On Almost Cosymplectic Manifolds with the Structure Vector Field ξ Belonging to the k-Nullity Distribution

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Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

Let M be an almost cosymplectic manifold whose structure vector field ξ belongs to the so-called k-nullity distribution. Then $k \leq 0$ and the local structure of M is completely described. For a positive constant λ, it can be defined a Lie group G_λ endowed with a left invariant almost cosymplectic structure whose structure vector field ξ belongs to the $(-\lambda^2)$-nullity distribution. It is proved that if $k < 0$, then M is locally isomorphic to G_λ with $k = -\lambda^2$; especially M is a locally homogeneous Riemannian manifold. We also show that in this case the manifold M is Ricci-pseudosymmetric. $k = 0$ if and only if M is locally a product of an open interval and an almost Kähler manifold.

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1 Preliminaries

An almost contact metric manifold M is by definition a $(2n+1)$-dimensional connected $\mathcal{C}^\infty$-manifold endowed with a quadruple (φ, ξ, η, g), where φ is a $(1,1)$-tensor field, ξ a vector field, η a 1-form and g a Riemannian metric, such that (Blair [2])

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$
$$\eta(X) = g(X, \xi), \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Here and in the sequel, $X, Y, \ldots$ denote arbitrary vector fields on M if it is not otherwise stated. Let Φ be the fundamental 2-form defined by $\Phi(X, Y) = g(\varphi X, Y)$.

An almost contact metric manifold is said to be almost cosymplectic if both forms η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$ (Goldberg and Yano [9]). If an almost cosymplectic manifold is additionally normal, then it is called cosymplectic. An almost

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contact metric manifold is cosymplectic if and only if $\nabla\varphi = 0$, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection of g.

Let M be an almost cosymplectic manifold. By Frobenius' Theorem, the $2n$-dimensional distribution $\mathcal{D} = Ker\,\eta$ is integrable. Let $\mathcal{F}$ be the foliation corresponding to $\mathcal{D}$. The almost cosymplectic structure of M induces an almost Kähler structure $(\widetilde{J}, \widetilde{g})$ on any leaf $\widetilde{M}$ of F. In fact, $\widetilde{J}\widetilde{X} = \varphi\widetilde{X}$ and $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g(\widetilde{X}, \widetilde{Y})$ for any vector fields $\widetilde{X}, \widetilde{Y}$ on $\widetilde{M}$. If this structure is Kähler for any leaf $\widetilde{M}$, then M is said to be almost cosymplectic with Kählerian leaves. An almost cosymplectic manifold has Kählerian leaves if and only if it fulfills the condition (Olszak [12])

$$(\nabla_X\varphi)Y = -g(\varphi AX, Y)\xi + \eta(Y)\varphi AX,$$

where A is a $(1,1)$-tensor field defined by

$$A = -\nabla\xi. \tag{1}$$

As it was already proved in [12],

$$\begin{aligned} &g(AX,Y) = g(AY,X) \quad \text{(i.e., } A \text{ is a symmetric linear operator)},\\ &A\varphi + \varphi A = 0, \quad A\xi = 0, \quad \eta\circ A = 0. \end{aligned} \tag{2}$$

For any leaf $\widetilde{M}$, the vector field $\xi|_{\widetilde{M}}$ is an orthonormal vector field of the leaf. Therefore, from (1) it follows that $A|_{\widetilde{M}}$ is the shape operator and $H(\widetilde{X}, \widetilde{Y}) = \widetilde{g}(A\widetilde{X}, \widetilde{Y})$ is the second fundamental form of $\widetilde{M}$.

Let k be a real constant. The k-nullity distribution $N(k)$ on M is defined as the assignment $M \ni p \to N_p(k)$, where $N_p(k)$ is the k-nullity space at p defined as

$$N_p(k) = \{z \in T_pM \mid R_{xy}z = k(g(y,z)x - g(x,z)y) \text{ for any } x, y \in T_pM\},$$

T_pM being the tangent space at $p \in M$ [10]. Contact metric manifolds with the structure vector field $\xi \in N(k)$ appeared and were investigated in the papers [15], [1], [13], [14], etc. H. Endo [6], [7], [8] studied almost cosymplectic manifolds with $\xi \in N(k)$.

The purpose of this paper is to find the local structure of almost cosymplectic manifolds with the structure vector field ξ belonging to the k-nullity distribution and to prove that they are locally homogeneous. Moreover, we describe completely the curvature of this kind of manifolds and show that they are Ricci-pseudosymmetric.

2 Auxiliary results

Throughout this section, we assume that M is an almost cosymplectic manifold with the structure vector field ξ fulfilling the following relation

$$R_{XY}\xi = f(\eta(Y)X - \eta(X)Y) \tag{3}$$

for any vector fields X, Y, where f is a certain function on M. In formula (3), f is not assumed to be a constant function. However, we shall show later that a function f realizing (3) must be constant.

We begin with the following lemmas.

Lemma 1 *For the almost cosymplectic manifold, we have*

$$R_{\xi Z}\xi = -(\nabla_\xi A)Z + A^2 Z, \tag{4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = 0 \quad \textit{if} \quad X, Y \perp \xi. \tag{5}$$

Proof. By direct calculations, using (1), we find

$$\begin{aligned} R_{XY}\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi \\ &= -\nabla_X(AY) + \nabla_Y(AX) + A[X,Y] \\ &= -(\nabla_X A)Y + (\nabla_Y A)X. \end{aligned} \tag{6}$$

¿From (6) with $X = \xi$, we have

$$R_{\xi Y}\xi = -(\nabla_\xi A)Y + (\nabla_Y A)\xi. \tag{7}$$

Applying (1) and (2), we compute

$$(\nabla_Y A)\xi = \nabla_Y A\xi - A\nabla_Y\xi = A^2 Y,$$

which substituted into (7) gives (4). Finally, if $X, Y \perp \xi$, then $R_{XY}\xi = 0$ by (3). Therefore, formula (5) follows from (6).

Lemma 2 *For the almost cosymplectic manifold, tensor field A satisfies the following relations*

$$A^2 X = -fX + f\eta(X)\xi, \tag{8}$$

$$(\nabla_\xi A)X = 0, \tag{9}$$

$$(\mathcal{L}_\xi A)X = 0, \tag{10}$$

where $\mathcal{L}_\xi$ indicates the Lie differentiation operator in the direction ξ. Moreover, function f is nonpositive.

Proof. By (3), it follows that

$$R_{\xi\varphi X\varphi Y\xi} + R_{\xi XY\xi} = -2f\eta(X)\eta(Y) + 2fg(X,Y). \tag{11}$$

On the other hand, we have in general ([11], equ. (4.10))

$$R_{\xi\varphi X\varphi Y\xi} + R_{\xi XY\xi} = -2g(\nabla_X\xi, \nabla_Y\xi),$$

which in view of (1) and (2) turns into

$$R_{\xi\varphi X\varphi Y\xi} + R_{\xi XY\xi} = -2g(A^2X, Y).$$

Comparing the last identity with (11), we obtain

$$g(A^2X, Y) = f\eta(X)\eta(Y) - fg(X,Y),$$

which implies (8). Note that (8) obviously implies $f \leq 0$.

From (3) and (8), it follows that

$$R_{\xi Y}\xi = A^2 Y,$$

which compared with (4) gives (9). Finally, making straightforward computations, we get

$$(\mathcal{L}_\xi A)X = [\xi, AX] - A[\xi, X] = (\nabla_\xi A)X = 0,$$

where we have also used (1) and (9).

Lemma 3 *For the almost cosymplectic manifold, we have*

$$
\begin{aligned}
(12) \qquad & df(\xi) &=& \; 0,\\
(13) \qquad & g((\nabla_X A^2)Y, Z) &=& \; -df(X)g(Y,Z),\\
(14) \qquad & g((\nabla_X A)Z, AY) - g((\nabla_Y A)Z, AX) &=& \; -df(X)g(Y,Z) + df(Y)g(X,Z),
\end{aligned}
$$

for any X, Y, Z *orthogonal to* ξ.

Proof. Differentiating covariantly (8) we obtain

$$(15) \qquad (\nabla_X A^2)Y = -df(X)Y + df(X)\eta(Y)\xi - fg(AX,Y)\xi - f\eta(Y)AX$$

for any $X, Y \in \mathcal{X}(M)$. With $X = \xi$, this becomes, using also (9) and (2)

$$df(\xi)Y = df(\xi)\eta(Y)\xi,$$

for any $Y \in \mathcal{X}(M)$, which leads to (12).

From (15) it follows

$$g((\nabla_X A^2)Y, Z) = -df(X)g(Y,Z)$$

for any X, Y, Z orthogonal to ξ, which is just (13).

Alternating (13) with respect to X, Y, we find

$$(16) \qquad g((\nabla_X A^2)Y, Z) - g((\nabla_Y A^2)X, Z) = -df(X)g(Y,Z) + df(Y)g(X,Z)$$

for any $X, Y, Z \perp \xi$. The left hand side of (16) equals

$$g((\nabla_X A)Z, AY) - g((\nabla_Y A)Z, AX) + g((\nabla_X A)Y - (\nabla_Y A)X, AZ),$$

which, by virtue of (5), reduces to

$$g((\nabla_X A)Z, AY) - g((\nabla_Y A)Z, AX).$$

Therefore, (16) implies (14).

Now we are going to prove that f is constant under the assumption (3). Since the proof in dimensions ≥ 5 differs from that in dimension 3, we present it in two parts.

Theorem 1 *If an almost cosymplectic manifold satisfies the condition*

$$R_{XY}\xi = f(\eta(Y)X - \eta(X)Y)$$

for any vector fields X, Y *and a certain function* f, *then* f *must be constant.*

Proof. *The case* $\dim M = 2n+1 \geq 5$. By (12), it is suffcient to show that $df(X) = 0$ for any $X \perp \xi$. Restrict consideration to an arbitrary fixed point $p \in M$.

If $f(p) = 0$, then $A^2(p) = 0$, by (8). Since $A(p)$ is symmetric, $A(p) = 0$ and consequently $(\nabla A^2)(p) = 0$. Therefore, from (13) it follows that $df(X)g(Y,Z) = 0$ for any $X, Y, Z \in T_pM, X, Y, Z \perp \xi$. Hence $df(X) = 0$ for any $X \in T_pM$, $X \perp \xi$.

Now, let $f(p) < 0$ (note that always $f(p) \leq 0$, see Proposition 2), and suppose $\lambda = \sqrt{|f(p)|}$. By (8), $A^2X = \lambda^2 X$ for any $X \in T_pM, X \perp \xi$. Since A anticommutes with φ (cf. (2)), both λ and $-\lambda$ are eigenvalues of the same multiplicity equal to n ($n \geq 2$ by the assumption). Let D_- and D_+ denote eigenspaces corresponding to the eigenvalues $-\lambda$ and λ, respectively.

Let X be an arbitrary vector from D_-. Take a unit vector $Y \in D_-$ such that $g(X, Y) = 0$. By (14) with $Z = Y$, we obtain

$$\begin{aligned} -df(X) &= -\lambda(g((\nabla_X A)Y, Y) - g((\nabla_Y A)Y, X)) \\ &= -\lambda g((\nabla_X A)Y - (\nabla_Y A)X, Y) = 0, \end{aligned}$$

where in the last equation we have used (5). In the same way we prove that $df(X) = 0$ if $X \in D_+$. Thus, the proof is complete in case of $\dim M \geq 5$.

The case $\dim M = 3$. Define $\lambda = \sqrt{|f|}$. Let suppose that $\lambda > 0$ at some $p \in M$. Then, on a connected neighbourhood U of p, there exist an orthonormal frame (E_0, E_1, E_2), such that

$$E_0 = \xi, \qquad AE_1 = -\nabla_{E_1} E_0 = -\lambda E_1, \qquad AE_2 = -\nabla_{E_2} E_0 = \lambda E_2. \tag{17}$$

In virtue of (17), we obtain

$$\nabla_{E_0} AE_1 = -(E_0\lambda)E_1 - \lambda \nabla_{E_0} E_1, \tag{18}$$

By (9) , we have

$$\nabla_{E_0} AE_1 = A\nabla_{E_0} E_1. \tag{19}$$

It follows from (12) that $E_0\lambda = 0$. Therefore, (18), (19) lead to

$$A\nabla_{E_0} E_1 = -\lambda \nabla_{E_0} E_1. \tag{20}$$

In the same way, we get

$$A\nabla_{E_0} E_2 = \lambda \nabla_{E_0} E_2. \tag{21}$$

The relation (20) (resp., (21)) shows that $\nabla_{E_0} E_1$ (resp., $\nabla_{E_0} E_2$) is an eigenvector field of A corresponding to the eigenvalue function $-\lambda$ (resp. λ). By (17), $\nabla_{E_0} E_1$ (resp. $\nabla_{E_0} E_2$) is proportional to E_1 (resp. E_2). Consequently, $\nabla_{E_0} E_1 = \nabla_{E_0} E_2 = 0$ on U. The last two formulas and (17) lead to

$$\begin{aligned} [E_0, E_1] &= \nabla_{E_0} E_1 - \nabla_{E_1} E_0 = -\lambda E_1, \\ [E_0, E_2] &= \nabla_{E_0} E_2 - \nabla_{E_2} E_0 = \lambda E_2, \end{aligned} \tag{22}$$

where $[\cdot, \cdot]$ indicates the Poisson bracket of vector fields. The Jacobi's identity and (22) give

$$\begin{aligned} [E_0, [E_1, E_2]] &= [[E_0, E_1], E_2] + [E_1, [E_0, E_2]] \\ &= d\lambda(E_2)E_1 + d\lambda(E_1)E_2. \end{aligned} \tag{23}$$

On the other hand, by (5)

$$(\nabla_{E_1} A)E_2 - (\nabla_{E_2} A)E_1 = 0,$$

or

$$\nabla_{E_1} AE_2 - A\nabla_{E_1} E_2 - \nabla_{E_2} AE_1 + A\nabla_{E_2} E_1 = 0.$$

With the help of (17), we can rewrite the above formula in the following way

$$d\lambda(E_1)E_2 + d\lambda(E_2)E_1 + \lambda\nabla_{E_1}E_2 + \lambda\nabla_{E_2}E_1 = A[E_1, E_2].$$

Projecting the last equation onto E_1 and taking into account (17) and the symmetry of A, we find

$$d\lambda(E_2) + \lambda g(\nabla_{E_1}E_2, E_1) = -\lambda g([E_1, E_2], E_1).$$

Because of

$$g(\nabla_{E_1}E_2, E_1) = g([E_1, E_2], E_1),$$

the previous formula reduces to

$$g([E_1, E_2], E_1) = -\frac{1}{2\lambda}d\lambda(E_2). \tag{24}$$

Similarly, one obtains

$$g([E_1, E_2], E_2) = \frac{1}{2\lambda}d\lambda(E_1). \tag{25}$$

We also claim the folowing

$$\begin{aligned} g([E_1, E_2], E_0) &= g(\nabla_{E_1}E_2 - \nabla_{E_2}E_1, \xi) \\ &= -g(\nabla_{E_1}\xi, E_2) + g(\nabla_{E_2}\xi, E_1) \\ &= -2d\eta(E_1, E_2) = 0. \end{aligned} \tag{26}$$

From (24), (25), (26), we deduce that

$$[E_1, E_2] = -d(ln\sqrt{\lambda})(E_2)E_1 + d(ln\sqrt{\lambda})(E_1)E_2.$$

Therefore,

$$\begin{aligned} [E_0, [E_1, E_2]] &= [E_0, -d(ln\sqrt{\lambda})(E_2)E_1 + d(ln\sqrt{\lambda})(E_1)E_2] \\ &= -E_0(E_2(ln\sqrt{\lambda}))E_1 + E_0(E_1(ln\sqrt{\lambda}))E_2 \\ &\quad - E_2(ln\sqrt{\lambda})[E_0, E_1] + E_1(ln\sqrt{\lambda})[E_0, E_2]. \end{aligned} \tag{27}$$

By (22) and $E_0(ln\sqrt{\lambda}) = 0$, we have

$$\begin{aligned} E_0(E_2(ln\sqrt{\lambda})) &= [E_0, E_2](ln\sqrt{\lambda}) = \lambda E_2(ln\sqrt{\lambda}), \\ E_0(E_1(ln\sqrt{\lambda})) &= [E_0, E_1](ln\sqrt{\lambda}) = -\lambda E_1(ln\sqrt{\lambda}). \end{aligned}$$

By virtue of these equalities and also (22), formula (27) reduces to $[E_0, [E_1, E_2]] = 0$. In this situation, (23) leads to $E_1\lambda = 0$, $E_2\lambda = 0$, which together with $E_0\lambda = 0$ yields $\lambda = const.$, and consequently $f = const.$ on the neighbourhood U. Since M is connected, this property implies the constancy of f on M. This completes the proof in case of $\dim M = 3$.

3 The local structure

In this section, we deal with almost cosymplectic manifolds M whose structure vector field ξ belongs to the k-nullity distribution, i.e., those satisfying the condition

$$R_{XY}\xi = k(\eta(Y)X - \eta(X)Y) \tag{28}$$

for any vector fields X, Y, where $k = const. \leq 0$.

At first, we consider the case $k = 0$, which is exceptional.

Theorem 2 *For an almost cosymplectic manifold M, the structure vector field ξ belongs to the 0-nullity distribution if and only if M is locally the product of an open interval and an almost Kähler manifold.*

Proof. Equality (8) shows that $k = 0$ if and only if $A = 0$. On the other hand, it is well-known that $A = 0$ ($\nabla\xi = 0$) if and only if M is locally the product of an open interval and an almost Kähler manifold.

In the rest of this section, we describe the local structure of almost cosymplectic manifolds satisfying (28) with $k < 0$.

Proposition 1 *Let M be an almost cosymplectic manifold with the structure vector field ξ belonging to the k-nullity distribution with $k < 0$. Then M is almost cosymplectic with Kählerian leaves.*

Proof. Using (28), we find that

$$\begin{aligned} R_{XY\varphi Z\xi} - R_{\varphi X\varphi Y\varphi Z\xi} &- R_{\varphi XYZ\xi} - R_{X\varphi YZ\xi} \\ &= -2k(\eta(Y)g(X,\varphi Z) - \eta(X)g(Y,\varphi Z)). \end{aligned}$$

The following curvature identity is already known for almost cosymplectic manifolds (see [11])

$$R_{XY\varphi Z\xi} - R_{\varphi X\varphi Y\varphi Z\xi} - R_{\varphi XYZ\xi} - R_{X\varphi YZ\xi} = -2(\nabla_{AZ}\Phi)(X,Y).$$

Comparing the last two relations, we obtain

$$k(\eta(Y)g(X,\varphi Z) - \eta(X)g(Y,\varphi Z)) = (\nabla_{AZ}\Phi)(X,Y).$$

Putting AZ instead of Z in this equation and taking into account (8), we obtain

$$\eta(Y)g(X,\varphi Z) - \eta(X)g(Y,\varphi Z) = -(\nabla_Z\Phi)(X,Y) + \eta(Z)(\nabla_\xi\Phi)(X,Y).$$

Hence, since $\nabla_\xi\Phi = 0$ (see [11]), we have

$$(\nabla_Z\varphi)X = -g(\varphi AZ, X)\xi + \eta(X)\varphi AZ. \tag{29}$$

Thus, by the result cited in the Introduction, M is almost cosymplectic with Kählerian leaves.

Formula (29) has already been obtained by Endo [6]. However, we included the proof for completness and since it differs in some details from that in [6]; for instance Endo does not use the notion of almost cosymplectic manifold with Kählerian leaves.

Lemma 4 *For an almost cosymplectic manifold with the structure vector field ξ belonging to the k-nullity distribution with $k < 0$, we have*

$$(30)\qquad (\mathcal{L}_\xi g)(X,Y) = -2g(AX,Y),$$

$$(31)\qquad (\mathcal{L}_\xi^2 g)(X,Y) = -4kg(X,Y) + 4k\eta(X)\eta(Y),$$

where $\mathcal{L}_\xi^2 = \mathcal{L}_\xi \circ \mathcal{L}_\xi$.

Proof. Equation (30) holds realy for any almost cosymplectic manifold, as it is shown in [4]. Using (30), we find

$$(\mathcal{L}_\xi^2 g)(X,Y) = (\mathcal{L}_\xi(\mathcal{L}_\xi g))(X,Y) = 4g(A^2X,Y) - 2g((\mathcal{L}_\xi A)X,Y).$$

This, in virtue of (8) and (10), turns into (31).

Theorem 3 *Let M be an almost cosymplectic manifold and (φ,ξ,η,g) its almost cosymplectic structure. The structure vector field ξ of the manifold M belongs to the k-nullity distribution with $k < 0$ if and only if for any point $p \in M$ there exists a coordinate neighbourhood $(U,(t,x^1\dots,x^{2n}))$, $p \in U$, on which*

$$(32)\qquad \xi = \frac{\partial}{\partial t}, \quad \eta = dt,$$

$$(33)\qquad g = dt\otimes dt + e^{2\lambda t}\sum_{\mu=1}^{n} dx^\mu \otimes dx^\mu + e^{-2\lambda t}\sum_{\mu=1}^{n} dx^{n+\mu}\otimes dx^{n+\mu},$$

$$(34)\qquad \varphi = e^{2\lambda t}\sum_{\mu=1}^{n} dx^\mu \otimes \frac{\partial}{\partial x^{n+\mu}} - e^{-2\lambda t}\sum_{\mu=1}^{n} dx^{n+\mu}\otimes \frac{\partial}{\partial x^\mu},$$

where $\lambda = \sqrt{|k|}$.

Proof. Following the general procedure due to Olszak [12] and our Proposition 1, the local shape of the almost cosymplectic structure (φ,ξ,η,g) can be described in the following way. Let $p \in M$. There is a neighbourhood U of p such that $U = (-a,a)\times\widetilde{U}$, where $(-a,a)$ is an open interval with coordinate t and $\widetilde{U}$ is a $2n$-dimensional manifold, $p = (0,\tilde{p})$, $\tilde{p} \in \widetilde{U}$. Moreover,on $\widetilde{U}$, there is a 1-parameter family of Kähler stuctures (J_t,G_t), $-a < t < a$, whose fundamental forms do not depend on the parameter t, and on U the almost cosymplectic structure is given by

$$(35)\qquad \xi = \frac{\partial}{\partial t}, \quad \eta = (dt,0), \quad \varphi = (0,J_t), \quad g = dt\otimes dt + G_t.$$

Treating $\widetilde{U}$ as a hypersurface given by $t = 0$, the Kähler structure $(J,G) = (J_0,G_0)$ is just the induced Kähler structure on $\widetilde{U}$. Let $\widetilde{A}$ denote the operator A restricted to $\widetilde{U}$. Thus, $\widetilde{A}$ is the shape operator of $\widetilde{U}$.

Note that by (2)

$$(36)\qquad \widetilde{A}J + J\widetilde{A} = 0.$$

This and (8) imply that $-\lambda$, λ are constant eigenvalues of $\widetilde{A}$ both of the same multiplicity. Let $\mathcal{D}_-,\mathcal{D}_+$ be the corresponding eigendistributions, then $\dim\mathcal{D}_- = \dim\mathcal{D}_+ = n$.

$\widetilde{A}$ as the shape operator must satisfy Codazzi equation

$$R_{\widetilde{X}\widetilde{Y}}\xi = -(\widetilde{\nabla}_{\widetilde{X}}\widetilde{A})\widetilde{Y} + (\widetilde{\nabla}_{\widetilde{Y}}\widetilde{A})\widetilde{X}.$$

By applying (28), the last identity turns into

$$0 = -(\widetilde{\nabla}_{\widetilde{X}}\widetilde{A})\widetilde{Y} + (\widetilde{\nabla}_{\widetilde{Y}}\widetilde{A})\widetilde{X}.$$

Being a Codazzi tensor field and having two different constant eigenvalues, the tensor field $\tilde{A}$ must be parallel. Equivalently, the distributions $\mathcal{D}_-, \mathcal{D}_+$ are parallel and the Riemann metric G is locally productable. We restrict $(\widetilde{U}, G)$ to the Riemann product neighbourhood of $\tilde{p}$, $\widetilde{U} = \widetilde{U}_1 \times \widetilde{U}_2$, $G = G_1 \times G_2$, $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$, $\tilde{p}_1 \in \widetilde{U}_1$, $\tilde{p}_2 \in \widetilde{U}_2$, $\widetilde{U}_1$, $\widetilde{U}_2$ being the integral submanifolds of $\mathcal{D}_-$, $\mathcal{D}_+$, respectively, and G_1, G_2 are the Riemann metrics induced on $\widetilde{U}_1$, $\widetilde{U}_2$.

The second fundamental form H has also the product shape, precisely

$$(37)\quad H(\widetilde{X}_1, \widetilde{Y}_1) = -\lambda G_1(\widetilde{X}_1, \widetilde{Y}_1), \quad H(\widetilde{X}_2, \widetilde{Y}_2) = \lambda G_2(\widetilde{X}_2, \widetilde{Y}_2), \quad H(\widetilde{X}_1, \widetilde{Y}_2) = 0.$$

In the above and in what follows, we denote by $\widetilde{X}_1, \widetilde{Y}_1, \widetilde{Z}_1 \ldots$ and $\widetilde{X}_2, \widetilde{Y}_2, \widetilde{Z}_2 \ldots$ vector fields tangent to $\widetilde{U}_1$ and $\widetilde{U}_2$, respectively.

Denote by $\widetilde{R}^1$ the curvature tensor of G_1. As it is well-known, for the product Riemann metric, we have

$$\widetilde{R}^1_{\widetilde{X}_1\widetilde{Y}_1}\widetilde{Z}_1 = \widetilde{R}_{\widetilde{X}_1\widetilde{Y}_1}\widetilde{Z}_1, \quad \widetilde{R}^1_{\widetilde{X}_1\widetilde{Y}_1}\widetilde{Z}_2 = 0.$$

By (36), $J\widetilde{Z}_1$ is tangent to $\widetilde{U}_2$ and consequently

$$\widetilde{R}^1_{\widetilde{X}_1\widetilde{Y}_1}\widetilde{Z}_1 = \widetilde{R}_{\widetilde{X}_1\widetilde{Y}_1}\widetilde{Z}_1 = -J\widetilde{R}_{\widetilde{X}_1\widetilde{Y}_1}J\widetilde{Z}_1 = 0.$$

This means that G_1 is locally flat. Analogously, we prove that G_2 is locally flat, too.

Restrict $\widetilde{U}_1$ to a neighbourhood of $\tilde{p}_1$ with coordinates $(x^1, \ldots, x^n)$, and $\widetilde{U}_2$ to a neighbourhood of $\tilde{p}_2$ with coordinates $(x^{n+1}, \ldots, x^{2n})$ such that

$$(38)\qquad G_1 = \sum_{\mu=1}^{n} dx^\mu \otimes dx^\mu, \quad G_2 = \sum_{\mu=1}^{n} dx^{n+\mu} \otimes dx^{n+\mu}.$$

Note that the coefficients of J with respect to the coordinates $(x^1, \ldots, x^{2n})$ on $\widetilde{U}$ are constant. This provides the orthonormal frame

$$\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, J\frac{\partial}{\partial x^1}, \ldots, J\frac{\partial}{\partial x^n}\right)$$

to be holonomic. We change linearly coordinates $(x^{n+1}, \ldots, x^{2n})$ to the new ones, which are denoted by the same letters, in the way that

$$(39)\qquad J\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^{n+\mu}},$$

and the local expression of G_2 given in (38) is still fulfilled. For the fundamental form Ω $(\Omega(\widetilde{X}, \widetilde{Y}) = G(J\widetilde{X}, \widetilde{Y}))$ of (J, G), by (39), we have

$$\Omega = 2\sum_{\mu=1}^{n} dx^{\mu} \wedge dx^{n+\mu}. \tag{40}$$

Now, we back to the metric g on $U = (-a,a)\times\widetilde{U}_1\times\widetilde{U}_2$. Let g_{ij} be the components of g with respect to the coordinates $(x^0 = t, x^1, \ldots, x^{2n})$. By (35), we have $g_{00} = 1$, $g_{i0} = 0$, $1 \le i \le 2n$. As concerns the functions g_{ij}, $1 \le i,j \le 2n$, by (31), they must satisfy the following 2-nd order differential equation

$$\frac{\partial^2 g_{ij}}{\partial t^2} = 4\lambda^2 g_{ij}, \tag{41}$$

which has the only solution of the form

$$g_{ij} = \alpha_{ij}\cosh(2\lambda t) - \frac{1}{\lambda}\beta_{ij}\sinh(2\lambda t) \tag{42}$$

where

$$\alpha_{ij}(x^1,\ldots,x^{2n}) = g_{ij}(0,x^1,\ldots,x^{2n}), \tag{43}$$

$$\beta_{ij}(x^1,\ldots,x^{2n}) = -\frac{1}{2}\frac{\partial g_{ij}}{\partial t}(0,x^1,\ldots,x^{2n}). \tag{44}$$

In view of (43), α_{ij} are just the components of the induced metric $G = G_0$ on $\widetilde{U}$. In virtue of (30), the second fundamental form H is given by

$$H(\widetilde{X},\widetilde{Y}) = g(\widetilde{A}\widetilde{X},\widetilde{Y}) = g(A\widetilde{X},\widetilde{Y}) = -\frac{1}{2}(\mathcal{L}_{\xi}g)(\widetilde{X},\widetilde{Y}).$$

By (44), β_{ij} are the components of H on $\widetilde{U}$.

On virtue of formulas (38), (37), we have

$$\alpha_{ij} = \delta_{ij}, \quad \beta_{\mu\nu} = -\lambda\delta_{\mu\nu}, \quad \beta_{(n+\mu)(n+\nu)} = \lambda\delta_{\mu\nu},$$

otherwise $\beta_{ij} = 0$. Consequently, with the help of (42), we obtain

$$g_{\mu\nu} = e^{2\lambda t}\delta_{\mu\nu}, \quad g_{(n+\mu)(n+\nu)} = e^{-2\lambda t}\delta_{\mu\nu}, \tag{45}$$

otherwise $g_{ij} = 0$. This proves formula (33).

Formulas (35), (40) lead to

$$\Phi_{\mu(n+\nu)} = -\Phi_{(n+\nu)\mu} = \delta_{\mu\nu} \tag{46}$$

and $\Phi_{ij} = 0$ in other cases. With the help of (45), (46) and $\varphi_i^j = \sum_s \Phi_{is}g^{sj}$, we compute the components of the operator φ

$$\varphi_{\nu}^{n+\mu} = e^{2\lambda t}\delta_{\nu}^{\mu}, \quad \varphi_{n+\nu}^{\mu} = -e^{-2\lambda t}\delta_{\nu}^{\mu}, \tag{47}$$

otherwise $\varphi_i^j = 0$. This gives (34).

Conversely, let us assume that an almost cosymplectic structure is given locally by (32)-(34). Define a local orthonormal frame $(E_0, E_1, \ldots, E_{2n})$ by putting

$$E_0 = \xi = \frac{\partial}{\partial t}, \quad E_\mu = e^{-\lambda t}\frac{\partial}{\partial x^\mu}, \quad E_{n+\mu} = e^{\lambda t}\frac{\partial}{\partial x^{n+\mu}}$$

for $\mu = 1, \ldots, n$. The non-zero Lie brackets $[E_i, E_j]$ are

$$[E_0, E_\mu] = -\lambda E_\mu, \qquad [E_0, E_{n+\mu}] = \lambda E_{n+\mu}. \tag{48}$$

Moreover, the non-zero covariant derivatives are

$$\begin{aligned} \nabla_{E_\mu} E_0 &= \lambda E_\mu, & \nabla_{E_\mu} E_\mu &= -\lambda E_0, \\ \nabla_{E_{n+\mu}} E_0 &= -\lambda E_{n+\mu}, & \nabla_{E_{n+\mu}} E_{n+\mu} &= \lambda E_0. \end{aligned} \tag{49}$$

With help of (48) and (49), we find

$$R_{E_i E_j} E_0 = 0, \qquad R_{E_0 E_j} E_0 = \lambda^2 E_j$$

for any $i, j = 1, \ldots, 2n$, $i < j$. Consequently, we have (28) with $k = -\lambda^2$.

4 Homogeneity

Let λ be a real positive number and G_λ be the solvable non-nilpotent Lie group whose underlying manifold is the Cartesian space $\mathbf{R}^{2n+1}$ and the multiplication is defined by the formula

$$\begin{aligned} &(t, x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}) * (s, y^1, \ldots, y^n, y^{n+1}, \ldots, y^{2n}) = \\ &\qquad (t + s, e^{-\lambda t} y^1 + x^1, \ldots, e^{-\lambda t} y^n + x^n, e^{\lambda t} y^{n+1} + x^{n+1}, \ldots, e^{\lambda t} y^{2n} + x^{2n}) \end{aligned}$$

for any $(t, x^1, \ldots, x^{2n}), (s, y^1, \ldots, y^{2n}) \in \mathbf{R}^{2n+1}$. As a basis of the Lie algebra of G_λ, we take the following left invariant vector fields

$$E_0 = \frac{\partial}{\partial t}, \quad E_\mu = e^{-\lambda t}\frac{\partial}{\partial x^\mu}, \quad E_{n+\mu} = e^{\lambda t}\frac{\partial}{\partial x^{n+\mu}}. \tag{50}$$

With respect to this base, we define a left invariant almost contact metric structure (φ, ξ, η, g) on G_λ as follows

$$\begin{aligned} &\xi = E_0, \qquad g(E_i, E_j) = \delta_{ij}, \qquad \eta(\cdot) = g(\xi, \cdot), \\ &\varphi E_0 = 0, \qquad \varphi E_\mu = E_{n+\mu}, \qquad \varphi E_{n+\mu} = -E_\mu. \end{aligned} \tag{51}$$

In terms of the global coordinates $(t, x^1, \ldots, x^{2n})$ on G_λ, the structure (φ, ξ, η, g) is given just by the formulas (32)–(34). By Theorem 3, the quadruple (φ, ξ, η, g) becomes an almost cosymplectic structure on G_λ whose structure vector field ξ belongs to the k-nullity distribution with $k = -\lambda^2$.

To formulate our next theorem, we need the following notion. Let M, M' be almost cosymplectic manifolds and (φ, ξ, η, g), $(\varphi', \xi', \eta', g')$ their almost cosymplectic structures, respectively. We say that M, M' are locally isomorphic if for any points $p \in M$, $p' \in M'$ there exist neighborhhoods U, U' of p, p', respectively, and a diffeomorphism $\psi : U \to U'$, $\psi(p) = p'$, such that

$$g = \psi^* g', \quad \psi_* \circ \varphi = \varphi' \circ \psi_*, \quad \psi_* \xi = \xi', \quad \eta = \psi^* \eta'.$$

By virtue of Theorem 3 and the above construction, the following theorem is true:

Theorem 4 *Let M be an almost cosymplectic manifold whose structure vector field ξ belongs to the k-nullity distribution with $k < 0$. Then M is locally isomorphic with G_λ endowed with the structure (φ, ξ, η, g) as in* (51) *and $k = -\lambda^2$.*

As a consequence of the last theorem, we have following:

Theorem 5 *An almost cosymplectic manifold M whose structure vector field ξ belongs to the k-nullity distribution with $k < 0$ is a locally homogeneous Riemaniann manifold. If M is additionally simply connected and complete, then it is Riemannian homogeneous.*

Remark. In the paper [3], there was defined a wider class of solvable non-nilpotent Lie groups $G(k_1, \dots, k_n)$ admitting left invariant non-cosymplectic almost cosymplectic structures. The importance of this class stem from the fact that these Lie groups admit discretes subgroups for which coset spaces are compact and inherit almost cosymplectic structures. It is worthwhile to notice that our Lie group G_λ endowed with the almost cosymplectic structure (51) belong to this class; indeed, the group G_λ is in fact the Lie group $G(k_1, \dots, k_n)$ from [3] with $k_1 = \dots = k_n = -\lambda$.

5 Curvature properties

In this section, we give a complete description of the curvature R of an almost cosymplectic manifold M satisfying (28) with $k < 0$, and next we prove that it is Ricci-pseudosymmetric.

First, let R_{XY} and $X \wedge Y$ be the endomorphisms acting on vector fields as follows

$$R_{XY}Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y.$$

We extend R_{XY} and $X \wedge Y$ to derivations of the tensor algebra on M, so that for the Ricci curvature tensor S, we have

$$\begin{aligned}(R_{XY}S)(Z,W) &= -S(R_{XY}Z,W) - S(Z,R_{XY}W),\\ ((X \wedge Y)S)(Z,W) &= -S((X \wedge Y)Z,W) - S(Z,(X \wedge Y)W),\end{aligned}$$

respectively.

Following R. Deszcz [5], M is said Ricci-pseudosymmetric if its Ricci tensor S satisfies the condition

$$R_{XY}S = L\,(X \wedge Y)S \tag{52}$$

for a certain function L and any vector fields X, Y.

Lemma 5 *For an almost cosymplectic manifold satisfying (28) with $k = -\lambda^2 < 0$, the curvature tensor R and the Ricci tensor S are given by*

$$R_{XY} = -(AX) \wedge (AY) - \lambda^2 \eta \otimes ((X \wedge Y)\xi) - \lambda^2 (\eta \circ (X \wedge Y)) \otimes \xi, \tag{53}$$
$$S = -2n\lambda^2 \eta \otimes \eta. \tag{54}$$

Proof. In virtue of the results of the previous section, it is sufficient to prove the assertion only in case of $M = G_\lambda$ and the almost cosymplectic structure is defined as in (51). We denote the Lie algebra of G_λ by g_λ. Note that (48)-(49) enable us to write the following

$$\begin{aligned} \nabla_X Y &= -\eta(Y)AX + g(AX,Y)\xi, \\ [X,Y] &= -\eta(Y)AX + \eta(X)AY, \end{aligned} \tag{55}$$

for $X, Y \in g_\lambda$. As a consequence of the above, we also have

$$\begin{aligned} \nabla_X AY = \nabla_Y AX &= \lambda^2(g(X,Y) - \eta(X)\eta(Y))\xi, \\ \nabla_{AX} Y &= -\lambda^2\eta(Y)X + \lambda^2 g(X,Y)\xi. \end{aligned} \tag{56}$$

For the curvature, using (55) and (56), we find

$$\begin{aligned} R_{XY}Z &= -g(AY,Z)AX + g(AX,Z)AY - \lambda^2\eta(Y)\eta(Z)X \\ &\quad + \lambda^2\eta(X)\eta(Z)Y + \lambda^2\eta(Y)g(Z,X)\xi - \lambda^2\eta(X)g(Z,Y)\xi, \end{aligned}$$

which is just equivalent to (53). Hence, we find (54).

Theorem 6 *Let M be an almost cosymplectic manifold whose structure vector field ξ belongs to the k-nullity distribution with $k = -\lambda^2 < 0$. Then M is Ricci-pseudosymmetric, precisely, it realizes* (52) *with $L = -\lambda^2$.*

Proof. Using (54), we compute $R_{XY}S$. Namely, we have

$$(R_{XY}S)(Z,W) = -2n\lambda^2(R_{XY}\eta)(Z)\eta(W) - 2n\lambda^2\eta(Z)(R_{XY}\eta)(W). \tag{57}$$

Note that

$$(R_{XY}\eta)(Z) = -\lambda^2 g(Z,X)\eta(Y) + \lambda^2 g(Z,Y)\eta(X),$$

which (57) turns into

$$\begin{aligned} (R_{XY}S)(Z,W) &= 2n\lambda^4 g(Z,X)\eta(Y)\eta(W) - 2n\lambda^4 g(Z,Y)\eta(X)\eta(W) \\ &\quad + 2n\lambda^4 g(W,X)\eta(Y)\eta(Z) - 2n\lambda^4 g(W,Y)\eta(X)\eta(Z) \\ &= -\lambda^2((X \wedge Y)S)(Z,W). \end{aligned}$$

This completes the proof.

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