On φ-Conformally Flat Contact Metric Manifolds

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

This paper presents a study of contact metric manifolds, namely φ -conformally flat ones under the condition that the charecteristic vector field ξ belongs to the (k, μ)-nullity distribution.

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1 Introduction

The conformally symmetric K-contact manifolds was considered in [7]. It has shown that conformally symmetric K-contact manifold is locally isometric to the unit sphere. However in [8] the ξ-conformally flat contact metric manifolds are considered, and it has shown that such manifolds are φ-Einstein-Sasakian. Although, in [3] it has shown that a compact φ-conformally flat K-contact manifold with regular contact vector field is a principal S^1-bundle over an almost Kaehler space of constant halomorphic sectional curvature (see Theorem 3.2 in [3]).

In this paper we shall therefore consider φ-conformally flat contact metric manifold under the condition that the characteristic vector field ξ belongs to the (k, μ)-nullity distribution (i.e. (k, μ)-contact manifold). We generalize the (Theorem 3.2) of [3].

This paper is organized as follows; In section 2 we give some preliminaries. In section 3 we give some known results. Finally in section 4 we prove the following theorem.

Theorem 4.3. *Let M be a contact metric manifold under the condition that ξ belongs to (k, μ)-nullity condition. If M is φ-conformally flat, then M is either a η-Einstein-Sasakian manifold or a (k, μ)-contact manifold with $\mu = 1$ and $k = \dfrac{\tau}{2n} - n + 2$.*

2 Preliminaries

Let M be an m-dimensional Riemannian manifold with metric g and let $\mathcal{X}(M)$ be the Lie algebra of differentiable vector fields in M. Denote by R the Riemannian curvature tensor on M. The Ricci operator Q of (M,g) is defined by

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$$g(QX,Y)=\sum_{i=1}^{m} g(R(e_i,X)Y,e_i), \tag{2.1}$$

where $\{e_1, e_2, ..., e_m\}$ is a local orthonormal basis of vector fields on M and $X, Y \in \mathcal{X}(M)$. Weyl introduced a generalized curvature tensor on a Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric; for this reason he called it the conformal curvature tensor at the metric. Schouten [8] showed that for $m > 3$ the converse is true. If τ denotes the scalar curvature of M, the Weyl conformal curvature tensor is defined as a map

$$C : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

such that
(2.2)

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{m-2}[g(QY,Z)X + g(Y,Z)QX - g(QX,Z)Y \\ &- g(X,Z)QY + \frac{\tau}{(m-1)(m-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

for any $X, Y, Z \in T(M)$.

Now let $(M, g, \varphi, \xi, \eta)$ be a $(2n+1)$-dimensional contact metric manifold (See [2]). Then φ is a (1,1)-tensor field, ξ is the contact vector field, η is the contact 1-form and g is the associated Riemannian metric. They are related by

$$\varphi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1, \varphi\xi = 0,\ d\eta(\xi, X) = 0, \tag{2.3}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),\ \eta(X) = g(X,\xi),\ d\eta(X,Y) = g(X,\varphi Y),$$

for any vector fields $X, Y \in \mathcal{X}(M)$. Denoting by l and R Lie derivation and the curvature tensor respectively, we define the operators l and h by

$$lX = R(X,\xi)\xi,\ hX = \frac{1}{2}(L_\xi \varphi)X. \tag{2.4}$$

The (1,1) tensors h and l are self-adjoint and satisfying

$$h\xi = 0,\ l\xi = 0,\ tr(h) = tr(h\varphi) = 0,\ h\varphi = -\varphi h. \tag{2.5}$$

A contact metric manifold for which ξ is a killing vector field is called K-contact manifold. It is well known that a contact manifold is K-contact if and only if $h = 0$. Moreover on a K-contact manifold it is valid $R(X,\xi)\xi = X - \eta(X)\xi$. A contact metric manifold is said to be a *Sasakian* manifold if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(X)Y. \tag{2.6}$$

Note that a Sasakian manifold is K-contact but the converse holds if $dimM = 3$.

A contact metric manifold M is said to be η-*Einstein* if the Ricci operator satisfies

$$Q = aI_d + b\eta \otimes \xi, \tag{2.7}$$

for the smooth functions a, b on M [2].

The sectional curvature $K(\xi, X)$ of a plane section spanned by ξ and a vector X orthogonal to ξ is called a *ξ-sectional curvature* while the sectional curvature $K(X,\varphi X)$ is called a *φ-sectional curvature.*

Let M be a contact metric manifold. The *(k,μ)-nullity distribution* of M for the pair (k,μ) is a distribution
(2.8)
$$\begin{aligned} N(k,\mu): p \to N_p(k,\mu) &= \{Z \in T_pM \mid R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ &+ \mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned}$$

where $k, \mu \in \mathbf{R}$ ([2], and [3]).

So if the characteristic vector field ξ belongs to the (k,μ)-nullity distribution we have:

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.9}$$

Let C be the Weyl conformal curvature tensor of M. Since at each point $p \in M$ the tangent space $T_p(M)$ can be decomposed into the direct sum $T_p(M) = \varphi(T_p(M)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a 1-dimensional linear subspace of $T_p(M)$ generated by ξ_p, we have a map:

$$C : T_p(M) \times T_p(M) \times T_p(M) \to \varphi(T_p(M)) \oplus \mathcal{L}(\xi_p).$$

It may be natural to consider the following particular cases:

(1) $C : T_p(M) \times T_p(M) \times T_p(M) \to \mathcal{L}(\xi_p)$, that is, the projection of the image of C in $\varphi(T_p(M))$ is zero.

(2) $C : T_p(M) \times T_p(M) \times T_p(M) \to \varphi(T_p(M))$, that is, the projection of the image of C in $\mathcal{L}(\xi_p)$ is zero.

(3) $C : \varphi(T_p(M)) \times \varphi(T_p(M)) \times \varphi(T_p(M)) \to \mathcal{L}(\xi_p)$, that is, when C is restricted to $\varphi(T_p(M)) \times \varphi(T_p(M)) \times \varphi(T_p(M))$, the projection of the image of C in $\varphi(T_p(M))$ is zero [3]. This condition is equivalent to

$$\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0. \tag{2.10}$$

A Riemannian manifold satisfying (2.10) is called φ-conformally flat.

The case (1) and (2) are considered in [6] and [7] respectively. The case (3) is considered in [3] for the case M is K-contact.

3 Known results

In this section we give some known results.

Proposition 3.1 (2). *Let M be a $(2n+1)$-dimensional contact metric manifold with*

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY),$$

for any vector fields X, Y. Then

$$Q\varphi - \varphi Q = 2[2(n-1) + \mu]h\varphi. \tag{3.1}$$

Proposition 3.2 (2). *Let M be a (k,μ)-contact metric manifold. Then*

i) $R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX)$,
ii) $Q\xi = 2nk\xi$.

Lemma 3.3 (2). *Let M be a (k,μ)-contact metric manifold, where $k<1$. For any vector field X, the Ricci operator Q is given by*

$$QX = [2(n-1) - n\mu]X + [2(n-1)+\mu]hX + [2(1-n) + n(2k+\mu)]\eta(X)\xi; n \geq 1.$$

Theorem 3.4 (2). *Let M be a (k,μ)-contact Riemannian manifold. Then $k \leq 1$. If $k=1$, then $h=0$ and M is a Sasakian manifold. If $k<1$, M admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$ determined by the eigenspaces of h, where $\lambda = \sqrt{1-k}$.*

In the proof of the main theorem we also use the following result.

Theorem 3.5 (4). *Let M be a $(2n+1)$-dimensional $(n>1)$ non-Sasakian (k,μ)-contact Riemannian manifold. Then M has constant φ-sectional curvature if and only if $\mu = k+1$.*

4 φ-conformally flat contact metric manifolds

In this section we generalize the following theorem;

Theorem 4.1 (3). *A compact φ-conformally flat K-contact manifold with regular contact vector field is a principal S^1-bundle over an almost Kaehler space of constant halomorphic sectional curvature.*

First we give

Lemma 4.2. *Let M be a $(2n+1)$-dimensional φ-conformally flat Riemannian manifold. If M is (k,μ)-contact, then*

$$\begin{aligned} S(Y,Z) &= \left(\frac{\tau}{2n} - k\right) g(Y,Z) + [2(2(n-1)+\mu)- \\ &- \mu(2n-1)]g(hY,Z) - \\ &- \left(\frac{\tau}{2n} - 2nk - k\right)\eta(Y)\eta(Z),\ (k \leq 1) \end{aligned} \tag{4.1}$$

Proof. It is easy to see that $\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(C(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any vector fields $X, Y, Z, W \in \mathcal{X}(M)$. So φ-conformally flat means

$$\begin{aligned} &g(R(\varphi X,\varphi Y)\varphi Z, \varphi W) = \\ &= \frac{1}{2n-1}[g(Q\varphi Y,\varphi Z)g(\varphi X,\varphi W) + g(\varphi Y,\varphi Z)g(Q\varphi X,\varphi W) - \\ &-g(Q\varphi X,\varphi Z)g(\varphi Y,\varphi W)\text{-}g(\varphi X,\varphi Z)g(Q\varphi Y,\varphi W)] \\ &-\frac{\tau}{2n(2n\text{-}1)}[g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)\text{-}g(\varphi X,\varphi Z)g(Q\varphi Y,\varphi W)]. \end{aligned} \tag{4.2}$$

Let $\{e_1, ..., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M. By using that $\{\varphi e_1,...,\varphi e_{2n},\xi\}$ is also a local orthonormal basis, if we put $X = W = e_\alpha$ in (4.2) and sum up with respect to α, then

(4.3)
$$\sum_{\alpha=1}^{2n} g(R(\varphi e_\alpha,\varphi Y)\varphi Z,\varphi e_\alpha) = \sum_{\alpha=1}^{2n}\frac{1}{2n-1}[g(Q\varphi Y,\varphi Z)g(\varphi e_\alpha,\varphi e_\alpha)$$
$$+g(\varphi Y,\varphi Z)g(Q\varphi e_\alpha,\varphi e_\alpha) - g(Q\varphi e_\alpha,\varphi Z)g(\varphi Y,\varphi e_\alpha) - g(\varphi e_\alpha,\varphi Z)g(Q\varphi Y,\varphi e_\alpha)]$$
$$-\frac{\tau}{2n(2n-1)}[g(\varphi Y,\varphi Z)g(\varphi e_\alpha,\varphi e_\alpha) - g(\varphi e_\alpha,\varphi Z)g(\varphi Y,\varphi e_\alpha)].$$

By the use of Proposition 3.2 ii) and the definition of the scalar curvature τ we have

$$\tau - 2nk = \sum_{\alpha=1}^{2n} g(Q\varphi e_\alpha,\varphi e_\alpha). \tag{4.4}$$

Substituting (4.4) into (4.3) we obtain

$$g(Q\varphi Y,\varphi Z) - g(R(\xi,\varphi Y)\varphi Z,\xi)$$
$$= \frac{1}{2n-1}[2(n-1)g(Q\varphi Y,\varphi Z) + (\tau - 2nk)g(\varphi Y,\varphi Z)]$$
$$-\frac{\tau}{2n(2n-1)}[(2n-1)g(\varphi Y,\varphi Z)].$$

The last equation can be also written as

$$g(Q\varphi Y,\varphi Z) - (2n-1)g(R(\xi,\varphi Y)\varphi Z,\xi) = (\frac{\tau}{2n} - 2nk)g(\varphi Y,\varphi Z)]. \tag{4.5}$$

Now, since M is (k,μ)-contact, then by Theorem 3.3 we find

$$g(R(\xi,\varphi Y)\varphi Z,\xi)) = kg(\varphi Y,\varphi Z) + \mu g(h\varphi Y,\varphi Z). \tag{4.6}$$

Using the relations (2.3) (2.5) we can also get

$$g(h\varphi Y,\varphi Z) = -g(hY,Z). \tag{4.7}$$

Further, by the use of (3.1) and (2.3) we obtain respectively

$$g(Q\varphi Y,\varphi Z) = g(\varphi QY,\varphi Z) + 2[2(n-1)+\mu]g(h\varphi Y,\varphi Z) \tag{4.8}$$

and

$$g(\varphi QY,\varphi Z) = g(QY,Z) - 2nk\eta(Y)\eta(Z). \tag{4.9}$$

Finally, substituting (4.8), (4.7) and (4.6) into (4.5) after some computations we find

$$g(QY,Z) - 2nk\eta(Y)\eta(Z) - 2[2(n-1)+\mu]g(hY,Z)-$$
$$-(2n-1)[k(g(Y,Z) - \eta(Y)\eta(Z)) - \mu g(hY,Z)] =$$
$$= \left(\frac{\tau}{2n} - 2nk\right)[g(Y,Z) - \eta(Y)\eta(Z)].$$

Since $S(Y,Z) = g(QY,Z)$ then the last equality can be written as (4.1). This completes the proof the Lemma.

By Lemma 4.1 we obtain the following result.

Theorem 4.3. *Let M be a contact metric manifold under the condition that ξ belongs to (k,μ)-nullity condition. If M is φ-conformally flat, then M is either a η-Einstein-Sasakian manifold or a (k,μ)-contact manifold with $\mu=1$ and $k=\frac{\tau}{2n}-n+2$.*

Proof. Since M is φ-conformally flat (k,μ)-contact manifold, then by the previous Lemma the relation (4.1) is satisfied. Also by the use of Theorem 2.3, we have $k\leq 1$.

Now, if $k=1$ then M is Sasakian and equation (4.1) reduces to

$$S(Y,Z)=(\frac{\tau}{2n}-1)g(Y,Z)-(\frac{\tau}{2n}-2n-1)\eta(Y)\eta(Z).$$

So by the use of (2.1) and (2.6) it is easy to show that M is a η-Einstein manifold.

If $k<1$, then by Lemma 3.2 we find

$$\begin{aligned} S(Y,Z) &= [2(n-1)-n\mu]g(Y,Z)+[2(n-1)+\mu]g(hY,Z) \\ &+ [2(1-n)+n(2k+\mu)]\eta(Y)\eta(Z). \end{aligned} \tag{4.10}$$

However, comparing the equations (4.9) and (4.1) we obtain the following system of equations:

$$\begin{aligned} &\frac{\tau}{2n}-k=2(n-1)-n\mu, \\ &2(2(n-1)+\mu)-\mu(2n-1)=2(n-1)+\mu, \\ &-\left(\frac{\tau}{2n}-2nk-k\right)=2(1-n)+n(2k+\mu). \end{aligned}$$

Solving this system we get $\mu=1$, $k=\frac{\tau}{2n}-n+2$. Hence by the virtue of Theorem 3.3 M must have constant φ-sectional curvature. This completes the proof of the theorem.

By Theorem 4.3 we have the following result.

Corollary. *Let M be a $(0,\mu)$-contact manifold. If M is φ-conformally flat manifold, then M has constant φ-sectional curvature.*

Proof. Let M be a $(0,\mu)$-contact φ-conformally flat manifold then by Theorem 4.3 $\mu=1$. Hence by the virtue of Theorem 3.3 M must have constant φ-sectional curvature.

References

[1] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, 509, Springer (1976).

[2] D. E. Blair, T. Koufogiorgos, Basil J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. 91 (1995), 189-214.

[3] J. L. Cabrerizo, L. M. Fernández and M. Fernández, *The structure of a class of K- contact manifolds*, Acta Math. Hungarica, 82 (4) (1999), 331-340.

[4] T. Koufogiorgos, *Contact Riemannian manifolds with constant φ-sectional curvature*, Geometry and Topology of Submanifolds VIII, World Scientific (1995), 195-197.

[5] B. J. Papantoniou, *Contact metric manifolds satisfying* $R(\xi, X) \cdot R = 0$ *and* $\xi \in (k, \mu)$*-nullity distribution*, Yokohama Math. J. 40 (1993), 149-161.

[6] G. Zhen, *On conformal symmetric K-contact manifolds*, Chinese Quart. J. of Math., 7 (1992), 5-10.

[7] G. Zhen, J.L. Cabrerizo, L.M. Fernández and M. Fernández, *On* ξ*-conformally flat contact metric manifods*, Indian J. Pure Appl. Math., 28 (1997), 725-734.

[8] J. A. Schouten, *Uber die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Maßbestimmung auf eine Mannigfaltigkeit mit euklidisher Maßbestimmung*, Math. Z., 11 (1921), 58-88.

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