The Manifold of Euclidean Inner Products of Sphere

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

We give an example for the manifold of all Euclidean inner products on various tangent spaces of a manifold, building the manifold $\mathcal{S}(S^n)$ of all Euclidean inner products for the sphere S^n. We obtain a representation of points of this manifold by matrices classes.

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1 Introduction

Let $\mathcal{S}(M)$ be the manifold of all Euclidean inner products given on various tangent spaces of a $n-$dimensional real smooth manifold M [5]. This is the $n+n(n+1)/2-$dimensional real smooth manifold $\mathcal{S}(M) = L(M) \times_{GL(n;R)} \mathcal{S}(R^n)$ -the total space of the fibre bundle over the base M, with standard fibre the homogeneous space $\mathcal{S}(R^n) = GL(n;R)/O(n)$ which is associated with the principal bundle $L(M)$ of linear frames of M. The homogeneous space $\mathcal{S}(R^n)$ is exactly the manifold of all Euclidean inner products on R^n. The manifold $\mathcal{S}(M)$ is different from the manifold $\mathcal{M}(M)$ of all Riemannian metrics of M which is an infinite smooth manifold [2], but $\mathcal{M}(M)$ is exactly the manifold of all global sections of the fibre bundle $\mathcal{S}(M)$.

We denote by $\pi : \mathcal{S}(M) \to M$ the canonical projection. The fibre $\pi^{-1}(x)$ over $x \in M$ may be considered as $\mathcal{S}(T_xM)$- the space of Euclidean inner products on the tangent space T_xM. If $\{x^1, ..., x^n\}$ are local coordinates in a neighborhood U of the point $x \in M$, let $\{\theta_1, ..., \theta_n\}$ be the canonical frame, $\theta_i = \dfrac{\partial}{\partial x^i}, i = 1, ..., n$. The vertical space in the point $g \in \mathcal{S}(M), \pi(g) = x$ is the space of all symmetric endomorphisms of T_xM related with g. If we denote by $\mathcal{V}_g\mathcal{S}(M)$ the vertical space at the point $g \in \mathcal{S}(M)$,then, locally,

$$\mathcal{V}_g\mathcal{S}(M) = \{\widehat{A}(g) \mid A \in gl(n;R), \widehat{A}(g) = Ag + g^tA\}, \pi(g) = x \in U,$$

where $gl(n;R)$ is the Lie algebra of endomorphisms of R^n and ${}^tA$ is the transposition of $A \in gl(n;R)$ related with the standard inner product e on R^n.

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If ∇ is a linear connection on M, the horizontal space $\mathcal{H}_g$ at the point $g \in \mathcal{S}(M)$ is locally spanned by $\{\widehat{\theta}_1, ..., \widehat{\theta}_n\}$, where for $i = 1, ..., n, \widehat{\theta}_i$ is the horizontal lift of θ_i and is given by $\widehat{\theta}_i = \theta_i + \widetilde{\Gamma_i(x)}(g), \Gamma_i = (\Gamma^k_{ij})_{jk} \in gl(n; R)$, with $\nabla_{\theta_i}\theta_j = \Gamma^k_{ij}\theta_k, (\forall) \quad i, j = 1, ..., n..., n$.

2 The principal bundle of linear frames of the sphere

Let M be a manifold of dimension n. A linear frame at a point $x \in M$ is an ordered basis $(X_1, ..., X_n)$ of a tangent space T_xM. We denote by $L(M)$ the set of all linear frames u at all points of M and let π' be the mapping of $L(M)$ onto M witch maps a linear frame u at x into x.The general linear group $GL(n; R)$ acts on $L(M)$ on the right as follows. If $a = (a_{ij})_{ij} \in GL(n; R)$ and $u = (X_1, ..., X_n)$ is a linear frame of $T_xM, x \in M$,then ua is, by definition, the linear frame $(Y_1, ..., Y_n)$ of T_xM defined by $Y_i = \sum_j a^j_i X_j$. In order to introduce a differentiable structure in $L(M)$, let $(x^1, ..., x^n)$ be a local coordinate neighborhood U in M. Every frame u at $T_xM, x \in M$, can be expressed uniquely in the form $u = (X_1, ..., X_n)$, with $X_i = \sum_k X^k_i \partial/\partial x^k, i = 1, ..., k$. Then, $(x_1, ..., x_n, X^k_i)_{ik}$ is a local coordinate system in $\pi'^{-1}(U)$. It is easy to verify that $L(M)$ is the total space of a principal fibre bundle over the base M with structure group $GL(n; R)$, denoted by $L(M)$ to [1].

Let $(R^{n+1}, <, >)$ be the standard Euclidean $n + 1-$dimensional space and let $S^n = \{x \in R^{n+1} \; |< x, x >= 1\}$ be the $n-$dimensional sphere. A frame at the point $x \in S^n$ is given by n independents vectors of R^{n+1}, orthogonals on the vector x.If $u = (X_1, X_2, ..., X_n)$ is a frame in $x \in S^n, x = (x_1, ..., x_{n+1})$,where $X_i = (x_{1i}, ..., x_{n+1i}), i = 1, ..., n$, then we consider the matrix

$$X = \begin{pmatrix} x_{11} & . & . & . & x_{1n} & x_1 \\ x_{21} & . & . & . & x_{2n} & x_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ x_{n+11} & . & . & . & x_{n+1n} & x_{n+1} \end{pmatrix} \in GL(n+1; R). \tag{1}$$

We observe initially that ${}^tXX \in \widetilde{GL(n; R)}$, where $\widetilde{GL(n; R)}$ is the Lie subgroup of the Lie group $GL(n + 1; R)$ with the elements the non–degenerated matrices

$$\widetilde{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}, \qquad M \in GL(n; R) \tag{2}$$

and ${}^tX$ is the transposed of the matrix X.

It is easy to check that the manifold of linear frames of the n-dimensional sphere $L(S^n)$, is just: $L(S^n) = \{X \in GL(n+1; R) \;|^t XX \in \widetilde{GL(n; R)}\}$, and, so, we give a representation of $L(S^n)$ as matrices. The Lie group $GL(n; R)$, identified with $\widetilde{GL(n; R)}$, acts naturally, freely, on the manifold $L(S^n)$ through matriceal multiplication. Let $\pi' : L(S^n) \to S^n$ be the projection $\pi'(X) = x, x = (x_1, ..., x_{n+1}) \in S^n$ where $X \in L(S^n)$ is give by (1). We observe that for every $\widetilde{M} \in \widetilde{GL(n; R)}$, we have $\pi'(X\widetilde{M}) = \pi'(X)$ and, moreover, for all $X, Y \in L(S^n)$ we have $\pi'(X) = \pi'(Y)$ if and only if there is a matrix $\widetilde{M} \in GL(n;R)$ so that $Y = X\widetilde{M}$. Consequently, $L(S^n)$ is the total space of the fibre bundle of linear frames of S^n.

Let $\mathcal{M}$ be the set of all matrices $X \in GL(n+1,R)$ with the property ${}^tXX = \lambda I_{n+1}$, $\lambda \in R$, I_{n+1} being the identity matrix. Then, $\mathcal{M}$ is a Lie subgroup of $GL(n+1;R)$ if and only if $\lambda = 1$, and so $\mathcal{M} = O(n+1)-$ the group of orthogonal matrices of order $n+1$.

3 Construction of the manifold of Euclidean inner products on the sphere

According with general theory [1], the fibre bundle over the base S^n, with standard fibre $\mathcal{S}(R^n)$ and structure group $GL(n;R)$ which is associated with the principal bundle $L(S^n)$ is obtained as follows. On the product manifold $L(S^n)X\mathcal{S}(R^n)$, $GL(n;R) \simeq \widetilde{GL(n;R)}$ acts on the right as follows: an element $\widetilde{M} \in \widetilde{GL(n;R)}$ maps $(X,g) \in L(S^n)X\mathcal{S}(R^n)$ into $(X,g)\widetilde{M} = (X\widetilde{M}, \widetilde{M^{-1}}\widetilde{g}\widetilde{M^{-1}}) \in L(S^n)X\mathcal{S}(R^n)$, with $\widetilde{g} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$, where we are denoted by g the associated matrix with the inner product g on R^n related to the canonical frame of R^n. The quotient space of $L(S^n)X\mathcal{S}(R^n)$ by this group action is the set $\mathcal{S}(S^n)$ of all Euclidean inner products on the sphere S^n.The mapping $L(S^n)X\mathcal{S}(R^n) \to S^n$ which maps (X,g) in $\pi'(X)$ induces a mapping π, called the projection, of $\mathcal{S}(S^n)$ onto S^n. We can introduce a differentiable structure in $\mathcal{S}(S^n)$ by the requirement that for every coordinates neighborhood U of the manifold S^n, the set $\pi^{-1}(U)$ is diffeomorphic with $UX\mathcal{S}(R^n)$. The projection π is then a differentiable mapping of $\mathcal{S}(S^n)$ onto S^n.

We observe now that every element $X \in L(S^n)$ is given by the conditions (1), (2) and therefore give rise at an inner product on the tangent space T_xS^n.Indeed, this inner product have his associated matrix M related to the frame $(X_1, X_2, ..., X_n)$ of T_xS^n , the matrix:

$$(3) \qquad M = \begin{pmatrix} <X_1,X_1> & <X_1,X_2> & \cdot\ \cdot\ \cdot & <X_1,X_n> \\ <X_2,X_1> & <X_2,X_2> & \cdot\ \cdot\ \cdot & <X_2,X_n> \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ <X_n,X_1> & <X_n,X_2> & \cdot\ \cdot\ \cdot & <X_n,X_n> \end{pmatrix}.$$

We have immediately that

$$(3') \qquad {}^tXX = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix},$$

with the matrix M given by (3). Moreover, if we have $X, Y \in L(S^n)$ with the same last columns $x = (x_1, ..., x_{n+1})$, then ${}^tXX = {}^tYY$ if and only if there is an element $O \in \widetilde{O(n)}$ so that $B = OA$ where $\widetilde{O(n)} = O(n+1) \cap \widetilde{GL(n;R)}$.Therefore, the matrices X and Y of $L(S^n)$, with the same last columns x, give rise at the same inner product of T_xS^n if and only if $Y = XO$ with $O \in \widetilde{O(n)}$. Following [5], we may conclude that

$$\mathcal{S}(S^n) = \{A^t\widetilde{\mathcal{S}(R^n)}A \mid A \in GL(n+1;R), ^tAA \in \widetilde{GL(n;R)}\} =$$
$$= \{A^tA \mid A \in GL(n+1;R), A^tA \in \widetilde{GL(n;R)}\}.$$

This last remark shows that the vertical space $\mathcal{V}_g$ of the fibre bundle $\mathcal{S}(S^n)$, in the point $g =^t AA \in \mathcal{S}(S^n)$, which is the tangent space at the fibre through g at g, is

$$\mathcal{V}_g = \{\widehat{M}(g) \mid \widehat{M}(g) = \widetilde{M}A^tA + A^tA^t\widetilde{M}, \widetilde{M} \in \widetilde{GL(n;R)}\}.$$

In order to obtain in every point g of the manifold $\mathcal{S}(S^n)$ an horizontal space $\mathcal{H}_g$, we introduce on the sphere S^n a linear connexion ∇. Then, the tangent space $T_g\mathcal{S}(S^n)$ at the point $g \in \mathcal{S}(S^n)$ splits, and we have $T_g\mathcal{S}(S^n) = \mathcal{V}_g \oplus \mathcal{H}_g$. We take now in consideration the property of S^n to be a Riemannian manifold and therefore the existence of an atlas of S^n with her Jacobi's matrices orthogonals. Then, the manifold of Euclidean inner product $\mathcal{S}(S^n)$ may be endowed in a natural way with a Riemannian metric $\mathcal{G}$ so that the horizontal and vertical spaces at every point g are orthogonal spaces and

$$\mathcal{G}(\widehat{M}(g), \widehat{N}(g)) = \frac{1}{2}Tr\widehat{M(g)}\widehat{N(g)} \tag{4}$$

$$\mathcal{G}(\widehat{X}_g, \widehat{Y}_g) = g(X,Y), \tag{4'}$$

where $\widehat{M}(g), \widehat{N}(g) \in \mathcal{V}_g$, $\widehat{X}_g, \widehat{Y}_g \in \mathcal{H}_g$ and, for every horizontal vector $\widehat{X}_g$, we have denoted by X his projection on S^n.

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