Cross Ratios of Points and Lines in Moufang Planes

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Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

In this paper, first we extend the known definition of cross-ratio of collinear points to whole Moufang plane. Later we introduce the cross-ratios for lines and the known results about the cross-ratios of points which are adapted to cross-ratios of lines without using the principle of duality. Finally, we give a theorem which describes the relation between the cross-ratios of points and lines.

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Key words: Cross ratio, Moufang planes

1 Introduction

Let M be a projective plane coordinated by an alternative field $\mathcal{A}$, $char\mathcal{A} \neq 2$. If $\mathcal{A}$ is associative, then M is Desarguesian and if $\mathcal{A}$ is non-associative then $\mathcal{A}$ is Cayley division algebra over its center $\mathcal{Z}$ and M is a Moufang plane (see [4]). Then, $\mathcal{A}$ is equipped with the involution $\gamma : x \to \bar{x}$, the norm form $n : x \to x\bar{x}$ and the trace form $t : x \to \frac{1}{2}(x + \bar{x})$ (see [1]). In this case, the ranges of the norm and trace forms are $\mathcal{Z}$, but the range of the γ is A. Also norm form is multiplicative and trace form is both symmetric and associative (i.e. $n(xy) = n(x)n(y)$, $t(xy) = t(yx)$, $t(x(yz)) = t((xy)z)$).

There is an equivalence relation $\equiv$ on $\mathcal{A}$ which is defined by "$a \equiv b \Leftrightarrow \exists c \in \mathcal{A}\backslash\{0\}, a = c^{-1}bc$" and this equivalence relation is called conjugate. For any element x of $\mathcal{A}$, the equivalence class of x is called the conjugacy class of x and it is denoted by $[x]$. It was shown in [5] and [3] that

$$\text{(1)} \qquad \text{“}n(x) = n(y), t(x) = t(y)\text{”} \Leftrightarrow \text{“}[x] = [y]\text{”}$$

and this property will be used frequently in this paper.

$\mathcal{A} \cup \{\infty\}$ is denoted by $\hat{\mathcal{A}}$, $\infty \notin \mathcal{A}$ and the transformations $t_u(x) = x + u$, $r_u(x) = xu$, $l_u(x) = ux$, $i(x) = x^{-1}, \infty \longleftrightarrow 0$, which are defined on $\hat{\mathcal{A}}$, are called translation with u (translation), right multiplication with u (right multiplication), left multiplication with u (left multiplication), and inverse transformation respectively.

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2 Definition and properties of the cross-ratio of points

Let M be a Moufang plane which is coordinated with an alternative field $\mathcal{A}$ such that $char\ \mathcal{A} \neq 2$. Any point of $l = [0,0]$ which is different from (0) is denoted by $X = (x,0) = x$ and $(0) := \infty$, $x, 0 \in \mathcal{A}$, $\infty \notin \mathcal{A}$. Let $A = (a,0)$, $B = (b,0)$, $C = (c,0)$, $D = (d,0)$ be four arbitrary affine points of l. The cross-ratio $(A, B : C, D)$ of A, B, C, D is defined by

$$(A, B : C, D) = \left[\left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}(a-c)\right)\right] = (a, b : c, d)$$

and $[x]$ denotes the conjugacy class $\{\ y^{-1}xy \mid y \in \mathcal{A}\}$ of x. If one of the A, B, C, D is ∞, then omit the factors containing it.

The proofs of Theorem 2.1 and Theorem 2.2 can be found in [3] with some calculating errors.

Theorem 2.1. *If a, b, c, d are distinct elements of $\mathcal{A}$, then*

$$(a, b : c, d) = \left[\left((a-b)^{-1} - (a-d)^{-1}\right)\left((a-b)^{-1} - (a-c)^{-1}\right)^{-1}\right].$$

Proof. Since $n(x)$ is multiplicative and $t(x)$ is associative

$$u' = \left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}(a-c)\right)$$

is conjugate to

$$u = \left(\left((a-d)^{-1}(b-d)\right)(b-c)^{-1}\right)(a-c).$$

Thus $(a-d)^{-1}(b-d) = \left(u(a-c)^{-1}\right)(b-c)$ so

$$\left((a-d)^{-1}(b-d)\right)(a-b)^{-1} = \left(\left(u(a-c)^{-1}\right)(b-c)\right)(a-b)^{-1} \tag{2}$$

The left hand side of this equation can be viewed as

$$\begin{aligned}\left((a-d)^{-1}(b-d)\right)(a-b)^{-1} &= \left((a-d)^{-1}((a-d)-(a-b))\right)(a-b)^{-1}\\ &= \left(1-(a-d)^{-1}(a-b)\right)(a-b)^{-1}\\ &= (a-b)^{-1} - \left((a-d)^{-1}(a-b)\right)(a-b)^{-1}\\ &= (a-b)^{-1} - (a-d)^{-1}\end{aligned}$$

substituting this into (2), we have

$$\begin{aligned}&(a-b)^{-1} - (a-d)^{-1} = \left(\left(u(a-c)^{-1}\right)(b-c)\right)(a-b)^{-1}\\ &\Rightarrow \left(\left(\left((a-b)^{-1} - (a-d)^{-1}\right)(a-b)\right)(b-c)^{-1}\right)(a-c) = u\\ &\Rightarrow [u] = \left[\left(\left(\left((a-b)^{-1} - (a-d)^{-1}\right)(a-b)\right)(b-c)^{-1}\right)(a-c)\right].\end{aligned}$$

And from (1),

$$\begin{aligned}[u] &= \left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left((a-b)\left((b-c)^{-1}(a-c)\right)\right)\right] \\ &= \left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left(\left((a-c)^{-1}(b-c)\right)(a-b)^{-1}\right)^{-1}\right] \\ &= \left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left(\left((a-c)^{-1}((a-c)-(a-b))\right)(a-b)^{-1}\right)^{-1}\right] \\ &= \left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left(\left(1-(a-c)^{-1}(a-b)\right)(a-b)^{-1}\right)^{-1}\right] \\ &= \left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left((a-b)^{-1}-\left((a-c)^{-1}(a-b)\right)(a-b)^{-1}\right)^{-1}\right],\end{aligned}$$

so

$$[u]=[u']=\left[\left((a-b)^{-1}-(a-d)^{-1}\right)\left((a-b)^{-1}-(a-c)^{-1}\right)^{-1}\right]$$

is obtained. □

If $[x]^{-1}$ and $1-[x]$ are defined to be $\left[x^{-1}\right]$ and $[1-x]$, respectively, the following statements are valid and cross-ratio is invariant under the identical permutation and (12)(34), (13)(24), (14)(23). Let a,b,c,d distinct elements of $\hat{\mathcal{A}}$ and $w\in(a,b:c,d)$. Then

$$(a,b:c,d)=(b,a:c,d)^{-1}, 1-(a,b:c,d)=(a,c:b,d)$$

$$(a,b:c,d)=[w], \qquad (b,a:c,d)=[w]^{-1}, \qquad (a,c:b,d)=1-[w]$$

$$(b,c:a,d)=1-[w]^{-1}, \qquad (c,a:b,d)=[1-w]^{-1}, (c,b:a,d)=\left[1-w^{-1}\right]^{-1}.$$

Theorem 2.2. *Let $r\in\mathcal{A}, r\neq 0, r\neq 1$. If $a,b,c\in\hat{\mathcal{A}}$ are distinct. Then there exists $d\in\hat{\mathcal{A}}$ such that $(a,b:c,d)=[r]$. If r is in the center $\mathcal{Z}$ of $\mathcal{A}$, then d is unique.*

Proof. Suppose first that $a,b,c\in\mathcal{A}$. Then we must determine $d\in\mathcal{A}$ such that

$$(a,b:c,d)=\left[\left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}(a-c)\right)\right]=[r].$$

Let $u=(b-c)^{-1}(a-c)$. For any $s\in[r]$, $s\neq u$, $d=\left(a\left(su^{-1}\right)-b\right)\left(u\left(s-u\right)^{-1}\right)$, if $s=u=(b-c)^{-1}(a-c)$, then $d=\infty$ is the desired element of $\mathcal{A}$.

If $s\in[r]$ and $c=\infty$, since $s\neq 1$, $d=(as-b)(s-1)^{-1}$ satisfies $(a,b:c,d)=[r]$. The remaining cases $b=\infty$, $a=\infty$ are reduced to the case $c=\infty$.

If $r\in\mathcal{Z}$, so $[r]=\{r\}$, $s=r$ and the solution $d\in\hat{\mathcal{A}}$ is unique. □

Now we give a theorem related to the transformations preserving cross-ratio

Theorem 2.3. *If $\sigma=t_u, r_u, i$ or γ, then $(a,b:c,d)=(\sigma(a),\sigma(b):\sigma(c),\sigma(d))$ for all $a,b,c,d\in\hat{\mathcal{A}}$ (see [3]).*

[redacted]

Fig. 1

A quadruple a, b, c, d of elements of $\hat{\mathcal{A}}$ is said to be *harmonic* if $(a, b : c, d) = [-1]$ and we let $h(a, b, c, d)$ represents the statement: a, b, c, d *are harmonic.*

It is trivial from Theorem 2.2 that if a, b, c are different elements of $\hat{\mathcal{A}}$, there is a unique element $d \in \hat{\mathcal{A}}$ such that $h(a, b, c, d)$ and the relation $h(a, b, c, d)$ is invariant under the elements of group which is generated by the permutations (12), (13), (24). Also when $\sigma = t_u, r_u, i$ or γ and $h(a, b, c, d)$ then by Theorem 2.3 it is easy to see that $h(\sigma(a), \sigma(b), \sigma(c), \sigma(d))$. And since $l_u = ir_{u^{-1}}i$, the transformation l_u also preserves harmonicity.

If A, B and C are distinct points of the line $l = [0, 0]$ and D is constructed from A, B, C, P_1 and P_2 via Fig. 1, then the point D is uniquely determined by A, B, C (i.e. independent of the choice of P_1 and P_2). The points A, B, C, D of l are called in *harmonic position* if they can be embedded as in Fig. 1. The distinct points a, b, c, d (possibly ∞) are in harmonic position if and only if $h(a, b, c, d)$ (see [3]).

In this paper we denote by $G_i(l)$ the group of all projectivities of l and by $T(l)$ the group which is generated by t_u, r_u and i. Since the transformation $\varphi : l \to l$ given by

$$\varphi = \begin{cases} r_{(b-a)^{-1}} t_{-a} & \text{if } c = \infty \\ r_{(b^{-1}-a^{-1})^{-1}} t_{-a^{-1}} i & \text{if } c = 0 \\ r_{\left((b-c)^{-1}-(a-c)^{-1}\right)^{-1}} t_{-(a-c)^{-1}} i t_{-c} & \text{otherwise} \end{cases}$$

transforms the points a, b, c to $0, 1, \infty$, respectively, $T(l\ )$ is transitive on ordered triples of distinct points of l.

In [3] Theorem 7, Ferrar shows that $G_i(l\ ) = T(l\ )$.

The cross-ratio definition of different points of l is extended to whole Moufang plane in [2].

3 The Cross-ratio of concurrent lines

Let $\mathcal{L}_{(0,0)}$ denote the set of lines which are passing through the point $(0, 0)$ in Moufang plane $\mathcal{M}$ which is coordinated by an alternative field $\mathcal{A}$ with $char\mathcal{A} \neq 2$. In this case,

$$\mathcal{L}_{(0,0)} = \{m := [m, 0] \mid m \in \mathcal{A}\} \cup \{\tilde{\infty} := [0]\}.$$

If p, q, r, s are distinct elements of $\mathcal{L}_{(0,0)}$, different from $\tilde{\infty}$, we denote the cross-ratio $\langle p, q : r, s\rangle$ as a conjugacy class as follows:

$$\langle p, q : r, s\rangle = \left[\left((p-s)^{-1}(q-s)\right)\left((q-r)^{-1}(p-r)\right)\right].$$

If one of the lines p, q, r, s is $\tilde{\infty}$, then omit the factors containing it.

After this definition every result about the cross-ratio of points on $l = [0,0]$ can be adapted to the cross-ratio of lines of $\mathcal{L}_{(0,0)}$ easily. For instance, if $p, q, r, s \in \mathcal{L}_{(0,0)}$ are different lines and $\langle p, q : r, s \rangle = [u]$ then the following equalities are valid:

$$\langle p, q : r, s \rangle = \left[\left((p-q)^{-1} - (p-s)^{-1} \right) \left((p-q)^{-1} - (p-r)^{-1} \right)^{-1} \right]$$

$$\langle p, q : r, s \rangle = \langle q, p : r, s \rangle^{-1}, \qquad 1 - \langle p, q : r, s \rangle = \langle p, r : q, s \rangle,$$

$$\langle p, q : s, r \rangle = [u]^{-1}, \ \langle p, r : q, s \rangle = [1-u], \ \langle p, r : s, q \rangle = \left[(1-u)^{-1} \right],$$

$$\langle p, s : r, q \rangle = \left[-u(1-u)^{-1} \right], \qquad \langle p, s : q, r \rangle = \left[1 - u^{-1} \right]$$

and elements of the group which is generated by the identic permutation and $(12)(34)$, $(13)(24)$, $(14)(23)$ preserve the cross-ratio of lines.

Theorem 3.1. *Let $u \in \mathcal{A}$, $u \neq 0, u \neq 1$. If $p, q, r, s \in \mathcal{L}_{(0,0)}$ are different elements, then there exist an $s \in \mathcal{L}_{(0,0)}$ such that $\langle p, q : r, s \rangle = [u]$ and if u is an element of center $\mathcal{Z}$ of $\mathcal{A}$, then s is unique.*

The proof of this theorem can be done by means of the process in the proof of Theorem 2.2.

Definition 3.1. A quadruple p, q, r, s of elements of $\mathcal{L}_{(0,0)}$ is said to be *harmonic* if $\langle p, q : r, s \rangle = [-1]$ and we let $H(p, q, r, s)$ for "the lines p, q, r, s are called *harmonic*". The distinct lines p, q, r, s are called to be in *harmonic position* if any quadrilateral l_1, l_2, l_3, l_4 exists as in Fig. 2.

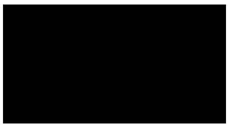

Fig. 2

The transformations

$$t_u : [x, 0] \to [x + u, 0], \quad \tilde{\infty} \to \tilde{\infty}$$

$$l_u : [x, 0] \to [ux, 0], \tilde{\infty} \to \tilde{\infty}$$

$$r_u : [x, 0] \to [xu, 0], \tilde{\infty} \to \tilde{\infty}$$

and

$$i : [x, 0] \to \left[x^{-1}, 0 \right], [0, 0] \to \tilde{\infty}$$

which are defined on $\mathcal{L}_{(0,0)}$ are called *translation* (by u), *left multiplication* (by u), *right multiplication* (by u) and *inverse transformations* respectively.

Now we can state a theorem which can be proved by using the methods of the proof of the Theorem 2.6 in [3].

Theorem 3.2. *Distinct lines p, q, r, s are in harmonic position iff $H(p,q,r,s)$.*

Proof. Let the lines p, q, r, s be in harmonic position with respect to the quadrilateral l_1, l_2, l_3, l_4 (Fig. 2). In this case without lose of generality, we may assume $l_1 = [\infty]$ and $l_2 = [p, 1]$ (since, if P is a point and l_1 and l_2 are lines not incident with P, then there is an elation fixing all lines passing through P and mapping l_1 to l_2). So we obtain

$$q \wedge l_2 = [q, 0] \wedge [p, 1] = \left((q-p)^{-1}\,,\ q\,(q-p)^{-1}\right),$$

$$r \wedge l_1 = [r, 0] \wedge [\infty] = (r)$$

and

$$l_3 = (q \wedge l_2) \vee \ (r \wedge l_1) = \left[r,\ (q-r)\,(q-p)^{-1}\right].$$

And since $s \wedge l_2 = \left((s-p)^{-1}\,,\ s\,(s-p)^{-1}\right)$ any line $[x, y]$ passing through $s \wedge l_2$ has a form

$$y = (s-x)\,(s-p)^{-1} \tag{3}$$

Similarly any line $[x, y]$ passing through $p \wedge l_3$ has the form

$$y = (p-x)\left((p-r)^{-1}\left((q-r)\,(q-p)^{-1}\right)\right) \tag{4}$$

and any line $[x, y]$ passing through $q \wedge l_1 = (q)$ has the form

$$x = q \tag{5}$$

Since $s \wedge l_2,\ p \wedge l_3$ and $q \wedge l_1$ are collinear, from the equations (3), (4) and (5)

$$(s-q)\,(s-p)^{-1} = (p-q)\left((p-r)^{-1}\left((q-r)\,(q-p)^{-1}\right)\right)$$

is obtained. Then

$$(p-q)^{-1}\left((s-q)\,(s-p)^{-1}\right) = (p-r)^{-1}\left((q-r)\,(q-p)^{-1}\right),$$

and substituting $s - q = (s-p) + (p-q)$ and $q - r = (q-p) + (p-r)$ by simple calculations we arrive at the equality

$$\left((p-q)^{-1} - (p-s)^{-1}\right)\left((p-q)^{-1} - (p-r)^{-1}\right)^{-1} = -1,$$

which is equivalent to $H(p,q,r,s)$.

If $s = \tilde{\infty}$ we utilize the same computations with the exception $s \wedge l_2 = (0, 1)$. In this case any line passing through $s \wedge l_2$ has the form $y = 1$ and using (4), (5) we obtain

$$1 = (p-q)\left((p-r)^{-1}\left((q-r)\,(q-p)^{-1}\right)\right).$$

So $(p-r)\,(p-q)^{-1} = (q-r)\,(q-p)^{-1}$ and then $(q-r)^{-1}\,(p-r) = -1$. Therefore $H(p,q,r,s)$. Other cases (i.e. $r = \tilde{\infty}$ or $q = \tilde{\infty}$ or $p = \tilde{\infty}$) can be shown by similar calculations, and the proof is complete, the converse following from Theorem 3.1.

□

Lemma 3.3. *The transformations t_u l_u $(u \neq 0)$ and i are projectivities of $\mathcal{L}_{(0,0)}$.*
Proof. By the calculations,

$$t_u = \sigma(0, [\infty], (1,0))\, \sigma((1,0), \tilde{\infty}, (1,-u))\, \sigma((1,-u), [\infty], 0)$$
$$l_u = \sigma(0, [\infty], (1,1))\, \sigma\left((1,1), [0,0], \left(1, u^{-1}\right)\right) \sigma\left(\left(1, u^{-1}\right), [\infty], 0\right)$$
$$i = \sigma(0, [\infty], (1,1))\, \sigma((1,1), [0,0], (1))\, \sigma((1), [1], 0)$$

are obtained and these complete the proof. □

We denote by $T\left(\mathcal{L}_{(0,0)}\right)$ the group of transformations of $\mathcal{L}_{(0,0)}$ generated by $\{t_u\} \cup \{l_u\} \cup \{i\}$. Note that since $r_u = i l_{u^{-1}} i$ the transformation

$$r_u : [x, 0] \to [xu, 0],\ \tilde{\infty} \to \tilde{\infty}$$

is a projectivity of $\mathcal{L}_{(0,0)}$ and element of $T\left(\mathcal{L}_{(0,0)}\right)$. And we denote by $G\left(\mathcal{L}_{(0,0)}\right)$ the group of all projectivities of $\mathcal{L}_{(0,0)}$.
Lemma 3.4. *$T\left(\mathcal{L}_{(0,0)}\right)$ is a triply transitive subgroup of $G\left(\mathcal{L}_{(0,0)}\right)$.*
Proof. By Lemma 3.3, $T\left(\mathcal{L}_{(0,0)}\right)$ is subgroup of $G\left(\mathcal{L}_{(0,0)}\right)$. Therefore it must be shown that there exists a transformation Ψ in $T\left(\mathcal{L}_{(0,0)}\right)$ which transforms the distinct lines $a, b, c \in \mathcal{L}_{(0,0)}$ to $0, 1, \tilde{\infty}$, respectively. We give the proof in three cases:

Case 1: If $c = \tilde{\infty}$, then $\Psi = l_{(b-a)^{-1}} t_{-a}$ since

$$l_{(b-a)^{-1}} t_{-a}(a) = l_{(b-a)^{-1}}(a-a) = l_{(b-a)^{-1}}(0) = 0$$
$$l_{(b-a)^{-1}} t_{-a}(b) = l_{(b-a)^{-1}}(b-a) = (b-a)^{-1}(b-a) = 1$$
$$l_{(b-a)^{-1}} t_{-a}(c) = l_{(b-a)^{-1}} t_{-a}(\tilde{\infty}) = l_{(b-a)^{-1}}(\tilde{\infty}) = \tilde{\infty}.$$

Case 2: If $c = 0$, then applying i we can return to the case 1.

Case 3: If $c \neq \tilde{\infty}$ and $c \neq 0$, then applying t_{-c} we can return to the case2. □.
Theorem 3.5. $G\left(\mathcal{L}_{(0,0)}\right) = T\left(\mathcal{L}_{(0,0)}\right)$
Proof. From Lemma 3.4 we must only show that $G\left(\mathcal{L}_{(0,0)}\right) \subset T\left(\mathcal{L}_{(0,0)}\right)$. Let

$$\mu = \prod_{i=0}^{n-1} \sigma(P_{i+1}, l_i, P_i),\ P_0 = P_n = (0,0).$$

There is a line l such that l$\neq l_i$ and l$\notin P_i$ for all i. Then

$$\mu = \prod_{i=0}^{n-1} \sigma((0,0), l, P_{i+1})\, \sigma(P_{i+1}, l_i, P_i)\, \sigma(P_i, l, (0,0))$$

and for this reason it suffices to show that

$$\sigma((0,0), l, P_{i+1})\, \sigma(P_{i+1}, l_i, P_i)\, \sigma(P_i, l, (0,0)) \in T\left(\mathcal{L}_{(0,0)}\right).$$

Thus we thus consider a general element

$$\sigma((0,0), l, P'')\, \sigma(P'', d, P')\, \sigma(P', l, (0,0)). \tag{6}$$

There are two cases, $d \notin (0,0)$ and d$\in (0,0)$.

Case 1: Let d$\notin$ $(0,0)$. Then

$$\sigma\left(P'',d,P'\right)=\sigma\left(P'',d,(0,0)\right)\sigma\left((0,0),d,P'\right)$$

and (6) becomes to

$$\sigma\left((0,0),l,P''\right)\sigma\left(P'',d,(0,0)\right)\sigma\left((0,0),d,P'\right)\sigma\left(P',l,(0,0)\right)$$

of which first two and last two factor forms to

$$\eta=\sigma\left((0,0),l,P''\right)\sigma\left(P'',d,(0,0)\right).$$

Thus it suffices to show that $\eta\in T\left(\mathcal{L}_{(0,0)}\right)$. Since $T\left(\mathcal{L}_{(0,0)}\right)$ is triply transitive on $\mathcal{L}_{(0,0)}$ there exists a $\sigma\in T\left(\mathcal{L}_{(0,0)}\right)$ such that $\sigma(r)=0,\ \sigma(s)=1$ and $\sigma(t)=\tilde{\infty}$. And now it suffices to show that $\eta\sigma\in T\left(\mathcal{L}_{(0,0)}\right)$. Thus we obtain a new mapping defined by the Fig. 3 where $r=0,\ s=1$ and $t=\tilde{\infty}$.

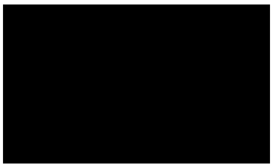

Fig. 3

This mapping will not be altered if the entire configuration in Figure 3 is acted upon by an elation with center $(0,0)$ mapping l to $[\infty]$. So without lose of generality we can take l= $[\infty]$. Thus, since l= $[\infty]$, s and d are concurrent, $d=[1,q]$, and since $P''\in t=[0]=\tilde{\infty}$, $P''=(0,s)$. Let $x=[a,0]$. Then

$$\begin{aligned} u &= (x\wedge d)\vee P''=([a,o]\wedge[1,q])\vee(0,s)\\ &= \left((a-1)^{-1}q,\ a\left(a-1\right)^{-1}q\right)\vee(0,s)\\ &= \left[a-s\left(q^{-1}a\right)+sq^{-1},\ s\right] \end{aligned}$$

and

$$\begin{aligned} u\wedge l &= \left[a-s\left(q^{-1}a\right)+sq^{-1},\ s\right]\wedge[\infty]\\ &= \left(a-s\left(q^{-1}a\right)+sq^{-1}\right). \end{aligned}$$

Finally

$$\begin{aligned} \eta(x) &= (0,0)\vee(u\wedge l)=(0,0)\vee\left(a-s\left(q^{-1}a\right)+sq^{-1}\right)\\ &= \left[a-s\left(q^{-1}a\right)+sq^{-1},\ 0\right]. \end{aligned}$$

Consequently, we have

$$\eta\left(x\right)=t_{sq^{-1}}l_{s}l_{s^{-1}-q^{-1}}\left(x\right).$$

which is the desired result.

Fig. 4

Case 2: Let $d \in (0,0)$. Then we must show that $\mu \in T\left(\mathcal{L}_{(0,0)}\right)$, where

$$\mu = \sigma\left((0,0), l, P''\right) \sigma\left(P'', d, P'\right) \sigma\left(P', l, (0,0)\right)$$

is the mapping defined by the Fig. 4. Because of the same reasons with Case 1, to take $d = 0$, $s = 1$, $t = \tilde{\infty} = [0]$ and $l = [\infty]$ is not a lose generality. Since $P' \in t = [0]$, $P' = (0,s)$ and since $P'' \in s = 1$, $P'' = (q,q)$. If $x = [a,0]$, $e = P' \vee (x \wedge l) = (0,s) \vee (a) = [a,s]$ and

$$f = P'' \vee (d \wedge e) = (q,q) \vee \left(-a^{-1}s, 0\right) = \left[q\left(q + a^{-1}s\right)^{-1}, \left(q\left(q + a^{-1}s\right)^{-1}\right)\left(a^{-1}s\right)\right].$$

So

$$\mu(x) = (f \wedge l) \vee (0,0) = \left(q\left(q + a^{-1}s\right)^{-1}\right) \vee (0,0) = \left[q\left(q + a^{-1}s\right)^{-1}, 0\right]$$

and therefore we have the desired result

$$\mu(x) = l_q i t_q r_s i(x).$$

□

Now we can extend the definition of the cross-ratio of lines which are passing through $(0,0)$ to whole Moufang plane as follows:

Let p, q, r, s be distinct lines passing through the point P. There are three cases:

i) If $P \notin [\infty]$, considering the perspectivity $\sigma(0, [\infty], P)$, we have

$$\langle p, q : r, s \rangle = \langle p', r' : s', q' \rangle$$

where

$$p' = \sigma(p) = (p \wedge [\infty]) \vee 0, \; q' = \sigma(q) = (q \wedge [\infty]) \vee 0,$$
$$r' = \sigma(r) = (r \wedge [\infty]) \vee 0, \; s' = \sigma(s) = (s \wedge [\infty]) \vee 0.$$

ii) If $P \in [\infty]$, $P \neq (\infty)$, then applying the perspectivity $\sigma(0, [1], P)$, we have

$$\langle p, q : r, s \rangle = \langle p', r' : s', q' \rangle,$$

where

$$p' = \sigma(p) = (p \wedge [1]) \vee 0, \; q' = \sigma(q) = (q \wedge [1]) \vee 0,$$
$$r' = \sigma(r) = (r \wedge [1]) \vee 0, \; s' = \sigma(s) = (s \wedge [1]) \vee 0.$$

iii) If $P \in [\infty]$, $P = (\infty)$, then considering the perspectivity $\sigma(0, [0,1], P)$, we have

$$\langle p, q : r, s \rangle = \langle p', r' : s', q' \rangle,$$

where

$$p' = \sigma(p) = (p \wedge [0,1]) \vee 0, \; q' = \sigma(q) = (q \wedge [0,1]) \vee 0,$$
$$r' = \sigma(r) = (r \wedge [0,1]) \vee 0, \; s' = \sigma(s) = (s \wedge [0,1]) \vee 0.$$

Now we can give the final theorem:

Theorem 3.6. *If P, Q, R, S are distinct collinear points and p,q,r,s are distinct concurrent lines such that $P \in p,\ Q \in q, R \in r, S \in s$, then $\langle p, q : r, s \rangle = (P, Q : R, S)$.*

Proof. We can denote, by a, the line incident with all of P, Q, R, S and, by A, the point on which all of the lines p, q, r and s pass through. Then there are four cases:

Case 1: If $A = (\infty)$ and $a = [0,0]$, then p, q, r, s are in form $p = [p]$, $q = [q]$, $r = [r]$, $s = [s]$ and $P = (p,0)$, $Q = (q,0)$, $P = (r,0)$, $P = (s,0)$. If one of the lines p, q, r, s is $\tilde{\infty}$ then one of the points P, Q, R, S is (0). Thus

$$\langle p, q : r, s \rangle = \left(p^{-1}, q^{-1} : r^{-1}, s^{-1}\right) = (p, q : r, s) = (P, Q : R, S)$$

Case 2: If $A = (\infty)$ and $a \neq [0,0]$ the proof follows by case 1, considering the perspectivity $\sigma([0,0], (\infty), a)$.

Case 3: If $A \neq (\infty)$ and $(\infty) \notin a$ the proof follows by previous two cases, considering the perspectivity $\sigma((\infty), a, A)$.

Case 4: If $A \neq (\infty)$ and $(\infty) \in A$, we can take a line b such that $(\infty), A \notin b$. Considering the perspectivity $\sigma(b, A, a)$, the proof follows by case 3. □

As a consequence of the last theorem we can give the following statement:

If P, Q, R and S are distinct collinear points and p, q, r and s are distinct concurrent lines such that $P \in p,\ Q \in q, R \in r, S \in s$, then $H(p, q, r, s) = h(P, Q, R, S)$.

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