On 2 - Framed Riemannian Manifolds with Godbillon - Vey Structure Form

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

In the last decade, contact, almost contact, paracontact cosymplectic, and conformal cosymplectic manifolds carrying $\kappa > 1$ structure vector fields ξ have been studied by many authors, e.g. [2], [7], [11], [15].

In the present paper we consider a $(2m+2)$-dimensional Riemannian manifold carrying two structure vector fields ξ^r ($r \in \{2m+1,\ 2m+2\}$), a $(1,1)$-tensor field Φ, and a structure 2 - form Ω of rank $2m$, such that for $\eta^r := (\xi_r)^\flat$

$$\begin{aligned} &\Phi^2 = -Id + \eta^r \otimes \xi_r \qquad \Phi\,\xi_r = 0, \qquad \eta^r\,(\xi_s) = \delta^r_s \\ &\Omega(Z,\ Z') = g(\Phi\, Z,\ Z'), \qquad \Omega^m \wedge \eta^{2m+1} \wedge \eta^{2m+2} \neq 0 \end{aligned} \tag{0.1}$$

holds. Here the $(2m)$-dimensional subspace $Im\Phi$ of the tangent space is supposed to be Kählerian (see eq. (2.12) below). If the 3-forms

$$\gamma^r = \eta^r \wedge d\eta^r \tag{0.2}$$

satisfy

$$d\gamma^r = 0\ , \tag{0.3}$$

they are called *Godbillon-Vey forms* [6]. On the other hand, if

$$\begin{aligned} &\nabla_X\, \xi_r = f_r\, X \\ &r = 2m+1,\ 2m+2 \end{aligned} \tag{0.4}$$

holds for all X orthogonal to ξ_r and for some $f_r \in \Lambda^0 M$, the structure vector fields define a *concircular pairing* [1]. It will turn out that (0.3) follows from (0.1) and (0.4). Therefore we call such manifolds $M(\Phi,\ \Omega,\ \eta^r,\ \xi_r)$ *2-framed Godbillon-Vey manifold* (abbreviated *2FG-V*). We shall prove that they have the following properties:

Any 2FG-V manifold is equipped with a conformal symplectic structure $CSp(m+1,\ I\!R)$ with $\xi := \sum f_r\, \xi_r$ as vector of Lee, i.e.

$$d\Omega = 2\xi^\flat \wedge \Omega \tag{0.5}$$

and M is the local *Riemannian* product

$$M = M^\perp \times M^\top$$

such that

Balkan Journal of Geometry and Its Applications, Vol.5, No.1, 2000, pp. 57-67

1. $M^{\perp}$ is a flat surface tangent to the structure vector fields ξ_r;
2. $M^{\top}$ is a $2m$-dimensional Kählerian submanifold, and the immersion $x : M^{\top} \to M$ has the following properties:
   (a) The mean curvature vector field H associated with x is $-\xi$ and satisfies $\| H \|^2 =$ const.
   (b) The immersion x is umbilical. In section 3, the existence of a horizontal skew symmetric conformal (abbreviated SC) vector field C is proved by an exterior differential system in involution (in the sense of E. Cartan [3]). Denote by K and R the scalar curvature of M and the Ricci tensor field of ∇, respectively. Then

   $$\mathcal{L}_C K = -\rho K\,; \quad \mathcal{L}_C R(Z, Z') = 0\,; \quad \rho = const.\,; \quad Z, Z' \in \mathcal{X}M$$

   and C is a module commuting vector field, i.e.

   $$[C, \nabla \| C \|^2] = 0\,, \qquad \nabla : \text{gradient of a scalar}\ .$$

   (c) C defines an infinitesimal homothety of all $(2q+1)$-forms $(C^{\flat})_q := C^{\flat} \wedge \Omega^q$, i.e.

   $$\mathcal{L}_C (C^{\flat})_q = (q+1)(C^{\flat})_q\,,$$

   and ΦC defines an infinitesimal automorphism of Ω:

   $$\mathcal{L}_{\Phi C}\, \Omega = 0\,.$$

Mathematics Subject Classification: 53C25
Key words: Riemannian manifold, 2-framed structure, concircular pairing, Godbillon - Vey form

1 Preliminaries

Let (M, g) be a Riemannian C^{∞}-manifold and ∇ the covariant differential operator with respect to the metric g. We assume that M is oriented and ∇ is the Levi-Civita connection.

Define $\Gamma(TM) =: \mathcal{X}M$ and let $TM \underset{\sharp}{\overset{\flat}{\rightleftarrows}} T^*M$ be the *musical isomorphism* defined by g and

$$\Omega^{\flat} \ : \ TM \to T^*M\,; \quad Z \to -i_Z \Omega =: \ {}^{\flat}Z$$

the *symplectic isomorphism* defined by Ω. Following Poor [10], we set

$$A^q(M, TM) := Hom(\Lambda^q TM, TM)$$

and notice that the elements of $A^q(M, TM)$ are vector valued q-forms. The local field of orthonormal frames on an n-dimensional Riemannian manifold is denoted by

$$\mathcal{O} = \{e_A;\ A = 1, \cdots, n\}$$

and the associated coframe by

$$\mathcal{O}^* = \{\omega^A;\ A = 1, \cdots, n\}\ .$$

The soldering form dp is expressed by

$$dp = \omega^A \otimes e_A \tag{1.6}$$

and Cartan's structure equations in index-free notation are written as

$$\nabla e = \theta \otimes e \tag{1.7}$$

$$d\omega = -\theta \wedge \omega \tag{1.8}$$

$$d\theta = -\theta \wedge \theta + \Theta \; . \tag{1.9}$$

Here the 1-forms θ and the 2-form Θ are the connection forms in the tangent bundle TM and the curvature form, respectively.

Now let W be a conformal vector field, i.e. a vector field satisfying the conformal version of Killing's equation

$$\mathcal{L}_W \, g = \rho \, g \; , \tag{1.10}$$

where the conformal scalar ρ is defined by

$$\rho = \frac{2}{dim M}(div W) \; . \tag{1.11}$$

We recall some basic formulas [14] which will be needed in the last section:

$$\mathcal{L}_W K = (n-1)\,\Delta\rho - K\,\rho \; ; \qquad n = dim\, M \tag{1.12}$$

$$2\,\mathcal{L}_W \, R(Z,\, Z') = g(Z,\, Z')\,\Delta\rho - (n-2)(Hess_\nabla\, \rho)(Z,\, Z'), \tag{1.13}$$

where

$$(Hess_\nabla\, \rho)(Z,\, Z') = g(Z,\, \nabla_{Z'}\, grad\, \rho) \; .$$

In these equations $\mathcal{L}_W$, K, Δ and R denote the Lie derivative with respect to W, the scalar curvature of M, the Laplacian and the Ricci tensor field of ∇ respectively.

2 2-Framed Godbillon - Vey manifolds

Let $M(\Phi,\, \Omega,\, \eta^r,\, \xi_r,\, g)$ be a $(2m+2)$ - dimensional Riemannian manifold carrying two structure vector fields ξ_r $(r \in 2m+1,\, 2m+2)$ and let η^r be their associated covectors. Suppose that the structure tensors $(\Phi,\, \Omega,\, \eta^r,\, \xi_r)$ satisfy (0.1). Then M carries a 2-framed structure in the sense of Yano and Kon [15]. We further assume that (0.4) holds. Defining $e_r := \xi_r$ and $\omega^r := \eta^r$, this yields

$$f_r\, \omega^a = \theta^a_r \; , \qquad f_r \in \Lambda^0 M \; , \qquad a = 1, \cdots, 2m \tag{2.1}$$

and

$$\begin{aligned} d\eta^{2m+1} &= \;\; u \wedge \eta^{2m+2} \\ d\eta^{2m+2} &= -u \wedge \eta^{2m+1} \end{aligned} \quad , \tag{2.2}$$

where u is some closed 1-form. In the same way, (0.4) ensures that $d\gamma^r = 0$ holds. (2.2) can be written as

$$u = \theta^{2m+2}_{2m+1} \quad . \tag{2.3}$$

Connections satisfying (2.1) are called *principal connections* [12].
One may split the soldering form dp in a unique manner as

$$dp = dp^{\top} \otimes dp^{\perp}, \tag{2.4}$$

where $dp^{\top} := \omega^a \otimes e_a$ and $dp^{\perp} := \eta^r \otimes \xi_r$ are called the *horizontal* and the *vertical* component of dp, respectively. From (2.3) and (2.1) one finds

$$\begin{aligned} \nabla \xi_{2m+1} &= f_{2m+1}\, dp^{\top} + u \otimes \xi_{2m+2} \\ \nabla \xi_{2m+2} &= f_{2m+2}\, dp^{\top} - u \otimes \xi_{2m+1} \quad . \end{aligned} \tag{2.5}$$

Hence we have

$$\begin{aligned} \nabla_{\xi_{2m+2}} \xi_{2m+1} &= \quad u(\xi_{2m+2})\, \xi_{2m+2} \\ \nabla_{\xi_{2m+1}} \xi_{2m+2} &= -u(\xi_{2m+1})\, \xi_{2m+1}, \end{aligned}$$

and referring to [1] one may say that the structure vector fields ξ_r define a *concircular pairing*. Then (2.5) and the well-known formula

$$div\, Z = tr(\nabla Z) = \sum_{a=1}^{2m} \omega^a \left(\nabla_{e_a} Z\right) + \sum_{r=2m+1}^{2m+2} \eta^r \left(\nabla_{\xi_r} Z\right) , \quad Z \in \mathcal{X} M$$

yield

$$\begin{aligned} div\, \xi_{2m+1} &= 2\, m\, f_{2m+1} + u(\xi_{2m+2}) \\ div\, \xi_{2m+2} &= 2\, m\, f_{2m+2} + u(\xi_{2m+1}) \quad . \end{aligned}$$

If u is a *basic form*, i.e. if $u(\xi_r) = 0$, then (2.2) entails

$$i_{\xi_r}\, d\eta^r = 0.$$

Therefore, according to a well known definition, we may say that ξ_r move to Reeb vector fields (in the large).

In the general case, i.e. $u(\xi_r) \neq 0$, we shall say that the manifold $M(\Phi, \Omega, \eta^r, \xi_r, g)$ is endowed with a *2-framed Godbillon - Vey structure*, (abbreviated 2FG-V structure). Referring to [11] we call the distribution $D^{\perp} := \{\xi_r; r = 2m+1, 2m+2\}$ the *vertical distribution*, and its orthogonal complement $D^{\top} := \{e_a, a = 1, \cdots, 2m\}$ the *horizontal distribution* on M. Similarly

$$\varphi^{\perp} := \eta^{2m+1} \wedge \eta^{2m+2}$$

and

$$\varphi^{\top} := \omega^1 \wedge \cdots \wedge \omega^{2m} \tag{2.6}$$

are called the *vertical* and the *horizontal form*, respectively. With these definitions, (2.2) gives immediately

$$d\varphi^{\perp} = 0\ .$$

Therefore it follows from *Frobenius'* theorem that the horizontal distribution $D^{\top}$ is involutive. Setting

$$\eta := \sum_{r=2m+1}^{2m+2} f_r\, \eta^r\ , \tag{2.7}$$

(2.6) and (2.1) yield

$$d\varphi^{\top} = 2\, m\, \eta \wedge \varphi^{\top}\ . \tag{2.8}$$

This shows that $\varphi^{\top}$ is an exterior recurrent form [5] and consequently $D^{\perp}$ is also involutive. Hence any 2FG-V manifold is the local Riemannian product

$$M = M^{\top} \times M^{\perp} ,$$

where $M^{\top}$ is a $2m$-dimensional manifold tangent to $D^{\top}$ and $M^{\perp}$ is a surface tangent to $D^{\perp}$.

Since η is the recurrence form of $\varphi^{\top}$ (see (2.8)), it is closed. (Generally, we shall call an exterior recurrent form *strictly recurrent*, if its recurrence form is closed.) This fact together with (2.7) and (2.2) give

$$\begin{aligned} df_{2m+1} &= \quad f_{2m+2}\, u \\ df_{2m+2} &= \ -f_{2m+1}\, u\,. \end{aligned} \tag{2.9}$$

Therefore the Poisson bracket $\{\ \ \}_P$ of the function f_r, i.e.

$$\{f_{2m+1},\, f_{2m+2}\}_P := \Omega(\nabla f_{2m+1},\, \nabla f_{2m+2})$$

vanishes. Defining

$$\xi := \sum_{r=2m+1}^{2m+2} f_r\, \xi_r\,; \qquad \eta := \sum_{r=2m+1}^{2m+2} f_r\, \eta^r = \xi^\flat$$

one easily deduces from (2.9) that

$$\|\xi\|^2 = (f_{2m+1})^2 + (f_{2m+2})^2 =: 2\, f = const. \tag{2.10}$$

and further from (2.9), (2.4), and (2.5):

$$\nabla\xi = 2\, f\, dp^{\top}. \tag{2.11}$$

On the other hand using (2.3), (2.1), (1.9), $du = 0$ (see (2.2)) and the fact that $\theta^a_{2m+2} = -\theta_a^{2m+2}$ holds because of $g(e_{2m+2}, e_a) = 0$, one finds

$$\Theta^{2m+2}_{2m+1} = 0\ .$$

It is easily seen that Θ^{2m+2}_{2m+1} is the curvature form of $M^{\perp}$. Therefore this surface is *flat*. Further, because of (0.1), the horizontal connection forms satisfy the Kähler relations

$$\theta^i_j = \theta^{i^*}_{j^*}\ ; \qquad \theta^{i^*}_j = \theta^{j^*}_i\ ; \qquad i = 1, \cdots, m\ ; \qquad i^* = i + m. \tag{2.12}$$

Recalling the standard expression for the structure 2-form Ω

$$\Omega = \sum_{i=1}^{m} \omega^i \wedge \omega^{i^*}\ ; \qquad i^* = i + m, \tag{2.13}$$

we find with the help of (2.1) and (2.7), after some calculation,

$$d\Omega = 2\,\eta\, \wedge\, \Omega\ . \tag{2.14}$$

This shows the important fact that the 2FG-V manifold under discussion is endowed with a *locally conformal symplectic structure* $CSp(m+1, \mathbb{R})$, with $\eta = \xi^\flat$ as covector of Lee. Since $i_\xi\, \Omega = 0$ and $f = const.$ (see (2.10)), one gets from (2.13):

$$\mathcal{L}_\xi\, \Omega = 2\, f\, \Omega\ , \tag{2.15}$$

which shows that ξ defines an *infinitesimal homothety* of Ω.

On the other hand, $\Omega_{|_{M^{\top}}}$ is of rank $2m$. Therefore it is the symplectic form of the Kähler submanifold $M^{\top}$ of M. Next let H be the mean curvature vector field associated with the immersion $x : M^\top \to M$. If γ^A_{BC} denote the coefficients of the connection θ, the vector field H is given by

$$H = \frac{1}{2m} \sum_{a=1}^{2m} \gamma^r_{aa} \, \xi^r \; .$$

(We denote the induced elements by the same letters.) Now using (2.1) and (2.10), an easy calculation gives

$$H = -\xi \quad \Rightarrow \quad \|H\|^2 = 2 f = const.$$

Hence one deduces the following important fact: $M^\top$ is a Kähler submanifold of M of *constant mean curvature.* Moreover, since $dp^\top$ is the soldering form of $M^\top$, it follows from (2.4) that the second quadratic forms associated with the immersion $x : M^\top \to M$ are

$$l_r = - < dp^\top , \nabla \xi_r >= -f_r \, g^\top \; .$$

This means that the immersion $x : M^\top \to M$ is *umbilical.*
Summing up we state

Theorem 1. *Let $M(\Phi, \Omega, \xi_r.\eta^r, g)$ be a $(2m+2)$-dimensional Riemannian manifold endowed with a 2 FG-V structure defined by (0.1) - (0.3). Such a manifold admits a locally conformal symplectic structure with $\xi^\flat$ as covector of Lee, i.e.*

$$d\Omega = 2 \, \xi^\flat \wedge \Omega \; .$$

Furthermore M is the local Riemannian product

$$M = M^\perp \times M^\top ,$$

where

1. *$M^\perp$ is a flat surface tangent to the structure vector fields ξ_r.*
2. *$M^\top$ is a $2m$-dimensional Kählerian submanifold, and the immersion $x : M^\top \to M$ has the following properties:*
   - (a) *$M^\top$ is of constant mean curvature.*
   - (b) *The immersion $x : M^\top \to M$ is umbilical.*

3 Skew symmetric conformal vector fields

In this section we assume that the 2FG-V manifold under consideration carries a *horizontal skew symmetric conformal* (abr. SSC) *vector field* C. The generative of C is assumed to be the Reeb vector field ξ. This means [9]

$$\nabla C = \lambda \, dp + C \wedge \xi \; . \tag{3.1}$$

Here $\wedge$ denotes the wedge product of vectors: $C \wedge \xi := \xi^\flat \otimes C - C^\flat \otimes \xi$. One may set

$$C = C^a \, e_a \; \in \; D^\top \; ; \qquad a, b \in \{1, \cdots, 2m\}.$$

Then it follows from (2.1), (3.1), and (1.7):

$$dC^a + C^b\,\theta^a_b = \lambda\,\omega^a + C^a\,\eta\ . \tag{3.2}$$

Clearly, from

$$C^\flat = \sum_{a=1}^{2m} C^a\,\omega^a \tag{3.3}$$

one obtains

$$dC^\flat = 2\,\eta \wedge C^\flat\ . \tag{3.4}$$

This agrees with Rosca's lemma [9]. As a simple consequence of (3.2), one derives

$$d\|C\|^2 = 2\,\lambda\,C^\flat - 2\,\|C\|^2\,\eta\ . \tag{3.5}$$

Denote now by Σ the exterior differential system which defines the vector field C. Then because of $d\eta = 0$, (3.4) and (3.5), the characteristic numbers of Σ are $r = 3,\ s_0 = 1,\ s_1 = 2$. Since $r = s_0 + s_1$ holds, it follows that Σ is in involution (in the sense of E. Cartan [3]). Therefore Cartan's test states that C exists and depends on two arbitrary functions of one argument. On the other hand, recall that the symplectic isomorphism (see also [8]) is expressed as

$$Z \to -i_Z\,\Omega = {}^\flat Z =: \Omega^\flat\,(Z)\ , \qquad \Omega(Z,\,Z') =: < Z',\,Z >\ . \tag{3.6}$$

So one may write

$$i_C\,\Omega = -{}^\flat C = \sum_{i=1}^{m} (C^i\,\omega^{i^*} - C^{i^*}\,\omega^i) =: \beta\ ,$$

where we have set $\beta := -{}^\flat C$. From (2.12), (2.14), and (3.2), one derives:

$$d\beta = 2\,\lambda\,\Omega + 2\,\eta \wedge \beta\ .$$

Again an exterior derivation yields $\lambda = const$ (remember $d\eta = 0$.) On the other hand, from

$$\mathcal{L}_Z\,g = \frac{2\,div\,Z}{dim\,M}\,g = \rho\,g\ ; \qquad Z \in \mathcal{X}(M)$$

(cf. (1.11)) and from (3.1), one quickly finds

$$\rho = 2\,\lambda. \tag{3.7}$$

This means that C defines an *infinitesimal homothety* of M, because using (2.13) and (2.15), one obtains at once

$$\mathcal{L}_C\,\Omega = \rho\,\Omega$$

and

$$\mathcal{L}_\xi\,\Omega = 2\,f\,\Omega$$

(remember $f = const.$). Furthermore, let L be the operator of type (1,1) given by

$$L\,u := u \wedge \Omega\ ; \qquad u\ \in\ \Lambda^1 M$$

and define (cf. [6])

$$L^q\,u := u_q := u \wedge \Omega^q\ \in\ \Lambda^{2q+1} M\ .$$

Coming back to the case under discussion, (3.4) yields

$$\mathcal{L}_C\, C^\flat = \rho\, C^\flat\ .$$

This shows that $C^\flat$ is a self-conformal form. A standard calculation gives

$$\mathcal{L}_C (C^\flat)_q = (q+1)(C^\flat)_q\ .$$

Therefore C defines an infinitesimal homothety of all these $(2q+1)$-forms.
With Yano's formulas (1.12) and (1.13), one finds

$$\mathcal{L}_C\, K = -\rho\, K$$

and

$$\mathcal{L}_C\, R(Z,\, Z') = 0\ ; \qquad Z,\, Z' \in \mathcal{X}(M)\ ,$$

where K and R denote the scalar curvarure of M and the Ricci tensor field, respectively. Now, for any vector field Z, one has

$$(\nabla\Phi)\, Z = \nabla(\Phi\, Z) - \Phi\, \nabla Z\ .$$

Therefore (0.1) and (3.1) yield

$$\begin{aligned}(\nabla\Phi)\, C &= (\frac{\rho}{2} - \lambda - \eta(C))\, \Phi\, dp - (\Phi\, C)^\flat \otimes \xi \\ &= \nabla(\Phi\, C) - \lambda\, \Phi\, dp - \eta\,(\Phi\, C).\end{aligned}$$

Hence

$$\begin{aligned}\nabla(\Phi\, C) &= \left(\frac{\rho}{2} - \eta(C)\right)\, \Phi\, dp + \eta\,(\Phi C) - (\Phi C)^\flat \otimes \xi \\ &= \left(\frac{\rho}{2} - \eta(C)\right)\, \Phi\, dp + \Phi\, C \wedge \xi\end{aligned} \tag{3.8}$$

($\wedge$: wedge product of vector fields). From the inner product $< Z,\, \Phi\, dp >= \Phi\, Z$, and from (3.8), one derives

$$< \nabla_Z \Phi C,\, Z' > + < \nabla_{Z'} \Phi C,\, Z >= 0\ ; \qquad Z,\, Z' \in \mathcal{X}(M)\ .$$

Furthermore, since C is a horizontal vector field, it is easily seen that

$$^\flat\Phi C = C^\flat$$

holds. So together with (2.13), this leads to

$$\mathcal{L}_{\Phi C} \Omega = 0\ .$$

Therefore $\Phi\, C$ defines an infinitesimal automorphism of Ω.
It should be noticed that (2.10), (3.1), and (3.8) entail

$$[\xi,\, \Phi C] = 0\ ; \qquad [C,\, \Phi C] = 0\ ; \qquad [C,\, \xi] = -\frac{\rho}{2}\, \xi\ .$$

So ξ and C commute with ΦC, and ξ admits an infinitesimal homothety of generators C [4].
Let now $\mathcal{C}\ :\ (M,\, g)\ \rightarrow\ (\tilde{M},\, \tilde{g})$ be a *conformal diffeomorphism* (abr. CD) of argument t, i.e.

$$\mathcal{C} : g \mapsto \tilde{g} := e^{2t} g .$$

One has (see also [10])

$$\tilde{\nabla} C = \nabla C + (\nabla t)^{\flat} \otimes C - C^{\flat} \otimes \nabla t + g(C, \nabla t)\, dp ,$$

and the scalar curvature $\tilde{K}$ of $\tilde{M}$ is given by

$$\tilde{K} = e^{-2t} \left(K + 2(2m+1)\, div\, \nabla t + (2m+1)\, 2m\, \|\nabla t\|^2 \right) .$$

Since $K = const.$, the manifold $\tilde{M}$ is homothetic to M, if it satisfies $\|\nabla t\|^2 = const.$ and $div\, \nabla t = const.$ Furthermore

$$d\|C\|^2 = \rho\, C^{\flat} + 2\, \|C\|^2\, \eta ,$$

and the gradient (which will also be denoted by ∇) of the function $\|C\|^2$ is expressed by

$$\nabla \|C\|^2 = \rho\, C + 2\, \|C\|^2\, \xi . \tag{3.9}$$

Thus from

$$div\, C = (m+1)\, \rho = const. ; \qquad div\, \xi = 4\, m\, f = const.$$

(see (2.5), (2.9), and (2.10)) one quickly derives

$$\Delta\, \|C\|^2 = -div\, \nabla\, \|C\|^2 = -\kappa\, f\, \|C\|^2 - (m+1)\, \rho^2 ; \quad \kappa \in I\!R . \tag{3.10}$$

Therefore as an extension of a well-known definition (see e.g. [13]), we may say that $\|C\|^2$ is an *almost eigenfunction* of Δ with $-\kappa\, f$ as eigenvalue. We notice that if C is a Killing vector field, i.e. if $\rho = 0$ (see (3.1) and (3.7)), then $\|C\|^2$ becomes an eigenfunction of Δ. Since the eigenvalue is negative definite, the corresponding manifold cannot be compact.

We recall that a function $\nu : I\!R \to I\!R$ is isoparametric, iff both, $\|\nabla \nu\|^2$ and $div\, (grad\, \nu)$ are functions of ν [13]. Then from (3.9) and (3.10), it is quickly seen that $\|C\|^2$ is an *isoparametric function*.

Finally, setting

$$\nabla^2 \|C\|^2 := \nabla\, grad\, \|C\|^2$$

in (3.1), one deduces after a short calculation

$$[C, \nabla \|C\|^2] = 0 .$$

This shows that C is a module commuting vector field. Thus we have proven

Theorem 2. *Let C be a horizontal skew symmetric conformal vector field on the 2FG-V manifold defined by conditions (0.1) - (0.3). Such a C always exists; it is determined by an exterior differential system in involution. C infinitesimal homothety on M, i.e.*

$$\mathcal{L}_C\, K = -\rho\, K ; \qquad \text{K: scalar curvature of } M;\ \rho = const.$$

Moreover:

1.

$$\mathcal{L}_C \, R(Z, Z') = 0 \, , \qquad Z, Z' \in \mathcal{X}M \, ,$$

where R denotes the Ricci tensor field, and

$$\mathcal{L}_C (C^\flat)_q = (q+1)(C)^\flat_q \, .$$

Here $L^q : C^\flat \to (C^\flat)_q := C^\flat \wedge \Omega^q$ is the (1,1) - Weyl operator.

2. *ΦC defines an infinitesimal automorphism of Ω, i.e.*

$$\mathcal{L}_{\Phi C} \Omega = 0 \, ,$$

and ξ and C commute with ΦC. In addition, ξ admits an infinitesimal homothety of generators C, i.e.

$$[\xi, \Phi C] = 0 \, ; \qquad [C, \Phi C] = 0 \, ; \qquad [C, \xi] = -\frac{\rho}{2} \xi \, .$$

3. *$\|C\|^2$ is an almost eigenfunction of Δ, as well as an isoparametric function, and C is a module commuting vector field.*

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