Theory of Conformally Berwald Finsler Spaces and Its Applications to (α, β)-Metrics

Shun-ichi Hojo, M.Matsumoto and K.Okubo

Dedicated to Prof.Dr. Constantin UDRIŞTE on the occasion of his sixtieth birthday

Abstract

The theory of conformal changes of Finsler metrics has been studied by M. Hashiguchi [2] in 1976 and some of the Japanese school have directed their efforts to find conformally invariant curvature tensors similar to the Weyl conformal curvature tensor of a Riemannian space and to establish the condition for a Finsler space to be conformally flat. Finally, about five years ago, S.Kikuchi [6] succeeded in finding a conformally invariant Finsler connection and giving the conformally flat condition.

We have, however, a strange and objectionable in Kikuchi's theory. His conformally invariant connection can be only defined on an essential assumption. Whether this assumption holds or not in a Finsler space under consideration poses newly a difficult problem. Since we have not a conformally invariant connection in the Riemannian case, the assumption is, of course, not satisfied by any Riemannian space.

About ten years ago, Y.Ichijyo and M.Hashiguchi [5] defined a conformally invariant Finsler connection in a Finsler space with (α, β)–metric, where $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form in y^i, on the assumption $b^2 = a^{ij}b_i b_j \neq 0$. They gave the condition for a Randers space with the metric $\alpha + \beta$ to be conformally flat based on their connection. M.Matsumoto [11] showed that their theory can be applied to a Kropina space with the metric α^2/β.

The main purpose of the present paper is to consider Kikuchi's conformally invariant Finsler connections of Finsler spaces with (α, β)–metric. Since our main interest is Kikuchi's assumption, it is sufficient to stop our studies halfway to Finsler spaces conformal to locally Minkowski spaces. Thus we shall propose a new notion of conformally Berwald Finsler space.

Mathematics Subject Classification: 53C05, 53C60
Key words: Berwald connection, (α, β)-metric, Kropina space

Balkan Journal of Geometry and Its Applications, Vol.5, No.1, 2000, pp. 107-118

1 Conformally Berwald connections

In this section, we give a conformally Berwald connection which is induced from a scalar field S with a regularity condition.

Let us consider a Finsler space $F^n = F^n(M^n, L)$ with the Berwald connection $B\Gamma = (G^i{}_j, G_j{}^i{}_k, 0)$ and a conformal change $L \to \bar{L} = e^{c(x)}L$. The quantities of the conformally changed space $\bar{F}^n$ will be denoted by putting a bar.

We have first the conformally invariant tensors B_{ij} and B^{ij};

$$B_{ij} = \left(\frac{2}{L^2}\right)(g_{ij} - 2l_i l_j), \quad B^{ij} = \left(\frac{L^2}{2}\right)(g^{ij} - 2l^i l^j).$$

The matrix (B^{ij}) is the inverse of (B_{ij}) [2]. In the following we denote by subscripts of B^{ij} the partial differentiations of B^{ij} by y^h : $B^{ij}{}_{h\ldots k} = \dot{\partial}\ldots\dot{\partial}_k B^{ij}$.

Putting $F = \dfrac{L^2}{2}$ and $2G^i = g^{ij}((\dot{\partial}_j \partial_r F)y^r - \partial_j F)$, we have $G^i{}_j = \dot{\partial}_j G^i$ and $G_j{}^i{}_k = \dot{\partial}_k G^i{}_j$. If we put $c_i = \partial_i c$, on account of the paper [2] we get $\bar{G}^h = G^h - B^{hr}c_r$ and

$$\bar{G}^h{}_i = G^h{}_i - B^{hr}{}_i c_r, \quad \bar{G}_i{}^h{}_j = G_i{}^h{}_j - B^{hr}{}_{ij} c_r. \tag{1.1}$$

Then we obtain the relations between the hv-curvature tensors

$$\bar{G}_i{}^h{}_{jk} = G_i{}^h{}_{jk} - B^{hr}{}_{ijk} c_r. \tag{1.2}$$

Assume that we have a conformally invariant scalar field $S(x, y)$ which is $(r)p$–homogeneous in (y^i). Denoting by $(;)$ the h-covariant differentiation in $B\Gamma$, (1.1) yields

$$\bar{S}_{;i} = \partial_i \bar{S} - (\dot{\partial}_r S)G^r{}_i = \partial_i S - (\dot{\partial}_r S)(G^r{}_i - B^{rs}{}_i c_s),$$

and hence

$$\bar{S}_{;i} - S_{;i} = W^r{}_i c_r, \quad W^j{}_i := (\dot{\partial}_r S)B^{rj}{}_i. \tag{1.3}$$

Along the lines of S.Kikuchi [6] and F.Ikeda [4] we shall suppose that

$$\det(W^j{}_i) \neq 0, \quad \text{and let } (V^i{}_j) \text{ be the inverse matrix of } (W^j{}_i). \tag{1.4}$$

$V^i{}_j(x, y)$ are $(-r)p$–homogeneous in y^i and (1.3) can be written in the form

$$\bar{V}_j - V_j = c_j, \quad V_j := S_{;r} V^r{}_j. \tag{1.5}$$

$V_j(x, y)$ are $(0)p$–homogeneous in y^i. Since c_j are functions of position, we must have

$$\dot{\partial}_i(\bar{V}_j - V_j) = 0. \tag{1.6}$$

Substituting from (1.5) in (1.1), we get the invariant quantities

$${}^cG^h{}_i = G^h{}_i + B^{hr}V_r, \quad {}^cG_i{}^h{}_j = G_i{}^h{}_j + B^{hr}V_r. \tag{1.7}$$

Consequently we obtain the conformally invariant Finsler connection ${}^cB\Gamma = ({}^cG^h{}_i, {}^cG_i{}^h{}_j, 0)$. This is called the *conformal Berwald connection* with respect to S. On the other hand, (1.2) yields a conformally invariant tensor

$$ {}^cG_i{}^h{}_{jk} = G_i{}^h{}_{jk} + B^{hr}{}_{ijk}V_r. \tag{1.8} $$

It is remarked that the hv-curvature tensor $(\dot{\partial}_k({}^cG_i{}^h{}_j)$ of ${}^cB\Gamma$ is conformally invariant, but it is different from ${}^cG_i{}^h{}_{jk}$:

$$ \dot{\partial}_k({}^cG_i{}^h{}_j) = {}^cG_i{}^h{}_{jk} + B^{hr}{}_{ij}\dot{\partial}_k V_r. \tag{1.9} $$

According to the Berwald expression (Theorem 3.4) of a Finsler connection given by T.Aikou and M.Hashiguchi [1], the set $(L_k, D^i{}_k, T_j{}^i{}_k, P^i{}_{jk}, C_j{}^i{}_k)$ of the essential tensor fields of ${}^cB\Gamma$ are

$$ L_k = -l_r B^{rs}{}_k V_s, \; D^i{}_k = 0, \; T_j{}^i{}_k = 0, \; P^i{}_{jk} = B^{ir}{}_j\dot{\partial}_k V_r, \; C_j{}^i{}_k = 0. \tag{1.10} $$

The conditions (1) and (2) mentioned in their theorem are satisfied because $V_i(x,y)$ are $(0)p-$homogeneous in y^i.

A Finsler space F^n is called a *Berwald space* if $G_j{}^i{}_k$ are functions of position alone, or $G_i{}^h{}_{jk} = 0$. In the Cartan connection $C\Gamma = (G^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k)$, F^n is a Berwald space if and only if $F_j{}^i{}_k$ are functions of position alone, or $C_j{}^i{}_{k/l} = 0$ in terms of the h-covariant differentiation in $C\Gamma$.

Definition. A Finsler space $F^n = (M^n, L)$ is called *conformally Berwald*, if there exists a conformal change $L \to \bar{L} = e^{c(x)}L$ such that the changed space $\bar{F}^n = (M^n, \bar{L})$ is a Berwald space.

We deal with a conformally invariant scalar S which satisfies $S_{;i} = 0$ for a Berwald space. Such an invariant S is called *of parallel type* [4], (Theorem 2.1). The supposition $\det(W^j{}_i) \neq 0$ with respect to S of parallel type is called the *Kikuchi condition.* Then we get

$$ c_j = \bar{V}_j - V_j, \quad V_j = S_{;r}V^r{}_j, \tag{1.11} $$

on the Kikuchi condition.

Now we consider a Finsler space F^n having S satisfying the Kikuchi condition and suppose that F^n is conformal to a Berwald space $\bar{F}^n$. Then $\bar{S}_{;i} = 0$ and $\bar{V}_j = 0$, and hence (1.11) is reduced to

$$ c_j = -V_j, \tag{1.12} $$

which implies that $V_j = V_j(x)$ is a gradient vector:

$$ (a) \quad \dot{\partial}_j V_i = 0, \qquad (b) \quad V_{i;j} - V_{j;i} = 0. \tag{1.13} $$

Next, since $\bar{F}^n$ is a Berwald space, we have $\bar{G}_i{}^h{}_{jk} = 0$ and hence (1.8) implies ${}^c\bar{G}_i{}^h{}_{jk} = 0$. Consequently we have

$$ {}^cG_i{}^h{}_{jk} = 0. \tag{1.14} $$

Therefore (1.13a), (1.14) and (1.9) lead to the fact that the hv–curvature tensor of ${}^cB\Gamma$ vanishes.

Conversely, we consider a Finsler space F^n having S satisfying the Kikuchi condition such that V_j with respect to S satisfies (1.13) and ${}^cB\Gamma$ has the vanishing hv–curvature tensor. (1.9) with (1.13a) show ${}^cG_i{}^h{}_{jk} = 0$. (1.13) gives the function

$c(x)$ satisfying (1.12) and hence we have the conformal change $L \to \bar{L} = e^{c(x)}L$. Then ${}^cG_i{}^h{}_{jk} = 0$ and (1.5) with (1.12) give $\bar{V}_j = 0$. Then (1.8) leads to $\bar{G}_i{}^h{}_{jk} = 0$ and thus the changed space $\bar{F}^n$ is a Berwald space.

We denote by ${}^c\nabla$ the h–covariant differentiation in ${}^cB\Gamma$, the v–covariant one in ${}^cB\Gamma$ is $\dot{\partial}$. Then (1.13) are written in terms of ${}^cB\Gamma$ as follows:

$$(1.13') \qquad (a) \quad \dot{\partial}_j V_i = 0, \quad (b) \quad {}^c\nabla_j V_i - {}^c\nabla_i V_j = 0.$$

Therefore we have

Theorem 1. *Let F^n be a Finsler space having an S satisfying the Kikuchi condition. F^n is a conformally Berwald space, if and only if its conformal Berwald connection with respect to S has the vanishing hv–curvature tensor and satisfies (1.13′).*

2 Kikuchi's assumption for (α, β)–metrics

To do justice Kikuchi's assumption (1.4), that is the condition $\det(W^i{}_j) \neq 0$, we shall be concerned with Finsler space with (α, β)–metrics.

Let us consider a Finsler space $F^n = (M^n, L(\alpha, \beta))$ with (α, β)–metric where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian fundamental form and $\beta = b_i(x)y^i$ is a 1–form in y^i. We put $\alpha_{i\ldots j} = \dot{\partial}_i \ldots \dot{\partial}_j \alpha$ and have

$$(2.1) \qquad \begin{aligned} \alpha\alpha_i &= Y_i, \quad Y_i := a_{ir}y^r, \\ \alpha\alpha_{ij} &= a_{ij} - Y_iY_j/\alpha^2 := k_{ij}, \end{aligned}$$

where k_{ij} is the angular metric tensor of the Riemannian space $R^n = (M^n, \alpha)$ associated with F^n. Next we have

$$(2.2) \qquad \alpha\alpha_{ijk} = -(k_{ij}Y_k + (ijk))/\alpha^2,$$

where $+(ijk)$ denotes cyclic permutations with respect to indices and their sum.

Further we put $F = L^2/2$ and the derivatives of F with respect to (y^i, α, β) are denoted by the subscripts $(i, 1, 2)$. Then $F_i = F_1\alpha_i + F_2 v_i$ and

$$(2.3) \qquad F_{ij} = F_1\alpha_{ij} + F_{11}\alpha_i\alpha_j + F_{12}(\alpha_i b_j + \alpha_j b_i) + F_{22}b_ib_j.$$

Since F_{ij} is the fundamental tensor g_{ij} of F^n, we have from (2.1) and $F_1 = F_{11}\alpha + F_{12}\beta$,

$$(2.3)' \qquad g_{ij} = (F_1/\alpha)a_{ij} + F_{22}b_ib_j + (F_{12}/\alpha)(b_iY_j + b_jY_i) - (\beta F_{12}/\alpha^3)Y_iY_j.$$

We have shown [8], [12].

$$(2.4) \qquad \begin{aligned} \det(g_{ij}) &= (F_1/\alpha)^{n-2}T\det(a_{ij}), \\ T &:= DB + 2FF_1/\alpha^3, \\ D &:= F_{11}F_{22} - F_{12}^2, \\ B &:= b^2 - (\beta/\alpha)^2. \end{aligned}$$

We suppose $F_1 \neq 0$, of course. As a consequence F^n has the irregular metric ($\det(g_{ij}) = 0$), if and only if $T = 0$ [7]. In the following we are concerned only with F^n having regular metric.

Then the inverse matrix (g^{ij}) of (g_{ij}) may be put

$$g^{ij} = (\alpha/F_1)a^{ij} - s_0 B^i B^j - s_{-1}(B^i y^j + B^j y^i) - s_{-2} y^i y^j, \tag{2.5}$$

where $B^i = a^{ir} b_r$. The condition $g^{ik} g_{ij} = \delta^k{}_j$ for g^{ij} leads to

$$\begin{aligned} T_{s_0} &= \alpha D/F_1, \quad T_{s_{-1}} = 2FF_{12}/\alpha^2 F_1, \\ T_{s_{-2}} &= -(F_{12}/\alpha^2 F_1)(BF_2 + 2F\beta/\alpha^2). \end{aligned} \tag{2.6}$$

From (2.3) and $F_{ijk} = 2C_{ijk}$ we have

$$\begin{aligned} 2C_{ijk} &= C_1 + C_2 + C_3 + C_4 + C_5, \\ C_1 &= F_1 \alpha_{ijk} + F_{11}(\alpha_{ij}\alpha_k + (ijk)) + F_{12}(\alpha_{ij} b_k + (ijk)), \\ C_2 &= (F_{111}\alpha_k + F_{112} b_k)\alpha_i \alpha_j, \quad C_3 = (F_{112}\alpha_k + F_{122} b_k) b_i \alpha_j, \\ C_4 &= (F_{112}\alpha_k + F_{122} b_k)\alpha_i b_j, \quad C_5 = (F_{112}\alpha_k + F_{222} b_k) b_i b_j. \end{aligned}$$

From (2.1) and (2.2), and putting

$$p_i := b_i - (\beta/\alpha^2) Y_i, \tag{2.7}$$

we have $C_1 = (F_{12}/\alpha)(k_{ij} p_k + (ijk))$. Next, from $F_{ab1}\alpha + F_{ab2}\beta = 0$, for $a, b = 1, 2$, we have $F_{122} = -(\beta/\alpha)F_{222}$, $F_{112} = (\beta/\alpha)^2 F_{222}$ and $F_{111} = -(\beta/\alpha)^3 F_{222}$. Then

$$\begin{gathered} C_2 = (\beta^2/\alpha^4) F_{222} Y_i Y_j p_k, \; C_3 = -(\beta/\alpha^2) F_{222} b_i Y_j p_k, \\ C_2 + C_3 = -(\beta/\alpha^2) F_{222} p_i Y_j p_k, \\ C_4 = -(\beta/\alpha^2) F_{222} Y_i b_j p_k, \; C_5 = F_{222} b_i b_j p_k, \\ C_4 + C_5 = F_{222} p_i b_j p_k, \\ C_2 + C_3 + C_4 + C_5 = F_{222} p_i p_j p_k. \end{gathered}$$

Consequently we have [12], [11]

$$C_{ijk} = (F_{12}/2\alpha)(k_{ij} p_k + (ijk)) + (F_{222}/2) p_i p_j p_k. \tag{2.8}$$

Now as a conformally invariant scalar S in the section 1, we shall study two cases, that is, $A^2 = g^{ij}(LC_i)(LC_j)$ in the case of a Berwald F^n in this section and β/α in the next section.

In the following of this section we study the condition (1.4) i.e. Kikuchi's assumption for (α, β)–metric.

Let us find $A^2 = g^{ij}(LC_i)(LC_j)$ for F^n with (α, β)–metric. From (2.5) and (2.8) we have

$$C_i = C_{ijk} g^{jk} = C_{ijk}(\alpha a^{jk}/F_1 - s_0 B^j B^k).$$

Also we have

$$p_k a^{jk} = B^j - (\beta/\alpha^2)y^j,\ k_{ij}p_k a^{jk} = p_i,\ p_j p_k a^{jk} = B,$$
$$C_{ijk}a^{jk} = ((n+1)F_{12}/2\alpha + F_{222}\beta/2)p_i,$$
$$k_{ij}B_j = p_i,\ p_k B^k = B,\ k_{jk}B^j B^k = B,$$
$$C_{ijk}B^j B^k = (3F_{12}B/2\alpha + F_{222}B^2/2)p_i.$$

Thus we get

(2.9)
$$C_i = Ep_i,$$
$$E = (F_{12}/\alpha)((n+1)\alpha/2F_1 - s_0 B) + (F_{222}B/2)(\alpha/F_1 - s_0 B).$$

Consequently we have

(2.10)
$$A^2 = 4F^2E^2B/\alpha^2 T.$$

For the later use we are concerned with two examples where we put $t = \beta/\alpha$.
Ex. 1. Randers metric $L = \alpha + \beta$,

$$T = (1+t)^3, \qquad \beta E = (n+1)t/2(1+t),$$
$$A^2 = (n+1)^2 B/4(1+t), \quad B = b^2 - t^2.$$

Ex.2 Kropina metric $L = \alpha^2/\beta$,

$$T = 2b^2/t^6,$$
$$\beta E = -n - 2 + t^2/b^2,$$
$$A^2 = (B/2b^2)(t^2/b^2 - n - 2)^2.$$

It is remarked that $\dot{\partial}_i t = p_i/\alpha$, $\dot{\partial}_i A^2 = (\partial A^2/\partial t)p_i/\alpha$ for the above two examples.

We now approach to our problem by another way from the homogeneity of $F(\alpha,\beta)$. If we put

$$F(\alpha,\beta) = \alpha^2 f(t),\ f(t) = F(1,t),\ t = \beta/\alpha,$$

then we have the following:

$$(F_1, F_2) = \alpha(\phi(t), f'(t)),\ \phi(t) = 2f - tf',$$
$$(F_{11}, F_{12}, F_{22}) = (\phi - t\phi', \phi', f''),$$
$$(F_{111}, F_{112}, F_{122}, F_{222}) = (f'''/\alpha)(-t^3, t^2, -t, 1).$$
$$D = \delta(t) := 2ff'' - (f')^2, \quad B = B(b,t) := b^2 - t^2,$$
$$T = \delta(t)B(b,t) + 2f(t)\phi(t) = T(b,t),$$
$$Ts_0 = \delta(t)/\phi(t), \quad E = \Psi(b,t)/\alpha,$$
$$\Psi(b,t) = [(n+1)f\phi' + B\{(n-1)\delta\phi/2\phi' + ff'''\}]/T,$$
$$A^2 = 4(f\Psi)^2 B/T := \Pi(b,t).$$

Consequently we have

$$\dot{\partial}_i A^2 = \Pi_t p_i/\alpha.$$

Putting $W^j{}_i = (\dot{\partial}_r A^2)B^{rj}{}_i$, $W_0{}^j{}_i = (\dot{\partial}_r t)B^{rj}{}_i$, we obtain

$$W^j{}_i = \Pi_t W_0{}^j{}_i.$$

As an example of this process we show a case of Kropina metric $L = \alpha^2/\beta$,

$$f(t) = 1/2t^2,\ \phi(t) = 2/t^2,\ \delta(t) = 2/t^6,\ T = 2b^2/t^6,$$

$$\Psi = (t^2 - (n+2)b^2)/b^2 t,\ A^2 = (B/2b^2)(t^2/b^2 - (n+2))^2.$$

Thus we have

Theorem 2. *If a non-Riemannian Finsler space F^n with $L(\alpha,\beta)$ has the non-zero Π_t, then $S = A^2$ satisfies the Kikuchi's condition and Theorem 1 can be applicable to F^n.*

3 Another assumption for (α,β)–metrics

As conformally changed $\bar{L}(\bar{\alpha},\bar{\beta}) = e^{c(x)}L(\alpha,\beta) = L(e^{c(x)}\alpha, e^{c(x)}\beta)$ by (1)p–homogeneity of L, β/α is conformally invariant [5]. Let us take $S = \beta/\alpha$ and put $W_{ij} = g_{ir}W^r{}_j = g_{ir}(\dot{\partial}_s S)B^{sr}{}_j$. Then we have $\dot{\partial}_s S = p_s/\alpha$ and

$$\begin{aligned} W_{ij} &= g_{ir}(p_s/\alpha)(y_j g^{rs} - \delta^s{}_j y^r - \delta^r{}_j y^s - L^2 C^{rs}{}_j) = \\ &= (p_i y_j - p_j y_i - 2Fp^r C_{rij})/\alpha. \end{aligned}$$

We put $P^r = a^{ri}p_i$, and have from (2.5), (2.7) and (2.6)

$$\begin{aligned} p^r &= g^{ri}p_i = (\alpha/F_1 - s_0 B)p^r - (s_0\beta/\alpha^2 + s_{-1})By^r \\ &= (2F/\alpha^2 T)P^r - (B/\alpha^2 FT)(\alpha\beta D + 2FF_{12})y^r. \end{aligned}$$

Since [7] shows $\alpha\beta D + 2FF_{12} = F_1F_2$, we find

$$p^r = (2FP^r - BF_2 y^r)/\alpha^2 T. \tag{3.1}$$

Thus (2.8) together with $P^i p_i = B$ and $P^i k_{ij} = p_j$ leads to

$$p^r C_{rij} = (F/\alpha^3 T)(F_{12}Bk_{ij} + (2F_{12} + \alpha F_{222}B)p_i p_j). \tag{3.2}$$

From (2.3′) we have

$$y_i = F_2 p_i + (2F/\alpha^2)Y_i. \tag{3.3}$$

Consequently we obtain

$$\begin{aligned} &W_{ij} = Qa_{ij} + Q_0 p_i p_j + Q_{-1}(p_i Y_j - p_j Y_i) + Q_{-2}Y_i Y_j, \\ &Q = -2F^2 F_{12}B/\alpha^4 T, \\ &Q_0 = -(2F^2/\alpha^4 T)(2F_{12} + \alpha BF_{222}), \\ &Q_{-1} = 2F/\alpha^3, \\ &Q_{-2} = -Q/\alpha^2. \end{aligned} \tag{3.4}$$

$Q = 0$, if and only if $F_{12} = 0$, that is, F is of the form $c_1\alpha^2 + c_2\beta^2$ with constant c's. Thus, suppose that F^n is not Riemannian, then $Q \neq 0$.

Next, we put $V^{jk} = V^j{}_r g^{rk}$. Then

$$W_{ij}V^{jk} = \delta_i{}^k. \tag{3.5}$$

Let us put

$$V^{jk} = a^{jk}/Q + R_0 P^j P^k + R_{-1}(P^j y^k - P^k y^j) + R_{-2} y^j y^k. \tag{3.6}$$

Then (3.5) yields as coefficients of the following terms,

$$\begin{aligned} p_i P^k &: \ (Q + Q_0 B)R_0 - \alpha^2 Q_{-1} R_{-1} = -Q_0/Q, \\ Y_i P^k &: \ -Q_{-1} B R_0 = Q_{-1}/Q, \\ p_i y^k &: \ (Q + Q_0 B)R_{-1} + \alpha^2 Q_{-1} R_{-2} = -Q_{-1}/Q, \\ Y_i y^k &: \ -Q_{-1} B R_{-1} = 1/\alpha^2. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &R_0 = -1/BQ, \ R_{-1} = -1/\alpha^2 B Q_{-1}, \\ &R_{-2} = -1/\alpha^2 Q + (Q + Q_0 B)/\alpha^4 (Q_{-1})^2 B. \end{aligned} \tag{3.7}$$

In an interesting paper [3] concerned with Finsler spaces equipped with a linear connection, M.Kashiguchi and Y.Ichijyo showed that if b_i of a Finsler space F^n with (α, β)–metric is parallel with respect to the Levi-Civita connection $\gamma = (\gamma_j{}^i{}_k(x))$ of the associated Riemannian space, then $F_j{}^i{}_k$ of the Cartan connection $C\Gamma = (F_j{}^i{}_k, G^i{}_j, C_j{}^i{}_k)$ coincide with $\gamma_j{}^i{}_k(x)$ and hence F^n is a Berwald space. This is also shown directly from the equation which gives the difference $B_j{}^i{}_k = G_j{}^i{}_k - \gamma_j{}^i{}_k$ [9].

$$L_\alpha B_j{}^k{}_i y^j y_k = \alpha L_\beta (b_{j;i} - B_j{}^k{}_i b_k) y^j.$$

If $b_{j;i} = 0$, then the uniqueness of the theorem leads to $B_j{}^k{}_i = 0$ immediately. The converse is not true; $b_{i;j} = 0$ is not necessary for F^n to be a Berwald space. For instance, as has been shown in [9], a Randers space with $L = \alpha + \beta$ is a Berwald space, if and only if $b_{i;j} = 0$, while a Kropina space with $L = \alpha^2/\beta$ is a Berwald space, if and only if there exists a vector field $f_i(x)$ satisfying $b_{i;j} = (f^r b_r)a_{ij} + b_i f_j - b_j f_i$.

Definition. Let a Finsler space F^n with $L(\alpha, \beta)$–metric (α, β) be a Berwald space. If b_i is necessarily parallel in the associated Riemannian space, then F^n is called a *parallel Berwald space* and $L(\alpha, \beta)$ is of *parallel type*.

We give here some example of parallel Berwald spaces.

Ex. 1 [13]

$$L = (\alpha^s + \ldots + c_k \alpha^{s-k}\beta^k + \ldots + \beta^s)^r,$$

where $rs = 1$ and const. c's, is of parallel type.

Ex. 2 [9], [10].

$$\begin{aligned} L &= c_1\alpha + c_2\beta + \beta^2/\alpha, \quad c_2 \neq 0, \\ L &= c_1\alpha + c_2\beta + \alpha^2/\beta, \quad c_1 \neq 0, \end{aligned}$$

where const. c's, are of parallel type.

We consider a Finsler space F^n with $L(\alpha,\beta)$ of parallel type and conformal to a Berwald space $\bar{F}^n$. Then $S=\beta/\alpha$ is conformally invariant and $\bar{S}_{;i}=0$ in the Levi-Civita connection $\bar{\gamma}$ of $\bar{R}^n$. Therefore we have

Theorem 3. *Let $F^n=(M^n,L(\alpha,\beta))$ be a Finsler space with (α,β)-metric of parallel type. F^n is a conformally Berwald space, if and only if the conformal Berwald connection with respect to β/α has the vanishing hv-curvature tensor and satisfies* (1.12′).

4 Conformally Berwald Randers spaces and Kropina spaces

The last section is devoted to the conditions for Finsler spaces of Randers type and Kropina type to be conformally Berwald. We shall use the symbols

$$r_{ij}=(b_{i;j}+b_{j;i})/2,\quad s_{ij}=(b_{i;j}-b_{j;i})/2,\quad s_j=b^i s_{ij},$$

where the covariant differentiation $(;)$ is the one with respect to the associated Riemannian space with the metric α. By a conformal change $L\to\bar{L}=e^{c(x)}L$ various quantities are changed as follows:

$$\bar{a}_{ij}=e^{2c}a_{ij},\quad \bar{b}_i=e^c b_i.$$

Putting $c_i=\partial_i c$ and $c^i=a^{ir}c_r$, the Christoffel symbols $\gamma_j{}^i{}_k$ constructed from a_{ij} are changed to

$$\bar{\gamma}_j{}^i{}_k=\gamma_j{}^i{}_k+\delta^i{}_j c_k+\delta^i{}_k c_j-c^i a_{jk},$$

and hence we obtain

$$\bar{b}_{i;j}=e^c(b_{i;j}-c_i b_j+b^r c_r a_{ij}).$$

First we are concerned with a Randers space with the metric $L=\alpha+\beta$. It is a Berwald space, if and only if $b_{i;j}=0$ [9]. Consequently the space is conformally Berwald, if and only if there exists a gradient $c_i(x)$ satisfying

$$b_{i;j}-c_i b_j+b^r c_r a_{ij}=0. \tag{4.1}$$

From (4.1) we get

$$b^j b_{i;j}=b^2 c_i-b^r c_r b_i,\quad a^{ij}b_{i;j}=-(n-1)b^r c_r.$$

Consequently we have

$$c_i=(b^r b_{i;r}-a^{rs}b_{r;s}b_i/(n-1))/b^2. \tag{4.2}$$

Since c_i is a gradient vector, we have

$$c_{i;j}-c_{j;i}=0. \tag{4.3}$$

(4.1) can be written as

$$r_{ij}=(c_i b_j+c_j b_i)/2-b^r c_r a_{ij},\quad s_{ij}=(c_i b_j-c_j b_i)/2.$$

These give respectively

$$a^{rs}r_{rs} = -(n-1)b^r c_r, \quad s_j = (b^r c_r b_j - b^2 c_j)/2.$$

Hence we have

(4.4) $$r_{ij} = (r^s{}_s/(n-1))(a_{ij} - b_i b_j/b^2) - (b_i s_j + b_j s_i)/b^2$$

(4.5) $$s_{ij} = (b_i s_j - b_j s_i)/b^2.$$

Now (4.2) can be written as

$$c_i = (b^r r_{ir} - s_i - a^{rs} r_{rs} b_i/(n-1))/b^2,$$

and (4.4) gives $b^r r_{ir} = -s_i$. Therefore we have

(4.6) $$c_i = -(2s_i + r^s{}_s b_i/(n-1))/b^2.$$

Therefore we have

Theorem 4. *A Randers space is conformally Berwald, if and only if (4.4) and (4.5) hold and c_i given by (4.6) is gradient, that is, satisfies (4.3).*

Let $F^n = (M^n,\ L = \alpha^2/\beta)$ be a Kropina space and $\bar{F}^n = (M^n, \bar{L})$ a conformally changed space with $\bar{L} = e^{c(x)}L$. The latter is a Berwald space [9], if and only if there exists f_i satisfying

$$\bar{b}_{i;j} = (\bar{b}^r f_r)(\bar{a}_{ij} + \bar{b}_i f_j - \bar{b}_j f_i.$$

From $\bar{b}_{i;j} = e^c(b_{i;j} - c_i b_j + b^r c_r a_{ij})$ and $\bar{b}^i = e^{-c} b^i$ the above is written as

(4.7) $$b_{i;j} - c_i b_j + b^r c_r a_{ij} = b^r f_r a_{ij} + b_i f_j - b_j f_i,$$

which is equivalent to

(4.8) $$r_{ij} - (b_i c_j + b_j c_i)/2 + b^r c_r a_{ij} = b^r f_r a_{ij}.$$

(4.9) $$s_{ij} + (b_i c_j + b_j c_i)/2 = b_i f_j - b_j f_i.$$

Multiplying b^i to (4.9) yields

(4.10) $$s_j = b^2(f_j - c_j/2) - b_j(f_i - c_i/2)b^i.$$

Consequently, eliminating f_i from (4.9) we obtain

(4.11) $$s_{ij} = (b_i s_j - b_j s_i)/b^2.$$

Next we deal with (4.8). Put

(4.12) $$u = b^r(c_r - f_r), \quad b^i r_{ij} = br_j \text{ and } b^j r_j = br.$$

(4.8), transvected with b^i yields

(4.13) $$br_j - (b^i c_i b_j + b^2 c_j)/2 + ub_j = 0.$$

Multiplying b^j and from $b^2 \neq 0$ for the Kropina space, we obtain $r + u = b^i c_i$. Then (4.13) gives

$$c_j = (2br_j + (u - r)b_j)/b^2. \tag{4.14}$$

As a consequence (4.8) may be written in the form

$$r_{ij} = (b_i r_j + b_j r_i)/b + (u - r)b_i b_j/b^2 - u a_{ij}. \tag{4.15}$$

(4.14) gives $b^j c_j = u + r$, and hence (4.12) yields $b^r f_r = r$. Consequently (4.10) yields

$$f_i = s_i/b^2 + r_i/b. \tag{4.16}$$

Conversely, we consider a Kropina space F^n such that (4.15) and (4.11) are satisfied and c_j of (4.14) is gradient ($c_j = \partial_j c(x)$). We make the conformally changed $\bar{F}^n$ from F^n by the conformal change $L \to \bar{L} = e^{c(x)}L$. Then (4.15), (4.11) and (4.14) lead to

$$b_{i;j} - c_i b_j + b^r c_r a_{ij} = r_{ij} + s_{ij} - c_i b_j + b^r c_r a_{ij} =$$
$$= r a_{ij} + b_i(r_j/b + s_j/b^2) - b_j(r_i/b + s_i/b^2).$$

Thus, (4.16) immediately leads to (4.7).

Theorem 5. *A Kropina space is conformally Berwald, if and only if (4.15) and (4.11) hold and c_j of (4.14) is gradient.*

References

[1] T.Aikou and M.Hashiguchi, *On the Cartan and Berwald expressions of Finsler connections*, Rep. Fac. Sci. Kagoshima Univ. 19(1986), 7–17.

[2] M.Hashiguchi, *On conformal transformations of Finsler metrics*, J. Math. Kyoto Univ. 16(1976), 25–50.

[3] M.Hashiguchi and Y.Ichijyo, *On some special (α, β)–metrics,* Rep. Fac. Sci. Kagoshima Univ. 8(1975), 39–46.

[4] F.Ikeda, *Criteria for conformal flatness of Finsler space*, Balkan J. Geom. and Its Applications, 2(1997), 63–68.

[5] Y.Ichijyo and M.Hashiguchi, *On the condition that a Randers space be conformally flat*, Rep. Fac. Sci. Kagoshima Univ. 22(1989), 7–14.

[6] S.Kikuchi, *On the condition that a FInsler space be conformally flat*, Tensor, N.S., 55(1994), 97–100.

[7] M.Kitayama, M.Asuma and M.Matsumoto, *On Finsler space with (α, β)–metric, regularity, geodesics and main scalars*, J. Hokkaido Univ. of Education, 46(1995), 1–10.

[8] M.Matsumoto, *Foundations of Finsler Geometry and Special Finsler spaces*, Kaiseisha Press, Saikawa, Otsu, Japan, 1986.

[9] M.Matsumoto, *The Berwald connections of a Finsler spaces with* (α, β)*–metric*, Tensor, N.S. 50(1991), 18–21.

[10] M.Matsumoto, *A special class of locally Minkowski spaces with* (α, β)*–metric and conformally flat Kropina spaces*, Tensor N.S., 50(1991), 202–207.

[11] M.Matsumoto, *Theory of Finsler spaces with* (α, β)*–metric*, Rep. on Math. Phys. 31(1992), 43–83.

[12] M.Matsumoto, *Reduction theorems of Landsberg spaces with* (α, β)*–metric*, Tensor N.S., 58(1997), 160–166.

[13] C.Shibata, *On Finsler space with an* (α, β)*–metric*, J. Hokkaido Univ. of Education, 35(1984), 1–16.

Shun-ichi Hojo
Dept. of Applied Math.
Konan University
Okamoto, Higashinada
Kobe, 658, Japan

Makoto Matsumoto
15 Zenbu-cho, Shomogamo
Sakyo-ku, 606-0815
Kyoto, Japan

Katsumi Okubo
Faculty of Education, Shiga University
2-5-1 Hiratsu, Otsu, 520, Japan