A Hyperbolic Twistor Space

David E. Blair

Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

In this lecture we introduce a twistor space over a manifold with an almost quaternionic structure of the second kind whose fibres are hyperbolic planes with a metric of constant curvature -1. We discuss almost complex structures and holomorphic functions on such a twistor space.

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This lecture is a preliminary announcement of continuing work with J. Davidov, B. Foreman and O. Mushkarov.

We begin with the following simple observation. In [8] P. Libermann introduced the notion of an *almost quaternionic structure of the second kind* (*presque quaternion-iennes de deuxième espèce*). This consists of an almost complex structure J_1 and an almost product structure, J_2 such that $J_1J_2 + J_2J_1 = 0$. Setting $J_3 = J_1J_2$ one has a second almost product structure which also anti-commutes with J_1 and J_2. Now on a manifold M with such a structure, set

$$j = y_1J_1 + y_2J_2 + y_3J_3.$$

Then j is an almost complex structure on M if and only if

$$-y_1^2 + y_2^2 + y_3^2 = -1.$$

This suggests a *hyperbolic twistor space* $\pi : Z \longrightarrow M$ with fibre the hyperbolic plane in the fibre of the subbundle E of $End(TM)$ spanned by $\{J_1, J_2, J_3\}$.

Recall that the classical twistor space over a quaternionic Kähler manifold of dimension ≥ 8 is a bundle over the manifold with the fibre being a sphere in the subbundle of the bundle of endomorphisms determined by the three underlying local almost complex structures defining the quaternionic Kähler structure.

There are a number of examples of almost quaternionic structures of the second kind including the paraquaternionic projective space as described by Blažić [1]. Under

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certain holonomy assumptions almost quaternionic structures of the second kind become paraquaternionic Kähler (see e.g. Garcia-Rio, Matsushita and Vazquez-Lorenzo [3]). Even more strongly one has neutral hyperkähler structures and Kamada [7] showed that every primary Kodaira surface admits a neutral hyperkähler structure. We remark that neutral hyperkähler surfaces are Ricci flat and self-dual (Kamada [7]).

Also the tangent bundle of a differentiable manifold carries an almost quaternionic structure of the second kind as studied by S. Ianuş and C. Udrişte [4,5]; this includes examples where the dimension of the manifold carrying the structure is not necessarily $4n$.

The most natural setting for this kind of structure is however on a manifold M of dimension $4n$ with a neutral metric g, i.e. a semi-Riemannian metric of signature $(2n, 2n)$. One reason for this is that such a metric may be given with respect to which J_1 acts as an isometry on tangent spaces and J_2, J_3 act as anti-isometries; the effect of this is that we may define three fundamental 2-forms Ω_a, $a = 1, 2, 3$, by $\Omega_a(X, Y) = g(X, J_aY)$. Riemannian metrics can be chosen such that $g(J_aX, J_aY) = g(X, Y)$, but then Ω_2 and Ω_3 are symmetric tensor fields instead of 2-forms.

The neutral metric g induces a metric on the fibres of E by $\frac{1}{4n} trA^tB$ where A and B are endomorphisms of T_pM and A^t is the adjoint of A with respect to g. This metric on the fibre is of signature $(+--)$, the norm of J_1 being $+1$ and the norms of J_2 and J_3 being -1.

Alternatively one may choose a Lorentz metric $\langle\, ,\rangle$ directly on the fibres of E such that $\langle J_1, J_1\rangle = -1$, $\langle J_2, J_2\rangle = +1$, $\langle J_3, J_3\rangle = +1$. This metric is of signature $(-++)$ and has the advantage of inducing immediately a Riemannian metric of constant curvature -1 on the hyperbolic planes defined by $-y_1^2 + y_2^2 + y_3^2 = -1$, $y_1 > 0$, in each fibre. We adopt this metric for its geometric attractiveness but keep its negative in mind.

We will also use the following notation. For the metric $\langle\, ,\rangle$ on the fibres of E we set $\epsilon_1 = -1$ and $\epsilon_2 = \epsilon_3 = +1$. For the neutral metric g of on the base, we set $\varepsilon_i = \pm 1$ according to the signature $(+\cdots+-\cdots-)$. Further, denoting also by π the projection of E onto M, if x_i are local coordinates on M, set $q_i = x_i \circ \pi$. For a section of E we denote its vertical lift to E as a vector field by the superscript ^{v} and frequently utilize the natural identifications of J_a^v with J_a itself and with $\frac{\partial}{\partial y_a}$ in terms of the fibre coordinates y_1, y_2, y_3.

We remark that for Z as a hypersurface in E with the $(-++)$ metric, the position vector is a time-like normal and the corresponding Weingarten map acts as the identity on vertical vectors and annihilates horizontal vectors.

As with the theory of twistor spaces over quaternionic Kähler manifolds, the theory of hyperbolic twistor spaces over paraquaternionic Kähler manifolds of dimension ≥ 8 developes nicely by virtue of the fact that the covariant derivatives of sections of the subbundle of the endomorphism bundle are again sections of the subbundle. To give this development we first need the natural machinery of horizontal lifts.

Let D denote the Levi-Civita connection of the neutral metric on M. Then the horizontal lift X^h of a vector field X to the bundle E is is given by

$$X^h = \sum_i X^i \frac{\partial}{\partial q^i} - \sum_{a,b=1}^{3} \epsilon_b y_a (\langle D_X J_a, J_b \rangle \circ \pi) \frac{\partial}{\partial y_b}.$$

It is straightforward to obtain the following at a point $\sigma \in Z \subset E$

$$[X^h, Y^h]_\sigma = [X,Y]^h_\sigma - (R_{XY}\sigma)^v.$$

For a section s of E, $[X^h, s^v] = (D_X s)^v$. Define a metric on E by $h = \pi^* g + \langle \, , \rangle$, and denote its Levi-Civita connection by $\bar{\nabla}$. Then

$$(\bar{\nabla}_{X^h} Y^h)_\sigma = (D_X Y)^h_\sigma - \frac{1}{2}(R_{XY}\sigma)^v, \quad (\bar{\nabla}_{X^h} s^v)_\sigma = \frac{1}{2}(\hat{R}_{\sigma s} X)^h + (D_X s)^v_\sigma$$

where $g(\hat{R}_{\sigma s} X, Y) = h((R_{XY}\sigma)^v, s^v)$, $(\bar{\nabla}_{s^v} X^h)_\sigma = \frac{1}{2}(\hat{R}_{\sigma s} X)^h, \quad \bar{\nabla}_{s^v} t^v = 0.$

We now define two almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on the hyperbolic twistor space Z as follows. Acting on horizontal vectors these are the same and given by $\mathcal{J}_1 X^h = \mathcal{J}_2 X^h = (jX)^h$ where as before $j = \sum y_a J_a$. For a vertical vector V tangent to Z, i.e. $\langle x, V \rangle = 0$, let

$$\mathcal{J}_1 V = (y_3 V^2 - y_2 V^3) \frac{\partial}{\partial y_1} + (y_3 V^1 - y_1 V^3) \frac{\partial}{\partial y_2} + (y_1 V^2 - y_2 V^1) \frac{\partial}{\partial y_3}$$

and let $\mathcal{J}_2 V$ be the negative of this expression. In particular $\mathcal{J}_k V = (-1)^{k-1} \sigma \times V$, $k = 1, 2$, $\sigma \in Z$ where $\times$ is the vector product determined by the paraquaternionic algebra.

Define a semi-Riemannian metric on Z by $h = \pi^* g + \langle \, , \rangle_v$, $\langle \, , \rangle_v$ being the restriction of $\langle \, , \rangle$ to the fibres (hyperbolic planes) of Z. It is easy to check that this metric is Hermitian with respect to both $\mathcal{J}_1$ and $\mathcal{J}_2$.

To get an idea as to what might be happening we first study the paraquaternionic Kähler case. The development of theory of paraquaternionic Kähler structures was carried out by Garcia-Rio, Matsushita and Vazquez-Lorenzo [3] when the dimension of the base manifold is $4n \geq 8$. As with the theory of quaternionic Kähler manifolds, dimension 4 is special. At the beginning however we will retain the 4-dimensional case and point out where the differences occur. The parallel to the present development in the quaternionic Kähler case can be found in Ishihara [6].

An almost quaternionic manifold of the second kind M of dimension $4n$ and neutral metric g is said to be *paraquaternionic Kähler* if the bundle E is parallel with respect to the Levi-Civita connection of g. This is equivalent to the existence of local 1-forms α, β and γ such that

$$\begin{aligned} D_X J_1 &= -\gamma(X) J_2 - \beta(X) J_3, \\ D_X J_2 &= -\gamma(X) J_1 - \alpha(X) J_3, \\ D_X J_3 &= -\beta(X) J_1 + \alpha(X) J_2. \end{aligned}$$

From the group theoretic point of view this structure corresponds to the linear holonomy group being a subgroup of $Sp(n, \mathbf{R}) \cdot Sp(1, \mathbf{R})$, just as a quaternionic Kähler structure corresponds to the linear holonomy group being a subgroup of $Sp(n) \cdot Sp(1)$. For $n = 1$ this is not a restriction.

Setting

$$A = 2(d\alpha - \beta \wedge \gamma), \quad B = 2(d\beta - \alpha \wedge \gamma), \quad C = 2(d\gamma + \alpha \wedge \beta)$$

one can easily obtain the following central relation of the action of the curvature tensor:

$$\begin{aligned} [R_{XY}, J_1] &= \qquad\qquad\quad -C(X,Y)J_2 - B(X,Y)J_3, \\ [R_{XY}, J_2] &= -C(X,Y)J_1 \qquad\qquad - A(X,Y)J_3, \\ [R_{XY}, J_3] &= -B(X,Y)J_1 + A(X,Y)J_2. \end{aligned}$$

The theory now develops as in the quaternionic Kähler case and we have the following result quite analogous to the classical twistor space theory. Let τ denote the scalar curvature of the base manifold.

Theorem 1. *On the hyperbolic twistor space of a paraquaternionic Kähler manifold of dimension* ≥ 8 *we have the following: The almost complex structure* $\mathcal{J}_1$ *is integrable and the almost Hermitian structure* $(\mathcal{J}_1, h)$ *is indefinite Kähler if and only if* $\tau = -4n(n+2)$. *The almost complex structure* $\mathcal{J}_2$ *is never integrable but* $(\mathcal{J}_2, h)$ *is indefinite almost Kähler if and only if* $\tau = 4n(n+2)$ *and indefinite nearly Kähler if and only if* $\tau = -2n(n+2)$.

These numbers are the negatives of what one would have in the usual twistor space over a quaternionic Kähler manifold of dimension ≥ 8. This sign change is due to our choice of metric on the fibres of E. If we take $<,>$ as the $(+--)$ metric we would have the other values, but the fibres of Z would then have a negative definite metric.

We now turn our attention to the 4-dimensional case. Let M^4 be an oriented 4-dimensional manifold with a semi-Riemannian metric g of signature $(++--)$ and $\mathbf{e}_1, \dots \mathbf{e}_4$ a local orthonormal basis with $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$ giving the orientation. The metric g induces a metric on bundle of bivectors, $\bigwedge^2 TM^4$, by

$$g(\mathbf{e}_i \wedge \mathbf{e}_j, \mathbf{e}_k \wedge \mathbf{e}_l) = \frac{1}{2} \begin{vmatrix} \varepsilon_i \delta_{ik} & \varepsilon_i \delta_{il} \\ \varepsilon_j \delta_{jk} & \varepsilon_j \delta_{jl} \end{vmatrix}, \quad \varepsilon_1 = \varepsilon_2 = 1, \ \varepsilon_3 = \varepsilon_4 = -1.$$

The Hodge star operator of the neutral metric acting on $\bigwedge^2 TM^4$ is given by

$$*(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_3 \wedge \mathbf{e}_4, \quad *(\mathbf{e}_1 \wedge \mathbf{e}_3) = \mathbf{e}_2 \wedge \mathbf{e}_4, \quad *(\mathbf{e}_1 \wedge \mathbf{e}_4) = -\mathbf{e}_2 \wedge \mathbf{e}_3.$$

Let $\bigwedge^-$ and $\bigwedge^+$ denote the subbundles of $\bigwedge^2 TM^4$ determined by the corresponding eigenvalues of the Hodge star operator. The metrics induced on $\bigwedge^-$ and $\bigwedge^+$ have signature $(+--)$.

Setting

$$\begin{aligned} s_1 &= -\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4, & \bar{s}_1 &= \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4, \\ s_2 &= -\mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4, & \bar{s}_2 &= \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4, \\ s_3 &= -\mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3, & \bar{s}_3 &= \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3, \end{aligned}$$

$\{s_1, s_2, s_3\}$ and $\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$ are local oriented orthonormal frames for $\bigwedge^-$ and $\bigwedge^+$ respectively.

Following the Riemannian construction of the twistor space of a 4-dimensional Riemannian manifold, define the *hyperbolic twistor space* $\pi : Z \longrightarrow M^4$ to be the bundle with fibres defined by $-y_1^2 + y_2^2 + y_3^2 = -1$, $y_1 > 0$, relative to the basis $\{s_1, s_2, s_3\}$ of $\bigwedge^-$.

On the other hand identifying $\{s_1, s_2, s_3\}$ with the matrices

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and noting that

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$J_i = s_i g$, $i = 1, 2, 3$ defines an almost quaternionic structure of the second kind. Specifically

$$J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and we can identify Z with the space of almost complex structures and define almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on Z as in the higher dimensional case.

The curvature operator $\mathcal{R} : \bigwedge^2 TM^4 \longrightarrow \bigwedge^2 TM^4$ admits an $SO(2,2)$-irreducible decomposition

$$\mathcal{R} = \frac{\tau}{6} I + \mathcal{B} + \mathcal{W}^+ + \mathcal{W}^-$$

similar to that in the 4-dimensional Riemannian case. Here $\mathcal{B}$ is the traceless Ricci tensor and $\mathcal{W} = \mathcal{W}^+ + \mathcal{W}^-$ is the decomposition of the Weyl conformal curvature tensor as a curvature operator on $\bigwedge^2 TM^4$ respecting $*\mathcal{W} = \mathcal{W}*$. The metric g is said to be *self-dual* if $\mathcal{W}^- = 0$.

Computing the Nijenhuis tensor of $\mathcal{J}_1$ and $\mathcal{J}_2$ we obtain the following.

Theorem 2. *On the hyperbolic twistor space of a 4-dimensional manifold with a neutral metric g we have the following: The almost complex structure $\mathcal{J}_1$ on the hyperbolic twistor space of M^4 in integrable if and only if the metric g is self-dual. The almost complex structure $\mathcal{J}_2$ is never integrable.*

When one has a development of a theory analogous to an existing one, I generally believe that the differences are more interesting than the similarities. Let us then turn to a difference between the hyperbolic twistor space and the classical one.

On the classical twistor space over a Riemannian 4-manifold with either almost complex structure, there are no non-constant holomorphic functions, even when the base manifold is non-compact [2].

Review of the proof. In the classical case, since the fibre is compact and its tangent spaces are $\mathcal{J}_k$-invariant, the restriction of a holomorphic function f on the twistor space to the fibre is a constant. Moreover $X^h f$ must also be constant on fibres.

Now given a local oriented orthonormal basis $\mathbf{e}_1, \dots \mathbf{e}_4$ about a point $p \in M$ and consider local sections of the twistor space $\{s_1, s_2, s_3\}$ defined by

$$s_1 = \mathbf{e}_1 \wedge \mathbf{e}_2 - \mathbf{e}_3 \wedge \mathbf{e}_4, \quad s_2 = \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_4, \quad s_3 = \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3.$$

Since f is holomorphic and $\mathbf{e}_k^h f$ is constant on fibres, one can compare $(\mathcal{J}_k \mathbf{e}_k^h)f = i\mathbf{e}_k^h f$ at $s_1(p)$ and $s_2(p)$ and see that df restricted to the horizontal subbundle vanishes. Therefore df vanishes on both vertical and horizontal vectors and hence is constant on Z.

In contrast we give an example of a hyperbolic twistor space with non-trivial holomorphic functions with respect to the (almost) complex structure $\mathcal{J}_1$. Let M be simply $\mathbf{R}^4 \cong \mathbf{C}^2$ with standard almost quaternionic structure of the second kind given by the matrices

$$J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and let g be the neutral metric given by $dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$. Then the twistor space $Z = \mathbf{R}^4 \times H^2$ where H^2 is the hyperboloid model of the hyperbolic plane defined by $-y_1^2 + y_2^2 + y_3^2 = -1$, $y_1 > 0$. $\mathcal{J}_1$ is integrable and its action is as follows. Since the horizontal lift of $\dfrac{\partial}{\partial x_i}$ is just $\dfrac{\partial}{\partial q_i}$, we identify x_i and q_i.

$$\begin{aligned}
\mathcal{J}_1\Big(\frac{\partial}{\partial x_1}\Big) &= y_1\Big(\frac{\partial}{\partial x_2}\Big) + y_2\Big(\frac{\partial}{\partial x_3}\Big) + y_3\Big(\frac{\partial}{\partial x_4}\Big),\\
\mathcal{J}_1\Big(\frac{\partial}{\partial x_2}\Big) &= -y_1\Big(\frac{\partial}{\partial x_1}\Big) - y_2\Big(\frac{\partial}{\partial x_4}\Big) + y_3\Big(\frac{\partial}{\partial x_3}\Big),\\
\mathcal{J}_1\Big(\frac{\partial}{\partial x_3}\Big) &= y_1\Big(\frac{\partial}{\partial x_4}\Big) + y_2\Big(\frac{\partial}{\partial x_1}\Big) + y_3\Big(\frac{\partial}{\partial x_2}\Big),\\
\mathcal{J}_1\Big(\frac{\partial}{\partial x_4}\Big) &= -y_1\Big(\frac{\partial}{\partial x_3}\Big) - y_2\Big(\frac{\partial}{\partial x_2}\Big) + y_3\Big(\frac{\partial}{\partial x_1}\Big).
\end{aligned}$$

The action of $\mathcal{J}_1$ on vertical vectors can be described by setting $U = y_2\dfrac{\partial}{\partial y_1} + y_1\dfrac{\partial}{\partial y_2}$; then

$$\mathcal{J}_1 U = -y_1 y_3 \frac{\partial}{\partial y_1} - y_2 y_3 \frac{\partial}{\partial y_2} + (1 + y_3^2)\frac{\partial}{\partial y_3}.$$

The map φ of H^2 onto the unit disk Δ in the $y_1 = 0$ plane given by

$$\varphi(y_1, y_2, y_3) = \frac{y_2 + iy_3}{1 + y_1}$$

maps the hyperboloid model H^2 of the hyperbolic plane onto the Poincaré disk model Δ. It is easy to check that $(U + i\mathcal{J}_1 U)\varphi = 0$ and hence that φ is a biholomorphism. Set

$$z = \frac{y_2 + iy_3}{1 + y_1}. \tag{1}$$

One may readily check that

$$(2) \qquad y_1 = \frac{1+|z|^2}{1-|z|^2}, \quad y_2 = \frac{z+\bar{z}}{1-|z|^2}, \quad y_3 = \frac{z-\bar{z}}{i(1-|z|^2)}.$$

Now the restriction of any $\mathcal{J}_1$-holomorphic function to the fibre is a holomorphic function on H^2. By the above identification of H^2 and Δ, we may identify Z with $\mathbf{C}^2 \times \Delta$. We will use the notation $x = (x_1, x_2, x_3, x_4)$ as well as $u = x_1 + ix_2$, $v = x_3 + ix_4$.

Suppose now that $f(x, z)$ is a $\mathcal{J}_1$-holomorphic function on Z, then we may write f as

$$f(x, z) = \sum_{k=0}^{\infty} f_k(x) z^k.$$

Moreover f satisfies

$$(3) \qquad \Big(\frac{\partial}{\partial x_i}\Big) f + i\Big(\mathcal{J}_1 \frac{\partial}{\partial x_i}\Big) f = 0.$$

Using (1) and (2), (3) yields the following conditions:

$$(4) \qquad \frac{\partial f_0}{\partial x_1} + i\frac{\partial f_0}{\partial x_2} = 0, \quad \frac{\partial f_0}{\partial x_3} + i\frac{\partial f_0}{\partial x_4} = 0$$

$$(5) \quad \frac{\partial f_{k+1}}{\partial x_1} + i\frac{\partial f_{k+1}}{\partial x_2} = -i\Big(\frac{\partial f_k}{\partial x_3} - i\frac{\partial f_k}{\partial x_4}\Big), \quad \frac{\partial f_{k+1}}{\partial x_3} + i\frac{\partial f_{k+1}}{\partial x_4} = -i\Big(\frac{\partial f_k}{\partial x_1} - i\frac{\partial f_k}{\partial x_2}\Big),$$

$k = 0, 1, 2, \ldots$.

Equation (4) shows that f_0 is a holomorphic function of u and v and (5) takes the form

$$(6) \qquad \frac{\partial f_{k+1}}{\partial \bar{u}} = -i\frac{\partial f_k}{\partial v}, \quad \frac{\partial f_{k+1}}{\partial \bar{v}} = -i\frac{\partial f_k}{\partial u}, \; k = 0, 1, 2, \ldots$$

Taking $f_0 = \alpha u + \beta v$ for complex constants α, β, $f_1 = -i(\beta\bar{u} + \alpha\bar{v})$ and the remaining f_k equal to zero, the conditions (6) are satisfied and hence

$$(7) \qquad f(x, z) = f_0 + f_1 z = \alpha u + \beta v - iz(\beta\bar{u} + \alpha\bar{v})$$

is a $\mathcal{J}_1$-holomorphic function. Also z itself is $\mathcal{J}_1$-holomorphic, though not of the form (7).

Theorem 3. *$(Z(\mathbf{R}^4), \mathcal{J}_1)$ is biholomorphic to $\mathbf{C}^2 \times \Delta$ with the standard complex structure.*

Proof. Set

$$F_1(x, z) = u - iz\bar{v}, \quad F_2(x, z) = v - iz\bar{u}, \quad F_3(x, z) = z$$

and define $F : \mathbf{R}^4 \times \Delta \longrightarrow \mathbf{C}^2 \times \Delta$ by

$$F(x, z) = (F_1(x, z), F_2(x, z), F_3(x, z)).$$

This mapping is $\mathcal{J}_1$-holomorphic and is 1:1 as can be seen by noting that if (w_1, w_2, ζ) is in $\mathbf{C}^2 \times \Delta$, then

$$u - iz\bar{v} = w_1, \quad v - iz\bar{u} = w_2, \quad z = \zeta$$

has a unique solution (u, v, z). Indeed

$$\bar{u} + i\bar{z}v = \bar{w}_1, \quad -iz\bar{u} + v = w_2$$

has a unique solution since $\begin{vmatrix} 1 & i\bar{z} \\ -iz & 1 \end{vmatrix} = 1 - |z|^2 > 0.$

As a consequence we have the following result.

Theorem 4. *Any $\mathcal{J}_1$-holomorphic function on $(Z(\mathbf{R}^4), \mathcal{J}_1)$ has the form*

$$h(u - iz\bar{v}, v - iz\bar{u}, z)$$

where $h(w_1, w_2, \zeta)$ is a holomorphic function on $\mathbf{C}^2 \times \Delta$.

When the base manifold is compact, the story is not quite as good; we give the following Theorem.

Theorem 5. *Let T^4 be the 4-torus with the flat metric of signature $(2,2)$. Then $Z(T^4) \cong T^4 \times \Delta$ and any $\mathcal{J}_1$-holomorphic function $f(x,z)$ on $Z(T^4)$ has the form $f(x,z) = h(z)$ where $h(z)$ is a holomorphic function on Δ.*

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Department of Mathematics
Michigan State University
East Lansing, MI 48824