The Laplace Spectrum and Hermitian Spaces

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

The spectrum $Spec^p\left(M^d, g\right)$ of eigenvalues of the Laplace operator on differential p-forms, $0 \leq p \leq d$, on compact Riemannian and Kähler manifolds $\left(M^d, g\right)$ determines the geometry to a considerable extent; though isospectral manifolds need not be isometric, even locally, and further the spectrum does not necessarily determine the topological invariants on a Riemannian manifold. This paper continues a study of the geometric consequences of isospectrality on different classes of almost Hermitian manifolds. We consider the implications of isospectrality on constant curvature and conformally flat Hermitian manifolds as well as on Bochner-flat spaces.

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1 Preliminaries

1.1 Almost Hermitian manifolds

Let (M, g, J) be an almost Hermitian manifold of real dimension $m = 2n$, with almost complex structure J on the tangent bundle $T(M)$, $J : T_p(M) \to T_p(M)$ for $p \in M$ with $J^2 = -id$, and Riemannian metric g such that J is an isometry, $g(JX, JY) = g(X, Y)$ for C^∞ vector fields X, Y on M. The Kähler form on M is then given by $\Omega(X, Y) = g(JX, Y)$.

An almost Hermitian manifold is almost Kähler if the differential form Ω is closed, that is, $d\Omega = 0$ and hence, Ω is harmonic since the codifferential $\delta\Omega = 0$ here as well. While an almost Hermitian manifold is nearly Kähler if Ω is a Killing (2-)form since $d\Omega = 3\nabla\Omega$ where ∇ is the covariant derivative with respect to the Riemannian connection on M. An almost Hermitian manifold is Kähler if $\nabla\Omega = 0$, and if M is Hermitian it suffices that $d\Omega = 0$. The Hermitian semi-Kähler manifolds [7] are complex Hermitian spaces on which the Kähler form is co-closed, $\delta\Omega = 0$. Within this class are the complex parallelizable spaces (that is, there exist on M, n holomorphic vector fields everywhere linearly independent), which are flat; and when compact are Kähler if and only if they are complex tori [12]. Three important classes of almost

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Hermitian manifolds (M,g,J) defined by identities on the Riemann curvature tensor are:

$AH_1 : R(X,Y,Z,W) = R(X,Y,JZ,JW)$,

$AH_2 : R(X,Y,Z,W) = R(JX,JY,Z,W)+R(JX,Y,JZ,W)+R(JX,Y,Z,JW)$,

$AH_3 : R(X,Y,Z,W) = R(JX,JY,JZ,JW)$.

While the Ricci curvature tensor is defined by

$$Ric(X_p,Y_p) = \sum_{i=1}^{m} R(e_i,X_p,Y_p,e_i)$$

for $p \in M$, the Ricci*-tensor is given by

$$Ric^*(X_p,Y_p) = \sum_{i=1}^{m} R(e_i,X_p,JY_p,Je_i),$$

where $\{e_i \mid i = 1...m\}$ is an arbitrary orthonormal basis of $T_p(M)$. Further, $Ric = Ric^*$ on an AH_1 space. The scalar curvature is defined by $\rho = \sum_{i=1}^{m} Ric(e_i,e_i)$ and the *-scalar curvature is $\rho^* = \sum_{i=1}^{m} Ric^*(e_i,e_i)$. On a Hermitian semi-Kähler manifold $\rho = \rho^*$.

1.2 Spaces of constant curvature, the Weyl conformal curvature tensor and conformally-flat spaces, and the Bochner curvature tensor and Bochner-flat spaces

The Weyl conformal curvature tensor $C = C(X,Y,Z,W)$ on a Riemannian manifold (M,g) of dimension $m \geq 3$ is defined by

$$\begin{aligned} C(X,Y,Z,W) &= R(X,Y,Z,W) - \\ &- \frac{1}{m-2}\left[g(X,W)Ric(Y,Z) - g(X,Z)Ric(Y,W) + \right. \\ &\left. + g(Y,Z)Ric(X,W) - g(Y,W)Ric(X,Z)\right] \\ &+ \frac{\rho}{(m-1)(m-2)}\left[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\right]. \end{aligned} \tag{1.2.1}$$

When $m \geq 4$ the manifold is conformally flat, that is, conformal to a flat metric, if and only if $C = 0$. A Riemannian manifold of constant curvature is conformally flat and Einstein, and conversely.

Since an almost Hermitian manifold of constant curvature is AH_3 [6], we have

Corollary 1. *A conformally flat almost Hermitian Einstein manifold is* AH_3.

The Bochner curvature tensor $B = B(X,Y,Z,W)$ is given by

(1.2.2)

$$B_{hijk} = R_{hijk} - \frac{1}{2n+4}\left(R_{ij}g_{hk} - R_{ik}g_{hj} + R_{hk}g_{ij} - R_{hj}g_{ik} + F_{ij}F_h^r R_{rk} - \right.$$
$$\left. - F_{ik}F_h^r R_{rj} + R_{rj}F_{hk}F_i^r - R_{rk}F_{hj}F_i^r - 2R_{ri}F_{jk}F_h^r - 2R_{rk}F_{hi}F_j^r\right) +$$
$$+ \frac{\rho}{(2n+2)(2n+4)}\left(g_{ij}g_{hk} - g_{ik}g_{hj} + F_{ij}F_{hk} - F_{ik}F_{hj} - 2F_{hi}F_{jk}\right).$$

and the manifold is said to be Bochner-flat if $B = 0$. Since on an AH_3 or Einstein space the Ricci tensor is J-invariant, that is, $Ric(JX, JY) = Ric(X, Y)$ (or in local coordinates $R_{ij} = R_{hk}F_i^h F_j^k$), a direct calculation gives the following

Lemma 2. *A Bochner-flat manifold that is either Einstein or AH_3 is AH_1.*

Lemma 3. *If (M, g, J) is an almost Hermitian Einstein manifold of dimension $m = 2n \geq 4$ with $B = 0$ and $\rho \neq 0$, then the manifold is Kähler with constant holomorphic curvature*

$$h = \frac{\rho}{n(n+1)}, \tag{1.2.3}$$

and conversely,

Lemma 4. *A Kähler manifold with constant holomorphic curvature is Einstein and Bochner-flat.*

Lemma 5. *On an almost Hermitian manifold with $Ric = Ric^*$, the square-length of the Bochner tensor is given by [3]*

$$\|B\|^2 = \|R\|^2 - \frac{8}{n+2}\|Ric\|^2 + \frac{2\rho^2}{(n+1)(n+2)}, \tag{1.2.4}$$

while on an almost Hermitian Einstein manifold [4]

$$\|B\|^2 = \|R\|^2 + \frac{\rho^2 - 3\rho\rho^*}{n(n+1)}, \tag{1.2.5}$$

and since on an almost Hermitian manifold with constant holomorphic sectional curvature h, $\rho + 3\rho^* = 4n(n+1)h$, then for the Bochner tensor on an almost Hermitian Einstein manifold with constant holomorphic sectional curvature h [5],

$$\|B\|^2 = \|R\|^2 + \frac{2\rho^2}{n(n+1)} - 4\rho h. \tag{1.2.6}$$

1.3 The Laplace spectrum

On a compact connected C^∞ Riemannian manifold (M, g) of dimension $m = 2n$, the Laplace operator on the space of differential p-forms $C^\infty(\Lambda^p(M))$, $0 \leq p \leq m$, is defined by $\Delta = d\delta + \delta d$, where δ is the adjoint of d with respect to the metric g and is defined by the Hodge star operator. The spectrum of the Laplacian, $Spec^p(M, g) = \{\lambda_{i,p} \mid 0 \leq \lambda_{1,p} \leq \lambda_{2,p} \leq ... \leq \lambda_{k,p} \leq ... \nearrow \infty\}$ is the set of eigenvalues $\lambda_{i,p}$ of Δ, thus satisfying $\Delta\omega_p = \lambda_{i,p}\,\omega_p$, $\omega_p \in C^\infty(\Lambda^p(M))$, where the eigenvalues are written with multiplicities. The eigenspace of $\lambda_{1,0} = 0$ is the harmonic functions on M.

Essential to the results here on the spectral geometry of almost Hermitian manifolds is the Minakshisundaram-Pleijel-Gaffney asymptotic formula for the trace of the heat operator or heat kernel determined by the spectrum given by

(1.3.1)
$$\sum_{i=1}^{\infty} \exp\left(\lambda_{i,p}\, t\right) \underset{t\to 0^+}{\widetilde{\quad}} \frac{1}{(4\pi t)^{\frac{m}{2}}} \sum_{k=0}^{\infty} a_{k,p}\, t^k,$$

where, for $p = 0, 1$, and 2 respectively, the first three coefficients are given by [11]:

(1.3.2)
$$p=0: \begin{cases} a_{0,0} = \int\limits_M dM = vol\,(M)\,, \\ a_{1,0} = \frac{1}{6}\int\limits_M \rho\, dM, \\ a_{2,0} = \frac{1}{360}\int\limits_M \left[5\rho^2 - 2\left\|Ric\right\|^2 + 2\left\|R\right\|^2\right] dM; \end{cases}$$

(1.3.3)
$$p=1: \begin{cases} a_{0,1} = 2n\int\limits_M dM = 2n vol\,(M)\,, \\ a_{1,1} = \frac{n-3}{3}\int\limits_M \rho\, dM, \\ a_{2,1} = \frac{1}{180}\int\limits_M \left[(5n-30)\,\rho^2 + (-2n+90)\left\|Ric\right\|^2 + (2n-15)\left\|R\right\|^2\right] dM; \end{cases}$$

(1.3.4)
$$p=2: \begin{cases} a_{0,2} = \left(2n^2-n\right)\int\limits_M dM = \left(2n^2-n\right) vol\,(M)\,, \\ a_{1,2} = \frac{2n^2-13n+12}{6}\int\limits_M \rho\, dM, \\ a_{2,2} = \frac{1}{360}\int\limits_M \left[\left(10n^2-125n+300\right)\rho^2 + \left(-4n^2+362n-1080\right)\left\|Ric\right\|^2 \right. \\ \left. + \left(4n^2-62n+240\right)\left\|R\right\|^2\right] dM. \end{cases}$$

2 Results on isospectrality

2.1 Bochner-flat manifolds and the spectrum

We shall concern ourselves here with the following spectral results on almost Hermitian Bochner-flat manifolds.

Theorem 6. *If (M, g, J) is a complex space form of constant holomorphic curvature $h \neq 0$ and $\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$ is an AH_3 Bochner-flat manifold with nonzero scalar curvature $\tilde{\rho}$ and $Spec^p\,(M, g, J) = Spec^p\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$, then $\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$ is a Kähler manifold and*

$\tilde{h} = h$ *for* $4 \leq m \leq 10$ *when* $p = 0$, *for* $10 \leq m \leq 102$ *when* $p = 1$, *and for* $m = 6, 8$ *and* $14 \leq m \leq 188$ *when* $p = 2$. *If, further,* $\tilde{\rho}$ *is constant then the results hold for* $m \neq 12$ *when* $p = 0$, $m \neq 6$ *when* $p = 1$ *and,* $m \geq 4$ *when* $p = 2$.

Proof. [3] This result follows immediately since $B = \tilde{B} = 0$ and $\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$ is AH_1.

Often the results in spectral geometry (e.g., the previous result) assume pairs of isospectral manifolds from two classes - one a subset of the other - such as a Riemannian space and a real space form, or two Kähler spaces one a complex space form, and then it is proved that isospectrality characterizes up to isometry the smaller class within the larger. The spectral methods may be applied to classes with no such relation. Here it may be proved that if the manifolds are isospectral then they are isometric and lie in a smaller class than that of either space without the isospectrality.

Theorem 7. *If* (M, g, J) *is Hermitian semi-Kähler Einstein and* $\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$ *is* AH_3 *Bochner-flat with constant scalar curvature* $\tilde{\rho} \neq 0$ *and*

$$Spec^p (M, g, J) = Spec^p \left(\tilde{M}, \tilde{g}, \tilde{J}\right)$$

for $p = 0$, 1 *or* 2, *then* (M, g, J) *and* $\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$ *are holomorphically isometric complex space forms for* $m \geq 14$ *when* $p = 0$, $m = 4$, $8 \leq m \leq 14$ *or* $m \geq 104$ *when* $p = 1$, *and* $m = 4$ *or* $m \geq 190$ *when* $p = 2$.

Proof. By Lemma 2, $\tilde{M}$ is AH_1. Further, $\rho = \tilde{\rho}$ from the first and second equations in each of (1.3.2)-(1.3.4). Further, from the third equations in each, and equations (1.2.4) and (1.2.5) in Lemma 5, we have

$$\int_M \left[\frac{\left(5n^2 + 4n + 3\right)\rho^2}{n(n+1)} + 2\left\|B\right\|^2\right] dM =$$

$$= \int_{\tilde{M}} \left[\frac{\left(5n^2 + 4n + 3\right)\tilde{\rho}^2}{n(n+1)} + \frac{-2n + 12}{n+2}\left\|\tilde{E}\right\|^2\right] d\tilde{M}, \tag{2.1.1}$$

$$\begin{aligned} &\int_M \left[\frac{\left(5n^3 - 26n^2 + 18n + 15\right)\rho^2}{n(n+1)} + (2n - 15)\left\|B\right\|^2\right] dM \\ &= \int_{\tilde{M}} \left[\frac{\left(5n^3 - 26n^2 + 18n + 15\right)\tilde{\rho}^2}{n(n+1)} + \frac{-2n^2 + 102n + 60}{n+2}\left\|\tilde{E}\right\|^2\right] d\tilde{M} \end{aligned} \tag{2.1.2}$$

(2.1.3)

$$\begin{aligned} &\int_M \left[\frac{\left(10n^4 - 117n^3 + 362n^2 - 183n - 60\right)\rho^2}{n(n+1)} + \left(4n^2 - 62n + 240\right)\left\|B\right\|^2\right] dM \\ &= \int_{\tilde{M}} \left[\frac{\left(10n^4 - 117n^3 + 362n^2 - 183n - 60\right)\tilde{\rho}^2}{n(n+1)}\right. \\ &\quad \left. + \frac{-4n^3 + 386n^2 - 852n - 240}{n+2}\left\|\tilde{E}\right\|^2\right] d\tilde{M} \end{aligned}$$

From, respectively, (2.1.1) in dimensions $m \geq 14$, (2.1.2) in dimensions $m = 4$, $8 \leq m \leq 14$ or $m \geq 104$, and (2.1.3) in dimensions $m = 4$ or $m \geq 190$, we have that $B = \tilde{E} = 0$, so that (M, g, J) and $\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$ are holomorphically isometric complex space forms.

2.2 Conformally-flat and constant curvature manifolds and the spectrum

A Kähler manifold (AH_1 is sufficient) of constant curvature is necessarily flat, while a constant curvature manifold that is either almost Kähler ([1], [8], [9], [10]), or nearly Kähler with $m \neq 6$ ([13]), is Kähler and flat. A similar result holds for still a different class of almost Hermitian manifolds. We prove this and then consider the spectral geometry of such spaces.

Lemma 8. *If (M, g, J) is Hermitian semi-Kählerian with constant curvature κ, then the manifold is flat and therefore H_1 and further, locally Kähler.*

Proof. We have that

$$\|R\|^2 = 4n(2n-1)\kappa^2$$

and

$$\rho = 2n(2n-1)\kappa, \quad \text{so } \|R\|^2 = \frac{\rho^2}{n(2n-1)}.$$

Further, by Lemma 5,

$$\|B\|^2 = \|R\|^2 + \frac{\rho^2 - 3\rho\rho^*}{n(n+1)}.$$

Since $\rho = \rho^*$, we obtain

$$\|B\|^2 = \frac{\rho^2}{n(2n-1)} - \frac{2\rho^2}{n(n+1)} = \frac{-3\rho^2(n-1)}{n(n+1)(2n-1)} \leq 0,$$

and therefore $B = 0$. Hence $\rho = \kappa = 0$ and the curvature tensor vanishes.

To complete the proof of the lemma we appeal to the following result [2]:

Theorem 9. *An even-dimensional Riemannian manifold with an analytic metric which is flat is locally Kählerian. Conversely, an even-dimensional Riemannian manifold with constant curvature is locally Kählerian only if the curvature is zero.*

Since a conformally flat Einstein space has constant curvature, we have that a conformally flat almost Kähler Einstein or nearly Kähler Einstein manifold is Kähler and flat, and

Corollary 10. *If (M, g, J) is Hermitian semi-Kähler Einstein and conformally flat, then the space is flat H_1 and locally Kähler.*

We turn to the spectral geometry of the spaces considered thus far.

Theorem 11. *If (M, g, J) is Hermitian with constant curvature κ, hence H_2 ([6] and [7]) and $\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$ is Hermitian semi-Kähler Einstein with $Spec^p(M, g, J) = Spec^p\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$ for $p = 0, 1$ or 2, then (M, g, J) and $\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$ are flat H_1 and locally Kähler for $m \geq 4$ when $p = 0$, $m \neq 6$ when $p = 1$, and $m \neq 16$ when $p = 2$.*

Proof. Since M has constant curvature κ,

$$\|R\|^2 = 4n(2n-1)\kappa^2 \text{ and } \rho = 2n(2n-1)\kappa,$$

so that

$$\|R\|^2 = \frac{\rho^2}{n(2n-1)}.$$

Further, M has constant holomorphic curvature $h = \kappa$, and (1.2.6) gives

$$\|B\|^2 = \|R\|^2 + \frac{2\rho^2}{n(n+1)} - 4\rho\kappa.$$

So

$$\|B\|^2 = \frac{\rho^2}{n(2n-1)} + \frac{2\rho^2}{n(n+1)} - \frac{2\rho^2}{n(2n-1)} = \frac{3(n-1)\rho^2}{n(n+1)(2n-1)}.$$

For the case $p = 0$ we consider, on the constant curvature space (M, g, J),
(2.2.1)

$$\begin{aligned} 5\rho^2 - 2\|Ric\|^2 + 2\|R\|^2 &= 5\rho^2 - \frac{\rho^2}{n} + \frac{2\rho^2}{n(2n-1)} \\ &= \frac{(10n^2 - 7n + 3)\rho^2}{n(2n-1)} \\ &= \frac{(5n^2 + 4n + 3)\rho^2}{n(n+1)} - 2\left(\frac{3(n-1)\rho^2}{n(n+1)(2n-1)}\right) \\ &= \frac{(5n^2 + 4n + 3)\rho^2}{n(n+1)} - 2\|B\|^2 . \end{aligned}$$

While for the Hermitian semi-Kähler Einstein space $\left(\tilde{M}, \tilde{g}, \tilde{J}\right)$ we have

(2.2.2) $$5\tilde{\rho}^2 - 2\left\|\tilde{Ric}\right\|^2 + 2\left\|\tilde{R}\right\|^2 = \frac{(5n^2 + 4n + 3)\tilde{\rho}^2}{n(n+1)} + 2\left\|\tilde{B}\right\|^2 .$$

Then from (1.3.2), (2.2.1) and (2.2.2)

(2.2.3) $$\begin{aligned} &\int_M \left[\frac{(5n^2 + 4n + 3)\rho^2}{n(n+1)} - 2\|B\|^2\right] dM = \\ &= \int_{\tilde{M}} \left[\frac{(5n^2 + 4n + 3)}{n(n+1)}\tilde{\rho}^2 + 2\left\|\tilde{B}\right\|^2\right] d\tilde{M}. \end{aligned}$$

The first and second equations of (1.3.2) give $\rho = \tilde{\rho}$ since they are constant and, consequently, (2.2.3) gives $B = \tilde{B} = 0$. Hence $\rho = \tilde{\rho} = 0$ and the curvature tensors R and $\tilde{R}$ vanish so the manifolds are flat H_1 and locally Kähler.

Similarly, for the case $p = 1$ we consider on M,

$$
\begin{aligned}
&(5n-30)\,\rho^2+(-2n+90)\,\|Ric\|^2+(2n-15)\,\|R\|^2\\
&=(5n-30)\,\rho^2+\frac{(-n+45)\,\rho^2}{n}+\frac{(2n-15)\,\rho^2}{n\,(2n-1)}\\
&=\frac{\left(10n^3-67n^2+123n-60\right)\rho^2}{n\,(2n-1)}\\
&=\frac{\left(5n^3-26n^2+18n+15\right)\rho^2}{n\,(n+1)}-(2n-15)\left(\frac{3\,(n-1)\,\rho^2}{n\,(n+1)\,(2n-1)}\right)\\
&=\frac{\left(5n^3-26n^2+18n+15\right)\rho^2}{n\,(n+1)}-(2n-15)\,\|B\|^2 .
\end{aligned}
\tag{2.2.4}
$$

While on $\tilde{M}$ we have

$$
\begin{aligned}
&(5n-30)\,\tilde{\rho}^2+(-2n+90)\left\|\tilde{R}ic\right\|^2+(2n-15)\left\|\tilde{R}\right\|^2\\
&=\frac{\left(5n^3-26n^2+18n+15\right)\tilde{\rho}^2}{n\,(n+1)}+(2n-15)\left\|\tilde{B}\right\|^2 .
\end{aligned}
\tag{2.2.5}
$$

Then from (1.3.3), (2.2.4) and (2.2.5)

$$
\begin{aligned}
&\int_M\left[\frac{\left(5n^3-26n^2+18n+15\right)\rho^2}{n\,(n+1)}-(2n-15)\,\|B\|^2\right]dM\\
&=\int_{\tilde{M}}\left[\frac{\left(5n^3-26n^2+18n+15\right)\tilde{\rho}^2}{n\,(n+1)}+(2n-15)\left\|\tilde{B}\right\|^2\right]d\tilde{M}.
\end{aligned}
\tag{2.2.6}
$$

The result then follows as in the previous case.

Lastly, for the case $p=2$ we consider on M,

(2.2.7)

$$
\begin{aligned}
&\left(10n^2-125n+300\right)\rho^2+\left(-4n^2+362n-1080\right)\|Ric\|^2+\\
&+\left(4n^2-62n+240\right)\|R\|^2=\left(10n^2-125n+300\right)\rho^2+\frac{\left(-2n^2+181n-540\right)\rho^2}{n}+\\
&+\frac{\left(4n^2-62n+240\right)\rho^2}{n\,(2n-1)}=\frac{\left(20n^4-264n^3+1093n^2-1623n+780\right)\rho^2}{n\,(2n-1)}\\
&=\frac{\left(10n^4-117n^3+362n^2-183n-60\right)\rho^2}{n\,(n+1)}-\left(4n^2-62n+240\right)\left(\frac{3\,(n-1)\,\rho^2}{n\,(n+1)\,(2n-1)}\right)\\
&=\frac{\left(10n^4-117n^3+362n^2-183n-60\right)\rho^2}{n\,(n+1)}-\left(4n^2-62n+240\right)\|B\|^2 .
\end{aligned}
$$

While on $\tilde{M}$ we have

$$
\begin{aligned}
&\left(10n^2-125n+300\right)\tilde{\rho}^2+\left(-4n^2+362n-1080\right)\left\|\tilde{R}ic\right\|^2\\
&+\left(4n^2-62n+240\right)\left\|\tilde{R}\right\|^2\\
&=\frac{\left(10n^4-117n^3+362n^2-183n-60\right)\tilde{\rho}^2}{n\,(n+1)}+\left(4n^2-62n+240\right)\left\|\tilde{B}\right\|^2 .
\end{aligned}
\tag{2.2.8}
$$

Then from (1.3.4), (2.2.7) and (2.2.8)

(2.2.9)

$$\int_M \left[\frac{\left(10n^4 - 117n^3 + 362n^2 - 183n - 60\right)\rho^2}{n\,(n+1)} - \left(4n^2 - 62n + 240\right)\|B\|^2 \right] dM$$

$$= \int_{\tilde M} \left[\frac{\left(10n^4 - 117n^3 + 362n^2 - 183n - 60\right)\tilde\rho^2}{n\,(n+1)} + \left(4n^2 - 62n + 240\right)\left\|\tilde B\right\|^2 \right] d\tilde M.$$

Once again the result follows as before.

Corollary 12. *If* (M, g, J) *is Hermitian and conformally flat and* $\left(\tilde M, \tilde g, \tilde J\right)$ *is Hermitian semi-Kähler Einstein with* $Spec^p\,(M, g, J) = Spec^p\left(\tilde M, \tilde g, \tilde J\right)$ *for any two values of* $p \in \{0,\,1, 2\}$ *then* (M, g, J) *and* $\left(\tilde M, \tilde g, \tilde J\right)$ *are flat* H_1 *and locally Kähler.*

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