

On Weak Symmetries of Kaehler Manifolds

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*Dedicated to Prof. Dr. Constantin UDRIȘTE
on the occasion of his sixtieth birthday*

Abstract

Weakly symmetric Riemannian manifolds are generalizations of the locally symmetric manifolds, spaces of recurrent curvature and pseudo symmetric manifolds. These are manifolds in which the covariant derivative ∇R of the curvature tensor R is a linear expression in R . The appearing coefficients of this expression are called associated 1-forms. They satisfy in the specified types of manifolds gradually weaker conditions. Weakly Ricci-symmetric Riemannian or Kaehler manifolds are defined by a similar representation of ∇S in place of ∇R , where S is the Ricci tensor.

We prove several relations that exist between the properties of the weakly symmetric or weakly Ricci-symmetric Kaehler manifolds and the associated 1-forms of these spaces. In these relations the Ricci tensor and its eigenvalues play the decisive role.

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1 Introduction

The notions of weakly symmetric and weakly Ricci symmetric manifolds were introduced by the first and third authors [7], [8]. A non-flat Riemannian manifold (M^n, g) ($n > 2$) is called *weakly symmetric* (denoted by $(WS)_n$) if the curvature tensor R of type $(0, 4)$ satisfies the condition

$$(1) \quad \begin{aligned} (\nabla_X R)(Y, Z, U, V) &= \alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) + \\ &+ \gamma(Z)R(Y, X, U, V) + \delta(U)R(Y, Z, X, V) + \\ &+ \rho(V)R(Y, Z, U, X), \quad \forall X, Y, Z, U, V \in \mathcal{X}(M), \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \rho$ are 1-forms called the *associated 1-forms* which are not zero simultaneously and ∇ denotes covariant differentiation.

A non-flat Riemannian manifold is called *weakly Ricci-symmetric* and denoted by $(WRS)_n$ if the Ricci tensor S is non-zero and satisfies the condition

$$(2) \quad (\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(Y)S(X, Z) + \gamma(Z)S(Y, X),$$

where α, β, γ are again 1-forms, not zero simultaneously. Weakly symmetric manifolds have been studied by M. Prvanović [6], T.Q. Binh [2], U.C. De and S. Bandyopadhyay [5] and others. If in (1) the 1-form α is replaced by 2α and ρ is equal to α , then the manifold is called a *generalized pseudo symmetric manifold* introduced and investigated by M. C. Chaki [3], and if in (2) the 1-form α is replaced by 2α , then the manifold is called a *generalized pseudo Ricci symmetric manifold* introduced by Chaki and Koley [4]. So the defining conditions of weakly symmetric and weakly Ricci symmetric manifolds are a little weaker than the generalized pseudo symmetric and generalized pseudo Ricci symmetric manifolds.

In a recent paper [5] U.C. De and S. Bandyopadhyay gave an example of $(WS)_n$ and showed that in (1) necessarily $\gamma = \beta$ and $\rho = \delta$. So (1) takes the form:

$$(3) \quad \begin{aligned} (\nabla_X R)(Y, Z, U, V) &= \alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) \\ &+ \beta(Z)R(Y, X, U, V) + \delta(U)R(Y, Z, X, V) + \delta(V)R(Y, Z, U, X). \end{aligned}$$

Let A, B and P be the vector fields associated with the 1-forms α, β and δ respectively i.e; $g(X, A) = \alpha(X)$, $g(X, B) = \beta(X)$ and $g(X, P) = \delta(X)$ for all X . A, B and P are called the *associated vector fields* corresponding to the 1-forms α, β and δ respectively.

In the present paper we study weakly symmetric and weakly Ricci symmetric Kaehler manifolds. In Section 2 we prove that in a weakly symmetric Kaehler manifold (a) if the scalar curvature is a non-zero constant, then the sum of the associated 1-forms is zero, and (b) the vector fields A, JA, B, JB, P and JP , with the almost complex structure J , are eigenvectors of the Ricci tensor S with the same eigenvalue $r/2$, where r is the scalar curvature of (M^n, g) . Finally, we prove that in dimension $n = 6$ if A, JA, B, JB, P and JP are linearly independent, then the manifold will be Ricci flat. In the last Section 3 we consider a weakly Ricci symmetric Kaehler manifold and prove that in a weakly Ricci symmetric Kaehler manifold of non-zero constant scalar curvature the associated 1-forms α, β, γ are all equal.

Before starting with our investigations we collect some properties of Kaehler manifolds which will be used in the sequel. A Kaehler manifold is an even-dimensional manifold M^{2k} with a complex structure J and a positive-definite metric g which satisfies the following conditions [1]

$$J^2 = -I, \quad g(\overline{X}, \overline{Y}) = g(X, Y), \quad \overline{X} = JX$$

and

$$(4) \quad \nabla J = 0,$$

where ∇ means the covariant derivation according to the Levi-Civita connection.

The formulas [1]:

$$(5) \quad R(X, Y) = R(\overline{X}, \overline{Y}),$$

$$(6) \quad S(X, Y) = S(\overline{X}, \overline{Y}),$$

$$(7) \quad S(X, \overline{Y}) + S(\overline{X}, Y) = 0$$

are well known for a Kaehler manifold.

2 Weakly symmetric Kaehler manifolds

In this section we suppose that (M^n, g) is a $(WS)_n$ and Kaehler manifold. Then from (3), (4) and (5) we find

$$(2.1) \quad (\nabla_X R)(Y, Z, U, V) = (\nabla_X R)(\bar{Y}, \bar{Z}, U, V)$$

and

$$(2.2) \quad (\nabla_X R)(Y, Z, U, V) = (\nabla_X R)(Y, Z, \bar{U}, \bar{V}).$$

From (3) and (2.1) we obtain

$$(2.3) \quad \begin{aligned} \beta(Y)R(X, Z, U, V) + \beta(Z)R(Y, X, U, V) = \\ = \beta(\bar{Y})R(X, \bar{Z}, U, V) + \beta(\bar{Z})R(\bar{Y}, X, U, V). \end{aligned}$$

Let $m \in M^n$, and in a neighbourhood N around m , let $e_i \in \mathcal{X}(M^n) : g(e_i, e_j)|_m = \delta_{ij}$, $\nabla e_i|_m = 0$. Letting $Z = U = e_i$ in (2.3) we have

$$\begin{aligned} \beta(Y)S(X, V) + g(B, e_i)g(R(Y, X)e_i, V) = \\ = \beta(\bar{Y})g(R(X, \bar{e}_i)e_i, V) + g(B, \bar{e}_i)g(R(\bar{Y}, X)e_i, V) \end{aligned}$$

or

$$\beta(Y)S(X, V) + g(R(X, Y)V, B) = \beta(\bar{Y})g(R(V, e_i)X, \bar{e}_i) + g(B, \bar{e}_i)g(R(\bar{Y}, X)e_i, V).$$

Putting $V = X = e_j$ in the above equation we obtain

$$(2.4) \quad \beta(Y)r - S(Y, B) = -\beta(\bar{Y})S(e_i, \bar{e}_i) - g(B, \bar{e}_i)S(\bar{Y}, e_i),$$

where r is the scalar curvature of (M^n, g) . From (7) it follows that $S(e_i, \bar{e}_i) = 0$. Hence, from (2.4) it follows

$$\beta(Y)r - S(Y, B) = g(\bar{B}, e_i)g(L\bar{Y}, e_i) = g(\bar{B}, L\bar{Y}) = S(\bar{B}, \bar{Y}) = S(B, Y),$$

where L , defined by the relation $S(X, Y) = g(LX, Y)$, is the symmetric endomorphism corresponding to the Ricci tensor S , which implies that

$$(2.5) \quad \beta(Y)r = 2S(Y, B).$$

Similarly, the formulas (3) and (2.2) imply

$$(2.6) \quad \delta(Y)r = 2S(Y, P), \quad \delta(X) = g(X, P).$$

Now from (3) we find

$$\begin{aligned} (\nabla_X S)(Z, V) = \alpha(X)S(Z, V) + \beta(R(X, Z)V) + \\ + \beta(Z)S(X, V) + \delta(V)S(Z, X) + \delta(R(X, V)Z). \end{aligned}$$

Let again $Z = V = e_i$. Then we obtain

$$(2.7) \quad X(r) = \alpha(X)r + 2S(X, B) + 2(X, P).$$

So, by (2.5) and (2.6)

$$(2.8) \quad X(r) = [\alpha(X) + \beta(X) + \delta(X)]r.$$

(3) can be written as

$$(2.9) \quad (\nabla_X R)(Y, Z)V = \alpha(X)R(Y, Z)V + \beta(Y)R(X, Z)V + \\ + \beta(Z)R(Y, X)V + \delta(V)R(Y, Z)X + g(R(Y, Z)V, X)P,$$

where $g(X, P) = \delta(X)$, $\forall X$. Contracting, from (2.9) we derive

$$(2.10) \quad (\operatorname{div} R)(Y, Z)V = \alpha(R(Y, Z)V) + \beta(Y)S(Z, V) - \\ - \beta(Z)S(Y, V) + R(Y, Z, V, P).$$

From the second Bianchi identity it follows that

$$(2.11) \quad (\operatorname{div} R)(Y, Z)V = (\nabla_Y S)(Z, V) - (\nabla_Z S)(Y, V)$$

and

$$(2.12) \quad (\operatorname{div} L)(Y) = \frac{1}{2}Y(r),$$

where $g(LX, Y) = S(X, Y)$. From (2.10) and (2.11) we deduce

$$(\nabla_Y S)(Z, V) - (\nabla_Z S)(Y, V) = \alpha(R(Y, Z)V) + \beta(Y)S(Z, V) - \\ - \beta(Z)S(Y, V) + R(Y, Z, V, P).$$

Letting $Y = V = e_i$ in the last equation, we obtain

$$(2.13) \quad (\operatorname{div} L)(Z) - Z(r) = -S(Z, A) + S(Z, B) - B(Z)Y - S(Z, P).$$

Using (2.5), (2.6) and (2.12) in (2.13) we get

$$(2.14) \quad Z(r) = 2S(Z, A) + 2S(X, B) + 2S(X, P).$$

From (2.7) and (2.14) it follows that

$$(2.15) \quad 2S(Z, A) = \alpha(Z)r = g(Z, A)r,$$

$$(2.16) \quad \text{i.e., } S(Z, A) = \frac{r}{2}g(Z, A), \quad \forall Z,$$

which implies that A is an eigenvector of S corresponding to the eigenvalue $r/2$.

Letting $A = \bar{A}$ in (2.16) we obtain

$$S(Z, \bar{A}) = \frac{r}{2}g(Z, \bar{A})$$

which implies that JA is also an eigenvector of S with the same eigenvalue $r/2$.

Similarly from (2.5) and (2.6) we find that B, JB, P and JP are eigenvectors of S corresponding to the same eigenvalue $r/2$.

Summing up, we can state the following theorem:

Theorem 1. *In a weakly symmetric Kaehler manifold,*

(a) *If the scalar curvature is a non-zero constant, then the sum of the associated 1-forms is zero.*

(b) *A, JA, B, JB, P and JP are the eigenvectors of the Ricci tensor S with the same eigenvalue $r/2$.*

Next we prove the following:

Theorem 2. *Let M be a weakly symmetric Kaehler manifold of dimension $n = 6$ and let A, JA, B, JP, P and JP be linearly independent. Then the manifold is Ricci flat.*

Proof.

$$Y = aA + a^*JA + bB + b^*JB + cP + c^*JP.$$

Now with appropriate scalars a, a^*, b, b^*, c, c^*

$$\begin{aligned} S(X, Y) &= g(X, L(aA + a^*JA + bB + b^*JB + cP + c^*JP)) = \\ &= g\left(X, \frac{r}{2}(Aa + a^*JA + Bb + b^*JB + cP + c^*JP)\right) = \\ &\quad \text{(by (2.15), (2.5) and (2.6))} \\ &= g\left(X, \frac{r}{2}Y\right) = \frac{r}{2}g(X, Y). \end{aligned}$$

So

$$S(X, Y) = \frac{r}{2}g(X, Y).$$

Letting $X = Y = e_i$ in the above equation, we get $r = 0$. Hence $S(X, Y) = 0$. This completes the proof.

3 Weakly Ricci symmetric Kaehler manifolds

In this section we suppose that the Kaehler manifold is a $(WRS)_n$. Then (2) holds. That is,

$$(3.1) \quad (\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(Y)S(X, Z) + \gamma(Z)S(Y, X).$$

From (4) and (6) it follows that

$$(3.2) \quad (\nabla_X S)(\bar{Y}, \bar{Z}) = (\nabla_X S)(Y, Z).$$

Letting $Y = \bar{Y}$ and $Z = \bar{Z}$ in (3.1) and using (3.2) and (6) we find

$$(3.3) \quad \beta(Y)S(X, Z) + \gamma(Z)S(Y, X) = \beta(\bar{Y})S(X, \bar{Z}) + \delta(\bar{Z})S(\bar{Y}, X)$$

Letting $X = Z = e_i$ in (3.3) gives

$$\beta(Y)r + \gamma(LY) = \beta(\bar{Y})S(e_i, \bar{e}_i) + \gamma(\bar{e}_i)S(\bar{Y}, e_i) = -\delta(LY),$$

$$\text{since } S(e_i, \bar{e}_i) = 0.$$

Hence

$$(3.4) \quad \beta(Y)r + 2\gamma(LY) = 0, \quad S(X, Y) = g(LX, Y).$$

Again putting $X = Y = e_i$ in (3.3) and proceeding in the same way as above, we get

$$(3.5) \quad \gamma(Y)r + 2\beta(LY) = 0$$

From (3.1) we obtain

$$(\nabla_X S)(Y, Z) - (\nabla_X S)(Z, Y) = [\beta(Y) - \gamma(Y)]S(X, Z) + [\gamma(Z) - \beta(Z)]S(X, Y),$$

which implies

$$(3.6) \quad [\beta(Y) - \gamma(Y)]S(X, Z) + [\gamma(Z) - \beta(Z)]S(X, Y) = 0.$$

Letting $X = Z = e_i$ in the above equation, it follows

$$(3.7) \quad [\beta(Y) - \gamma(Y)]r + [\gamma - \beta](LY) = 0.$$

Using (3.4) and (3.5) in (3.7) we have

$$(\beta - \gamma)r = 0.$$

Hence we can state the following

Theorem 3. *In a weakly Ricci symmetric Kaehler manifold with non-zero scalar curvature the 1-forms β and γ are equal.*

Putting $Y = Z = e_i$, the relation (3.1) gives

$$X(r) = \alpha(X)r + \beta(LX) + \gamma(LX).$$

Using (3.4) and (3.5) in the above equation we can write

$$(3.8) \quad X(r) = \alpha(X)r - \frac{r}{2}(\beta(X) + \gamma(X))$$

From (3.8) and Theorem 3 we find

$$X(r) = [\alpha(X) - \beta(X)]r.$$

Hence we get the following

Theorem 4. *In a weakly Ricci symmetric Kaehler manifold with non-zero constant scalar curvature, the 1-forms of $(WRS)_n$ are all equal.*

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