y^j} y^k \right); \tag{9}$$

g^{ij} are the components of the inverse matrix of $\|g_{ij}\| = \left\| \frac{\partial^2 L}{\partial y^i \partial y^j} \right\|$.

A tensor field g of type $(0, 2)$ on TM is called a *d-tensor field* of type $(0, 2)$ on M if the components $g_{ij}(x, y)$ of g change by the relations

$$g_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} g_{ij}, \tag{10}$$

with respect to a local coordinate transformation $x^i = x^i(x^{1'}, \ldots, x^{n'})$ on M, i.e., g is a vertical tensor field of type $(0,2)$ on M or $g = g_{ij}(x,y)dx^i \otimes dx^j$.

A pair (M,g) where g is a nondegenerate symmetric d -tensor field of type $(0,2)$ on M is called a *generalized Lagrange space* with the *generalized Lagrange metric* g ([4]). A Riemannian metric, a Finslerian metric and the fundamental tensor $g = \frac{\partial^2 L}{\partial y^i \partial y^j} dx^i \otimes dx^j$ of a Lagrange space supply such generalized Lagrange metrics.

3 Determination of Lagrangians for a dynamical system

In the sequel we shall present a geometrical method for the determination of a Lagrangian of the form (2) in the general case of the dynamical systems which are not necessarily Lagrangian in classical sense.

Proposition 1. *Any dynamical system of the form* (1) *defined on a generalized Lagrange space admits a Lagrangian of the form* (2) *such that the corresponding Euler-Lagrange equations were linear combinations of the equations* (1) *and their derivatives.*

Proof. Let (M,g) be a generalized Lagrange space and let $g_{ij}(x,y)$ be the components of g in a local chart. Also let S be the spray which corresponds to the system (1). S is of the form (5), where the components $S^i(x,y)$ are the right-hand sides of (1).

Lemma. (M,g,S) *defines a canonical indefinite Riemannian metric on* TM,

$$G = 2g_{ij}dx^i \otimes \delta y^j, \tag{11}$$

where $\{\delta y^i, dx^i\}$ *is the co-basis adapted to the nonlinear connection* N *associated to* S *via* (6).

Proof. G is globally defined on TM and nonsingular. The quadratic form defined by G is $d\sigma = 2g_{ih}N^h_j dx^i dx^j + 2g_{ij}dx^i dy^j$ and the matrix of the components of G is

$$\|G_{ab}\| = \left\| \begin{matrix} g_{ih}N^h_j + g_{jh}N^h_i & g_{ij} \\ g_{ij} & 0 \end{matrix} \right\|. \tag{12}$$

G may be considered as the horizontal lift of g with respect to N.

Going back to our problem let us consider a first order system on TM associated to (1), namely

$$\frac{dy^i}{dt} = S^i(x,y), \quad \frac{dx^i}{dt} = y^i, \quad i = \overline{1,n}, \tag{13}$$

or, in a homogeneous writing,

$$\dot{z}^a = F^a(z), \quad a = \overline{1,2n}, \tag{14}$$

where $(z^a) = (x^i, y^i)$, $F^i = S^i$, $F^{n+i} = y^i$, $i = \overline{1,n}$. Obviously the solutions of (1) are the projections of the solutions of (13). Using the structure of TM and the result from [5] we obtain for the system (14) a Lagrangian $L : TTM \to R$ of the form

$$L(z,\dot{z}) = \frac{1}{2}G_{ab}(z)(\dot{z}^a - F^a)(\dot{z}^b - F^b), \quad a,b = \overline{1,2n}, \tag{15}$$

where G_{ab} are given by (12). Thanks to the special form of the matrix $\|G_{ab}\|$, the expression of L in the initial variables is (2). (TM, L) is a Lagrange space whose geodesics are combinations of the solutions of (13), respectively the Euler-Lagrange equations associated to L are linear combinations of the equations (1) and their derivatives.

Let (M, L) be an n -dimensional Lagrange space and let S be the canonical Lagrangian spray associated to L. S is determined by the relations (9) and the corresponding second order differential equations (1) coincide with the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad i = \overline{1,n}, \tag{16}$$

where $y^i = \dfrac{dx^i}{dt} = \dot{x}^i$. The trajectories of S, respectively the solutions of (1) are the geodesics of (M, L). Finally, the tangent manifols TM of the Lagrange space (M, L) is endowed with a canonical indefinite Riemannian metric. This metric is defined by $\left(M, \dfrac{\partial^2 L}{\partial y^i \partial y^j} dx^i \otimes dx^j, S\right)$ via Lemma.

Example. The Morse-Feshbach equations

$$\begin{cases} \ddot{x}^1 + b\dot{x}^1 + kx^1 = 0, \\ \ddot{x}^2 - b\dot{x}^2 + kx^2 = 0, \quad (b, k \in R,\ k > 0) \end{cases}$$

on a 2-dimensional manifold M_2, supply a Lagrangian spray

$$S = y^1\frac{\partial}{\partial x^1} + y^2\frac{\partial}{\partial x^2} - (by^1 + kx^1)\frac{\partial}{\partial y^1} + (by^2 - kx^2)\frac{\partial}{\partial y^2}$$

on M_2. The Lagrangian which generates S is $L : TM \to R$, $L(x,y) = y^1y^2 + \frac{b}{2}(x^1y^2 - x^2y^1) - kx^1x^2$. The fundamental tensor of (M, L) is $g = 2dx^1 \otimes dx^2$ and the metric on TM has the components

$$\|G_{ab}\| = \left\| \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \right\|.$$

Now we consider the most general case when a second order system of the form (1) is given on a n-dimensional manifold M. In general the solutions of such a system satisfy so called Euler-Lagrange equations of type II,

$$\frac{d}{dt}\left(\frac{\partial L_0}{\partial y^i}\right) - \frac{\partial L_0}{\partial x^i} = Q_i(x,y), \quad y^i = \frac{dx^i}{dt},\ i = \overline{1,n}, \tag{17}$$

where $L_0 \in C^\infty(TM)$ is a nondegenerate first order Lagrangian and $Q_i \in C^\infty(TM)$, $i = \overline{1,n}$.

Proposition 2. *Any dynamical system of the form* (1) *on* M *satisfying the above condition admits a Lagrangian of the form* (2) *such that the corresponding Euler-Lagrange equations were linear combinations of the equations* (1) *and their derivatives.*
Proof. Under the supposed conditions, the spray S defined by (1) on M admits a decomposition $S = S_0 + S'$, where S_0 is the Lagrangian spray associated to L_0. (M, L_0) is a Lagrange space and $g = \frac{\partial L_0}{\partial y^i \partial y^j} dx^i \otimes dx^j$, $i, j = \overline{1, n}$, defines the fundamental tensor field of (M, L_0). According to Lemma there is an indefinite Riemannian metric G_0 on TM, determined by g and the nonlinear connection associated to S_0, therefore intrinsically jointed to S. From now on the proof is similar to that of Proposition 1.
Example. The spray defined on a 2-dimensional manifold M_2 by the equations

$$\left\{ \begin{array}{l} \ddot{x}^1 + \varepsilon x^1 = 0 \\ \ddot{x}^2 + \varepsilon \dot{x}^1 = 0, \quad \varepsilon \in \{-1, 1\}, \end{array} \right. \tag{18}$$

is not variational in classical sense. For $\varepsilon = -1$, (18) represents the Whittaker equations and for $\varepsilon = 1$ we obtain the Urrutia-Hojmann equations. Both systems can be studied with the same Lagrangian spray if (18) is expressed like that

$$\left\{ \begin{array}{l} \ddot{x}^1 + x^1 = (1 - \varepsilon) x^1, \\ \ddot{x}^2 + x^2 = x^2 - \varepsilon \dot{x}^1. \end{array} \right. \tag{19}$$

The spray defined by the system

$$\left\{ \begin{array}{l} \ddot{x}^1 + x^1 = 0, \\ \ddot{x}^2 + x^2 = 0, \end{array} \right. \tag{20}$$

is variational $\left(S_0 = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial y^1} - x^2 \frac{\partial}{\partial y^2} \right)$ and the coresponding Lagrangian of (20) is $L_0 = y^1 y^2 - x^1 x^2$. For example if $\varepsilon = -1$ the metric of TM_2 has the nonzero components $G_{14} = G_{41} = G_{23} = G_{32} = \frac{1}{2}$. The first order system associated to (19) is $\dot{y}^1 - x^1 = 0$, $\dot{y}^2 - y^1 = 0$, $\dot{x}^1 - y^1 = 0$, $\dot{x}^2 - y^2 = 0$ and the corresponding Lagrangian is $L = (\dot{x}^2 - y^2)\dot{y}^1 + (\dot{x}^1 - y^1)\dot{y}^2 + x^1 y^2 + (y^1)^2 - x^1 \dot{x}^2 - y^1 \dot{x}^1$.
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