

# On Conformally Flat Pseudosymmetric Spaces

U. C. De and S.K. Ghosh

*Dedicated to Prof. Dr. Constantin UDRIȘTE  
on the occasion of his sixtieth birthday*

## Abstract

In a recent paper [1] M. C. Chaki introduced and studied a type of non-flat Riemannian space  $(M^n, g)$  ( $n \geq 2$ ) whose curvature tensor  $R_{ij}^h$  satisfies the condition

$$(1) \quad R_{ijk,l}^h = 2\lambda_l R_{ijk}^h + \lambda^h R_{lij}^k + \lambda_i R_{ilk}^h + \lambda_k R_{ijl}^h$$

where  $\lambda_i$  is a non-zero vector and comma denotes covariant differentiation with respect to the metric  $g_{ij}$ . Such a space was called by him a pseudo symmetric space, the vector  $\lambda_i$  was called its associated vector and an  $n$ -dimensional space of this kind has been denoted  $(PS)_n$ . Tarafder[2] proved that a conformally flat  $(PS)_n$  ( $n \geq 3$ ) with non-zero constant scalar curvature is a subprojective space in the sense of Kagan[3], if the associated vector is gradient. In the present paper we obtain the above result without assuming any restriction on the scalar curvature. Among others it is shown that a conformally flat  $(PS)_n$  can be expressed as a warped product  $I \times e^q M^*$  where  $M^*$  is an Einstein space and such space is a space of quasi-constant curvature [4].

**Mathematics Subject Classification:** 53C25

**Key words:** conformally flat space, subprojective space, Einstein space

## 1 Conformally flat $(PS)_n$ ( $n \geq 3$ )

It is known [1] that a conformally flat  $(PS)_n$  ( $n \geq 3$ ) can not be of zero scalar curvature and also it is known [2] that in a conformally flat  $(PS)_n$

$$R_{ij} = \frac{R-t}{n-1} g_{ij} + \frac{nt-R}{(n-1)\lambda_p \lambda^p} \lambda_i \lambda_j$$

where  $R$  denotes the scalar curvature and  $t$  is a scalar.

The above expression can be written as

$$(1.1) \quad R_{ij} = \alpha g_{ij} + \beta v_i v_j$$

where  $\alpha = \frac{R-t}{n-1}$ ,  $\beta = \frac{nt-R}{n-1}$  are two scalars and  $v_i = \frac{\lambda_i}{\sqrt{\lambda_i \lambda^i}}$  is a unit vector. On the otherhand, a conformally flat space is conformally symmetric, that is,  $C_{ijk,l}^h = 0$ . The above equation is equivalent to

$$(1.2) \quad R_{jl,k} - R_{jk,l} = \frac{1}{2(n-1)}(g_{jl}R_{,k} - g_{jk}R_{,l}).$$

The relation (1.1) implies

$$(1.3) \quad R_{ij,k} = \alpha_k g_{ij} + \beta_k v_i v_j + \beta(v_j v_{i,k} + v_i v_{j,k}), \text{ where } \alpha_{,k} = \alpha_k \text{ and } \beta_{,k} = \beta_k.$$

Substituting (1.3) into (1.2) we obtain

$$(1.4) \quad \begin{aligned} & \alpha_k g_{jl} + \beta_k v_j v_l + \beta(v_l v_{j,k} + v_j v_{l,k}) - \alpha_l g_{jk} - \beta_l v_j v_k - \beta(v_k v_{j,l} + v_j v_{k,l}) \\ & = \frac{1}{2(n-1)}(g_{jl}R_{,k} - g_{jk}R_{,l}) \end{aligned}$$

where  $R_k = R_{,k}$ .

Since  $v^i v_i = 1$  and  $(v_{i,k})v^i = 0$ , so by transvecting with  $g^{jl}$ , (1.4) reduces to

$$(1.5) \quad (n-1)\alpha_k + \beta_k - (\beta_a v^a)v_k - \beta(v_k v_{,a}^a + v^a v_{k,a}) = \frac{1}{2} R_k.$$

Transvecting (1.4) with  $v^j$  we obtain

$$(1.6) \quad (\alpha_k v_l - \alpha_l v_k) + (\beta_k v_l - \beta_l v_k) + \beta(v_{l,k} - v_{k,l}) = \frac{1}{2(n-1)}(v_l R_k - v_k R_l).$$

Transvecting again with  $v^l$  we have

$$(\alpha_k + \beta_k) - (\alpha_a v^a)v_k - (\beta_a v^a)v_k - \beta v^a v_{k,a} = \frac{1}{2(n-1)} \{R_k - (v^a R_a)v_k\}$$

Substituting this into (1.5) we find

$$(1.7) \quad (n-2)\alpha_k - \beta v_k v_{,a}^a + (\alpha_a v^a)v_k - \frac{1}{2(n-1)}v_k(v^a R_a) = \frac{1}{2} \frac{n-2}{(n-1)}R_k$$

Transvecting (1.7) with  $v^k$ , we get

$$(n-1)(\alpha_a v^a) - \beta v_{,a}^a = \frac{1}{2}(R_a v^a).$$

Thus (1.7) reduces to

$$(1.8) \quad R_k = \lambda v_k + 2(n-1)(\alpha_k - \mu v_k)$$

where  $\lambda = R_a v^a$  and  $\mu = \alpha_a v^a$ .

Substituting this into (1.6), we obtain

$$(1.9) \quad (\beta_k v_l - \beta_l v_k) + \beta(v_{l,k} - v_{k,l}) = 0.$$

Now if  $v_i$  is gradient, that is,  $v_{l,k} - v_{k,l} = 0$ , then

$$(1.10) \quad \beta_k v_l - \beta_l v_k = 0. \text{ That is,}$$

$$(1.10a) \quad \beta_k = a v_k \text{ where } a \text{ is a scalar.}$$

Now by (1.8), (1.9) and (1.10) the equation (1.4) reduces to

$$\beta(v_l v_{j,k} - v_k v_{j,l}) = \frac{1}{2(n-1)} \phi(v_k g_{jl} - v_l g_{jk})$$

where  $\phi = \lambda - 2(n-1)\mu$ .

Transvecting the above equation with  $v^l$  and using  $v_{j,l} = v_{l,j}$ , we get

$$(1.11) \quad v_{j,k} = \frac{1}{2(n-1)} \frac{\phi}{\beta} (v_k v_j - g_{jk}).$$

Let us consider the scalar function

$$f = \frac{1}{2(n-1)} \frac{\phi}{\beta} \neq 0.$$

We have

$$f_k = -\frac{1}{2(n-1)} \frac{\phi}{\beta^2} \beta_k + \frac{1}{2(n-1)} \frac{\phi_k}{\beta} \text{ where } f_k = f_{,k} \text{ and } \phi_k = \phi_{,k}.$$

Again (1.8) implies

$$R_{k,j} = \phi_j v_k + \phi v_{k,j} + 2(n-1)(\alpha_{k,j} - \alpha_{j,k})$$

from which we get  $\phi_j v_k = \phi_k v_j$ , that is,  $\phi_k = A v_k$ , where  $A$  is a scalar function.

Thus from (1.10a) and (1.12)  $f_k = B v_k$  where

$$B = \frac{1}{2(n-1)\beta} \left( -\frac{\phi_a}{\beta} + A \right)$$

Using (1.12), it is easy to show that  $\omega_i = \frac{1}{2(n-1)\beta} \phi v_i$  is a gradient vector field.

In fact,  $\omega_{i,j} = v_i f_j + f v_{ij} = \beta v_j v_i + f v_{ij} = \omega_{j,i}$ . Thus (1.11) can be written as follows:  $v_{j,k} = -f g_{jk} + \omega_k v_j$  where  $\omega_k$  is gradient.

Hence  $v_i$  is a concircular vector field. Since  $f \neq 0$ ,  $v_i$  is a proper concircular vector.

Hence  $\lambda_i$  is a proper concircular vector field.

It is known [3] that if a conformally flat space admits a proper concircular vector field, then the space is a subprojective space in the sense of Kagan. Thus we can state **Theorem 1.** *If the associated vector of a conformally flat  $(PS)_n$  is gradient, then the space is a subprojective space.*

In [6] K. Yano proved that in order that a Riemannian space admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = (dx^1)^2 + e^q g_{\alpha\beta}^* dx^\alpha dx^\beta$$

where  $g_{\alpha\beta}^* = g_{\alpha\beta}^*(x^\nu)$  are the functions of  $x^\nu$  only ( $\alpha, \beta, \gamma = 2, 3, \dots, n$ ) and  $q = q(x^1) \neq \text{constant}$  is a function of  $x^1$  only. Since conformally flat  $(PS)_n$  admits proper concircular vector field  $v_i$  the space under consideration is the warped product  $1 \times e^q M^*$  where  $(M^*, g^*)$  is an  $(n-1)$ -dimensional Riemannian space. Gebarowski [6] proved that the warped product  $1 \times e^q M^*$  satisfies (1.2) iff  $M^*$  is an Einstein space. Thus we state the following theorem :

**Theorem 2.** *A conformally flat  $(PS)_n$  is the warped product  $1 \times e^q M^*$  where  $M^*$  is an Einstein space.*

*A conformally flat Riemannian space is said to be of quasi-constant curvature [4] if the curvature tensor  $R_{hijk}$  is given by*

$$(1.11) \quad R_{hijk} = a(g_{hj}g_{ik} - g_{hk}g_{ij}) + b(g_{hj}\theta_i\theta_k - g_{hk}\theta_i\theta_j - g_{ij}\theta_h\theta_k + g_{ik}\theta_h\theta_j)$$

where  $a$  and  $b$  are differentiable functions and  $\theta_i$  is a unit vector. Since our space is conformally flat, the curvature tensor is given by

$$R_{hijk} = \frac{1}{n-2} (R_{hk}g_{ij} - R_{hj}g_{ik} + R_{ij}g_{hk} - R_{ik}g_{hj}) - \frac{R}{(n-1)(n-2)} (g_{ij}g_{hk} - g_{ik}g_{hj})$$

Now on account of (1.1) the above equation reduces to (1.11), where

$$\theta_i = v_i, a = \frac{R}{(n+1)(n-2)} - \frac{2\alpha}{(n-2)} \text{ and } b = -\frac{\beta}{n-2}$$

Hence we obtain

**Theorem 3.** *A conformally flat  $(PS)_n$  is a space of quasi-constant curvature.*

## References

- [1] M.C. Chaki, *On Pseudosymmetric Manifolds*, Analele Stiintifice Ale Universitatii "Al. I. Cuza" Iasi, 33 (1987), 53-58.
- [2] M. Tarafder, *On Conformally flat Pseudosymmetric Manifolds*, Analele Stiintifice Ale Universitatii "Al. I. Cuza" Iasi, 1995, 237-241.
- [3] T. Aditi, *On Subprojective space - III*, Tohoku Math J. 3 (1951), 343-358.
- [4] B.Y. Chen, and K. Yano, *Hypersurface of a Conformally Flat Space*, Tensor, N. S., 26 (1972), 318-322.
- [5] K. Yano, *On the Torsion direction in Riemannian Spaces*, Proc. Imp. Acad. Tokyo, 20 (1944), 340-345.
- [6] A. Gebarowski, *Nearly Conformally Symetric Wraped Product Manifolds*, Bulletin of the Institute of Mathematics, Academia Sinica, 20, No. 4 (1992), 359-371.

Department of Mathematics  
University of Kalyani  
Kalyani - 741235, W. B., India  
Email:ucde@klyuniv.ernet.in