

A Classification Theorem for Connections

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*Dedicated to Prof.Dr. Constantin UDRIȘTE
on the occasion of his sixtieth birthday*

Abstract

In this paper we put together ideas and results related to the geometric theory of connections, with the hope that, on such bases, the applications to Physics will become a little bit more conceptual.

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1 Introduction

The theory of connections arose in Riemannian geometry by the grace of Levi Civita and became soon a part of Physics in the works of A.Einstein, H.Weyl, E.Cartan, C.T.Yang and Mills. The geometrical spirit dominated the works of these giants and of their followers, until the hightech penetrated the field, leading to important achievements, especially in Differential Topology. These trends shadowed the geometrical intuition, but recent work, especially due to E.Witten, brought the geometrical thinking into Physics once more.

It is the first goal of this Note to remind the origins of the geometrical theory of connections, which can be found in the work of H.Poincaré, who invented covering spaces and the fundamental group.

The second goal is to generalize Poincaré's construction of universal coverings and fundamental groups. As a result, we produce *the universal principal bundle* associated with a connected manifold. This principal bundle is named universal because *every differentiable finite dimensional bundle with structure group is associated to the universal bundle.*

2 Connections on complex vector line bundles

Let M be an orientable, connected, compact manifold and consider a complex vector line bundle $L = (E, p, M)$ endowed with a connection C . Denote by $\Omega \in \Lambda^2(M)$ the curvature form of the connection C . Since Ω is closed, for each contractible neighbourhood $U \subset M$ there exists a 1-form $\varphi_U \in \Lambda^1(U)$ such that

$$\Omega|_U = d\varphi_U.$$

When \mathcal{U} is an open covering of M consisting in contractible sets with contractible double intersections $U \cap V$, $U \in \mathcal{U}$, $V \in \mathcal{U}$, we can find a system of 1-forms $\{\varphi_U\}_{U \in \mathcal{U}}$ and a system of functions $F_{UV} : U \cap V \rightarrow \mathbf{C}$ such that

$$\Omega = d\varphi_U \text{ on } U, \quad \varphi_U - \varphi_V = df_{UV} \text{ on } U \cap V.$$

Let $(h_1, \dots, h_m, k_1, \dots, k_r)$ be a system of generators of the homology group $H_1(M, \mathbf{Z})$ and denote by $x_1, \dots, x_m, y_1, \dots, y_r$ smooth closed paths representing the homology classes h_j, k_s . We can suppose that all linear relations between x_j, y_s are consequences of torsion relations

$$p_s k_s = 0, \quad s = 1, \dots, r, \quad p_s \in \mathbf{N}.$$

Denote by $c_1 = \rho(x_1), \dots, c_m = \rho(x_m)$, $d_1 = \rho(y_1), \dots, d_r = \rho(y_r)$ the (non vanishing) complex numbers representing the holonomy automorphisms corresponding to the paths x_j, y_s . According to one of the famous de Rham theorems, there exists a closed complex-valued 1-form α such that

$$\exp \left(\int_{x_j} \alpha \right) = c_j, \quad j = 1, \dots, m.$$

Then we have this almost obvious

Proposition. *The bundle L and the connection C determined, up to isomorphisms, by the system*

$$(\Omega, \alpha, x, y, d),$$

where

$$x = (x_1, \dots, x_m), \quad y = (y_1, \dots, y_r), \quad d = (d_1, \dots, d_r).$$

The numbers $(d_j)^{p_j}$ are known when Ω and y are given.

Proof. We make use of the fact that the holonomy group of the connection C is abelian and that, for each closed path x in M there exist integers $z_1, \dots, z_m, u_1, \dots, u_r$ and 2-chains $\sigma, \sigma_1, \dots, \sigma_r$ such that

$$x = z_1 x_1 + \dots + z_m x_m = u_1 y_1 + \dots + u_r y_r + d\sigma, \quad p_s y_s = d\sigma_s.$$

Then we shall have, by using Stokes' theorem,

$$(d_1)^{p_1} = \exp \left(\int_{\sigma_1} \Omega \right), \dots, (d_r)^{p_r} = \exp \left(\int_{\sigma_r} \Omega \right)$$

and the holonomy automorphism $\rho(x)$ will be represented by the complex number:

$$c(x) = \exp \left(\int_{\sigma} \Omega \right) (c_1)^{z_1} \dots (c_m)^{z_m} (d_1)^{u_1} \dots (d_r)^{u_r}.$$

The Proposition is now a direct consequence of the following general Theorem, which will be proved in the next sections:

Theorem I . A pair (E, C) , consisting in a differentiable vector bundle E and in a linear connection C in this bundle, is determined, up to isomorphisms, by the holonomy representation of the connection.

Corollary 1. When the group $H_1(M, \mathbf{Z})$ is torsion-free, the isomorphism class $[E, C]$ of the pair (E, C) is determined by the system (Ω, α, x) .

Corollary 2. When the manifold M is simply connected, the isomorphism class $[E, C]$ is determined by the curvature form Ω alone.

Remark. For each integer 2-cycle Z of M , the period $\int Z\Omega$ is an integer.

3 A generalized fundamental group

Let M be a connected differentiable manifold and let a, b be points in M . We denote by $P(a, b)$ the set of continuous paths of M with endpoints a, b and which are smooth excepting finite sets of points. Then introduce an equivalence relation in the set $P(a, b)$ by considering equivalent two paths c_1, c_2 which define the same holonomy isomorphism $h : E_a \rightarrow E_b$ for all connexions C in all bundles $E \rightarrow M$. Let $P'(a, b)$ the set of equivalence classes. There is a natural composition law for paths $P(a, b) \times P(b, q) \rightarrow P(a, q)$, which induces a multiplication $P'(a, b) \times P'(b, q) \rightarrow P'(a, q)$. Let $Q(a), Q'(a), G(a), G'(a)$ be the sets

$$Q(a) = \bigcup \{P(a, b); b \in M\}, \quad Q'(a) = \bigcup \{P'(a, b); b \in M\}$$

$$G(a) = P(a, a), \quad G'(a) = P'(a, a).$$

Then it is a question of routine to prove the following statements, by repeating the guidelines leading to Poincaré's universal coverings:

1. $G'(a)$ is a group and the multiplication of equivalence classes induces a left action

$$G'(a) \times Q'(a) \rightarrow Q'(a).$$

2. When (E, C) is a bundle-connection pair, the holonomy construction provides natural maps

$$\rho(C, a) : G'(a) \rightarrow G(E_a), \quad \lambda(C, a) : E_a \times Q'(a) \rightarrow E,$$

where $G(E_a)$ is the group of automorphisms of the fibre E_a .

3. The map $\rho(C, a)$ is a morphism of groups and is related to $\lambda(C, a)$ in a way which is expressed by the equality:

$$\lambda(C, a)(e, \omega \alpha) = (\rho(C, a)(\omega))(\lambda(C, a)(e, \alpha))$$

$$e \in E_a, \quad \omega \in G'(a), \quad \alpha \in Q'(a).$$

4. The group $G'(a)$ is canonically embedded as a dense subgroup of the projective limit of the groups $G(E_a)$ and $Q'(a)$ is canonically embedded as a dense subset of the projective limit of the bundles $E \rightarrow M$. These embeddings induce topologies on $G'(a)$ and $Q'(a)$ such that $G'(a)$ becomes a topological group acting continuously on $Q'(a)$.

5. The map

$$\lambda_{C,a} : E_a \times Q'(a) \rightarrow E$$

is onto and E arises as coset space of $E_a \times Q'(a)$ and of the equivalence relation given by

$$(e, \omega \alpha)(\rho(C, a)(\omega)(e), \alpha).$$

6. There is a canonical projection $p_a : Q'(a) \rightarrow M$ defined as follows:

$$p_a(\alpha) = b \text{ if } \alpha \in P'(a, b)$$

and the triple $\pi(M, a) = (Q'(a), p_a, M)$ is a topological, locally trivial, principal bundle with structure group $G'(a)$.

7. Every differentiable bundle ξ over M , with structure group G , is associated with the principal bundle $\pi(M, a)$ and with a continuous homomorphism $\rho : G'(a) \rightarrow G$. Such a homomorphism defines a connection on the bundle ξ .

Theorem I is a direct consequence of these properties, from the topological background view. In order to recover the differentiable structures of the bundle-connection pairs, we have to introduce differentiable structures in the triple $\pi(M, a)$.

Definition. The bundle $\pi(M, a)$ will be named *the universal bundle of the pair* (M, a) and the group $G'(a)$ will be named *the generalized fundamental group of* (M, a) .

4 Generalized connections

Let $\xi = (E, p, M)$ be a differentiable bundle with fibres $E_b = p^{-1}(b)$.

We denote by $C(M)$ the category whose objects are points of M and whose sets of morphisms are the sets $P'(a, b)$. And we denote by $C(\xi)$ the category whose objects are the fibres E_b and whose morphisms are the diffeomorphisms $f : E_a \rightarrow E_b$.

When ξ is a bundle with structure group G , each fibre E_a is endowed with a group G_a of automorphisms of E_a , which is isomorphic to the group G .

When ξ is a bundle with structure group G , we denote by $C(\xi, s)$ the subcategory of $C(\xi)$ having the same objects as $C(\xi)$ and whose morphisms $f : E_a \rightarrow E_b$ are the diffeomorphisms with the property

$$g \in G_a \Rightarrow f g f^{-1} \in G_b.$$

The categories $C(\xi)$, $C(\xi, s)$ have canonical differentiable structures.

Suppose C is a given connection on ξ . Then *the holonomy construction provides differentiable morphisms of categories*

$$F_C : C(M) \rightarrow C(\xi), C(\xi, s).$$

Conversely, suppose $F : C(M) \rightarrow C(\xi, s)$ is a given differentiable morphism. In this case, by restriction, we get a homomorphism of groups

$$\rho'_G(a) \rightarrow G_a$$

and an associated bundle η with a connection C . The bundle η is canonically isomorphic to ξ and the canonic isomorphism $\varphi : \eta \rightarrow \xi$ allows us to transport the connection C and get a connection C' on ξ . We proved:

Theorem II. *Each differentiable morphism of categories*

$$F : C(M) \rightarrow C(\xi, s)$$

defines a connection on ξ .

5 Subgroups of $G'(m)$ associated with acyclic open coverings

Let H be a set of pairs (U, h_U) the first terms of which form an open acyclic covering \mathcal{U} of M and such that each h_U is a diffeomorphism from U to the open unit ball $B = \{x \in br^n : |x| < 1\}$, where n is the dimension of M .

Let further m be a fixed point in M . For each $U \in \mathcal{U}$, we define $m_U = (h_U)^{-1}(0)$ and select a path λ_U with end points m, m_U . Then for each point $x \in U$ we denote $x_U = h_U(x)$. For $a \in B$ we denote by $c_a : [0, 1] \rightarrow B$ the straight path in B with endpoints $0, a$. When $a = x_U$, we denote the path c_a by $C_{x,U}$ and consider the path $l_{x,U} = (h_U)^{-1}c_{x_U}$.

Finally, we suppose that, for each couple (x, y) of points in M , which belong at least to a set $U \in \mathcal{U}$, we selected a path $m_{x,y}$, with endpoints x, y , depending differentiably on x and y , but not depending on U . This is always possible, using for instance a Riemannian structure on M and minimal geodesic arcs.

The paths

$$\lambda_U \in P(m, m_U), l_{x,U} \in P(m_U, x), m_{x,y} \in P(x, y)$$

define equivalence classes

$$[\lambda_U] \in P'(m, m_U), [l_{x,U}] \in P'(m_U, x), [m_{x,y}] \in P'(x, y).$$

We define further:

$$\text{for } x \in U, y \in U, \omega_{U,x,y} = [\lambda_U][l_{x,U}][l_{y,U}]^{-1}[\lambda_U]^{-1} \in G'(m)$$

$$\text{for } x \in U \cap V, \omega_{U,V,x} = [\lambda_U][l_{x,U}][l_{x,V}]^{-1}[\lambda_V]^{-1}$$

$$\text{for } x \in U, y \in U, z \in U, \omega_{U,x,y,z} = \omega_{U,x,y}\omega_{U,y,z}\omega_{U,z,x} \in G'(m).$$

Then we have the identities:

$$\omega_{U,V,y} = \omega_{U,y,x}\omega_{U,V,x}\omega_{V,x,y}$$

$$\omega_{U,x,y} = \omega_{U,y,x}$$

$$\omega_{U,V,x}\omega_{V,W,x} = \omega_{U,W,x}.$$

For each non empty double or triple intersection, let us select points

$$x_{U,V} \in U \cap V, x_{U,V,W} \in U \cap V \cap W$$

not depending on the order of the intersected sets. Denote

$$\omega_{U,V} = \omega_{U,V,x_{U,V}}, \quad \omega_{U,V,W} = \omega_{U,x_{U,V},x_{U,V,W}}.$$

Then, for $x \in U \cap V$, we have $\omega_{U,V,x} = \omega_{x,x_{U,V}}\omega_{U,V}\omega_{V,x_{U,V},x}$ and, in particular, $\omega_{U,V,x_{U,V},x_{U,V,W}} = (\omega_{U,V,W})^{-1}\omega_{U,V}\omega_{V,U,W}$. Let us denote

$$p_{U,V,W} = (\omega_{U,V,W})^{-1}\omega_{U,V}\omega_{V,U,W}.$$

Then

$$p_{U,V,W}p_{V,W,U}p_{W,U,V} = 1.$$

Let us denote by $G(m, \mathcal{U})$ the subgroup of $G'(m)$ generated by the elements $\omega_{U,x,y}$, $\omega_{U,V,x}$; the group $G(m, \mathcal{U})$ carries a differentiable structure consisting in the family of maps $f_U : U \times U \rightarrow G(m, \mathcal{U})$, $f_U(x, y) = \omega_{U,x,y}$, $f_{UV} : U \cap V \rightarrow G(m, \mathcal{U})$, $f_{UV}(x) = \omega_{U,V,x}$. The group $G(m, \mathcal{U})$ is generated by the elements $\omega_{U,x,y}$ and ω_{UV} .

6 Computation of the coefficients of a connection

Suppose it is given a linear connection in a vector bundle $\xi = (E, p, M)$ with fibre $F = E_m$. Then we have a homomorphism of groups $\rho : G'(a) \rightarrow GL(F)$ and we can consider the maps:

$$g_U : U \times U \rightarrow GL(F), \quad g_U(x, y) = \rho(\omega_{U, x, y})$$

$$g_{U, V} : U \cap V \rightarrow GL(F), \quad g_{U, V}(x) = \rho(\omega_{U, V, x}).$$

Using the holonomy along the paths λ_U , we can identify canonically each set $p^{-1}(U)$, with $U \times F$ and, considering the charts h_U , $u \in U$, the connection form $A_i dx^i$ will be defined, on each U , by the relation

$$A_i(x) = \frac{\partial g_U(x, y)}{\partial y^i} \Big|_{(x, x)}.$$

The maps $g_U, g_{U, V}$, enjoy some properties that are consequences of the relation verified by the ω 's.

Theorem II. *Let F be a real vector space of finite dimension. Then with each differentiable vector bundle-connection pair (E, C) , there exists a family of differentiable maps*

$$\mathcal{G} = (g_U : U \times U \rightarrow GL(F), g_{UV} : U \cap V \rightarrow GL(F))$$

subject to the relations

$$\begin{aligned} g_{UV}(y) &= g_U(y, x) g_{UV} g_V(x, y) \\ g_{UV}(x) g_{VW}(x) &= g_{VW}(x). \end{aligned}$$

Conversely, given a family \mathcal{G} enjoying the properties above, there exists a unique class of isomorphism of bundle-connection pairs (E, C) such that \mathcal{G} is associated with (E, C) .

When we want to compare this theorem with the Proposition given in Section 1, we recognize that the role of Ω is played by the system $\{g_U\}$, while the role of α is played by the system $\{g_{U, V}\}$.

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