On Contact Metric R-Harmonic Manifolds

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Dedicated to Prof.Dr. Constantin UDRIŞTE
on the occasion of his sixtieth birthday

Abstract

In this paper we consider contact metric R-harmonic manifolds M with ξ belonging to (κ,μ)-nullity distribution. In this context we have $\kappa \leq 1$. If $\kappa < 1$, then M is either locally isometric to the product $\mathbf{E}^{n+1} \times S^n(4)$, or locally isometric to $E(2)$ (the group of the rigid motions of the Euclidean 2-space). If $\kappa = 1$, then M is an Einstein-Sasakian manifold.

Mathematics Subject Classification: 53C05, 53C20, 53C21, 53C25.
Key words: contact metric manifold, Einstein manifold, (k,μ)-nullity distribution, R- harmonic manifold

1 Introduction

Throughout this paper we use the notations and terminology of [1] and [2]. Let M be a $(2n+1)$–dimensional Riemannian C^∞ manifold. M^{2n+1} is said to be *contact manifold*, if it admits a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$, everywhere on M^{2n+1}. Given a contact form η, we have a unique vector field ξ, which is called the *characteristic vector field*, satisfying $\eta(\xi) = 1$, $d\eta(\xi, X) = 0$, for any vector field X.

It is well–known that, there exists a Riemannian metric g and a (1,1)–tensor field φ such that

(1) $$\eta(X) = g(X,\xi),\ d\eta(X,Y) = g(X,\varphi Y) \text{ and } \varphi^2 X = -X + \eta(X)\xi,$$

where X and Y are vector fields on M^{2n+1}.

From (1) it follows that $\eta \circ \varphi = 0$, $\varphi(\xi) = 0$, $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$.

A Riemannian manifold M^{2n+1} equipped with structure tensors (φ, ξ, η, g) satisfying (2) is said to be a *contact metric manifold* and denoted by $M = (M^{2n+1}, \varphi, \xi, \eta, g)$.

Given a contact metric manifold M we can define a $(1,1)$-tensor field h by $h = \frac{1}{2} L_\xi \varphi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies $h\xi = 0$ and $h\varphi = -\varphi h, \nabla_X \xi = -\varphi X - \varphi h X$, where ∇ is Levi-Civita connection [2]. A contact metric manifold for which ξ is Killing vector field is called K

Balkan Journal of Geometry and Its Applications, Vol.5, No.1, 2000, pp. 1-6

-contact manifold. It is well-known that a contact manifold is K-contact if and only if $h = 0$.

We denote by R the *Riemannian curvature tensor field* defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z,$$

for all vector fields X, Y, Z.

For a contact metric manifold M one may define naturally an almost complex structure on $M \times \mathbf{R}$. If this almost complex structure is integrable, M is said to be a *Sasakian manifold* [1]. A Sasakian manifold is characterized by the condition $(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(X)Y$, for all vector fields X and Y on the manifold [1].

Let M be a contact metric manifold. It is well known that M is *Sasakian* if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2}$$

for all vector fields X and Y [1].

A contact metric manifold M is said to be $\eta-$ *Einstein* if

$$Q = aId + b\eta \otimes \xi, \tag{3}$$

where Q is the Ricci operator and a, b are smooth functions on M [2].

2 Contact metric manifolds with ξ belonging to (κ, μ)–nullity distribution

In this section we give some well-known results.

Let M be a contact metric manifold. The (κ, μ)- *nullity distribution* of M for the pair (κ, μ) is a distribution

$$\begin{aligned} N(\kappa,\mu) : p \to N_p(\kappa,\mu) &= \{Z \in T_pM \mid R(X,Y)Z = \\ &= \kappa[g(Y,Z)X - g(X,Z)Y] + \\ &+ \mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned} \tag{4}$$

where $\kappa, \mu \in \mathbf{R}$ (see [5]). So if the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution we have

$$R(X,Y)\xi = \kappa\left[\eta(Y)X - \eta(X)Y\right] + \mu\left[\eta(Y)hX - \eta(X)hY\right].$$

Lemma 2.1 [**2**]. *If M is a contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution, then*

$$(\nabla_X h)Y = \left[(1-\kappa)g(X,\varphi Y) - g(X,h\varphi Y)\right]\xi + \eta(Y)h(\varphi X + \varphi hX) - \mu\eta(X)\varphi hY,$$

where X and Y are any vector fields on M.

Theorem 2.2 [**2**]. *Let M be a contact metric manifold with ξ belonging to a (κ, μ)-nullity distribution. Then $\kappa \leq 1$. If $\kappa = 1$, then $h = 0$ and M is Sasakian manifold. If $\kappa < 1$, M admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$ determined by the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$.*

Lemma 2.3 [**2**]. *Let M be a contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution ($\kappa < 1$). For any vector field X, the Ricci operator Q is given by*

$$QX = [2(n-1) - n\mu]X + [2(n-1) + \mu]hX + \\ +[2(1-n) + n(2\kappa + \mu)]\eta(X)\xi;\ n \geq 1. \tag{5}$$

A consequence of Lemma 2.3 is the following

Lemma 2.4. *Let M be a contact metric manifold. If ξ belonging to the (κ, μ)-nullity distribution, then*

$$(\nabla_X S)(Y, Z) = [2(n-1) + \mu]g((\nabla_X h)Y, Z) + \\ + [2(1-n) + n(2\kappa + \mu)]\left\{g(Y, \nabla_X \xi)\eta(Z) + g(Z, \nabla_X \xi)\eta(Y)\right\}. \tag{6}$$

3 R–Harmonic manifolds

Let M be a (2n+1)–dimensional Riemannian C^∞ manifold, ∇ and R denote its Levi–Civita derivative and curvature tensor respectively.

A tensor field R of type (1,3) on M is called *algebraic curvature tensor field* if it has symmetric properties of the curvature tensor field of Riemannian manifolds.

The curvature tensor R satisfies the second Bianchi identity if

$$(\nabla_X R)(Y, Z, W) + (\nabla_Y R)(X, Z, W) + (\nabla_Z R)(X, Y, W) = 0.$$

Proposition 3.1 [**4**]. *Let R be an algebraic curvature tensor field which satisfies the second Bianchi identity. If S is the associated Ricci tensor field, then*

$$(divR)(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z).$$

Definition 3.1. An algebraic curvature tensor field R is harmonic (or Codazzi type in the sense of [3]) if

$$(divR)(X, Y, Z) = 0.$$

A Riemannian manifold M is called *R-harmonic* if its curvature tensor field R is harmonic.

It is obvious that every Ricci-symmetric manifold (i.e. $\nabla S = 0$) is R-harmonic.

Corollary 3.2 [**4**]. *An algebraic curvature tensor field satisfying the second Bianchi identity is harmonic if and only if the associated Ricci tensor Q (related to S by $S(X, Y) = g(QX, Y)$) is a Codazzi tensor field i.e., $(\nabla_X Q)Y - (\nabla_Y Q)X = 0$, for every $X, Y \in \chi(M)$.*

Now we state our main results.

Theorem 3.3. *Let M be a contact metric R-harmonic manifold with ξ belonging to (κ, μ)-nullity distribution.*

i) If $\kappa < 1$, then M is either a) locally isometric to the product $\boldsymbol{E}^{n+1} \times S^n(4)$, or b) locally isometric to $E(2)$ (the group of the rigid motions of the Euclidean 2-space).

ii) If $\kappa = 1$, then M is an Einstein-Sasakian manifold.

Proof. i) Since M is a contact metric manifold with ξ belonging to (κ, μ)-nullity distribution, with $\kappa < 1$, then by the covariant differentiation of the relation (13) we have

$$(7)\qquad \begin{aligned}(\nabla_X Q)Y-(\nabla_Y Q)X&=[2(n-1)+\mu]\,[(\nabla_X h)Y-(\nabla_Y h)X]+\\ &+[2(1-n)+n(2\kappa+\mu)]\,[g(Y,\nabla_X\xi)\xi+\eta(Y)\nabla_X\xi]+\\ &-[2(1-n)+n(2\kappa+\mu)]\,[g(X,\nabla_Y\xi)\xi+\eta(X)\nabla_Y\xi]\,.\end{aligned}$$

By Lemma 3.1 iv) in [2] it can be seen that

$$(8)\qquad \begin{aligned}(\nabla_X h)Y-(\nabla_Y h)X &= (1-k)\,[2g(X,\varphi Y)\xi+\eta(X)\varphi Y-\eta(Y)\varphi X]\\ &+ (1-\mu)\,[\eta(X)\varphi hY-\eta(Y)\varphi hX]\,.\end{aligned}$$

Substituting (8) into (7) and using R-harmonic property we obtain

$$(9)\qquad \begin{aligned}&0=(\nabla_X Q)Y-(\nabla_Y Q)X=\\ &=[2(n-1)+\mu]\,\{(1-k)\,[2g(X,\varphi Y)\xi+\eta(X)\varphi Y-\eta(Y)\varphi X]\\ &+(1-\mu)\,[\eta(X)\varphi hY-\eta(Y)\varphi hX]\}+\\ &+[2(1-n)+n(2\kappa+\mu)]\,[g(Y,\nabla_X\xi)\xi+\\ &+\eta(Y)\nabla_X\xi-g(X,\nabla_Y\xi)\xi-\eta(X)\nabla_Y\xi\end{aligned}$$

Taking the product of both sides of the equation (9) by ξ and using the fact that φ is antisymmetric, h is symmetric, $\varphi\xi=0$, $\nabla_X\xi=-\varphi X-\varphi hX$, after some computation we find $[\kappa(2-\mu)+\mu(n+1)]\,g(X,\varphi Y)=0$. Since $g(X,\varphi Y)=d\eta(X,Y)\neq 0$, we have $\kappa(2-\mu)+\mu(n+1)=0$.

Taking $X=\xi$ into (9) and using the fact that φ is antisymmetric, h is symmetric, $\varphi\xi=0$, $\nabla_X\xi=-\varphi X-\varphi hX$, after some computations we obtain

$$(10)\qquad [\kappa(2-\mu)+\mu(n+1)]\,\varphi Y+[2n\kappa+\mu(3-n-\mu)]\,\varphi hY=0.$$

Since $\kappa(2-\mu)+\mu(n+1)=0$, the relation (10) becomes $[2n\kappa+\mu(3-n-\mu)]\,\varphi hY$=0. So we have two possible cases:

Case I. $\kappa(2-\mu)+\mu(n+1)=0$ and $[2n\kappa+\mu(3-n-\mu)]=0$.

Case II. $\varphi hY=0$.

Let us consider these in turn.

(**Case I**). Suppose $\kappa(2-\mu)+\mu(n+1)=0$ and $[2n\kappa+\mu(3-n-\mu)]=0$. Then solving this system we obtain the following solutions:

$$\kappa=\mu=0,\ \kappa=\mu=3+n \text{ or } \kappa=\frac{(n-1)(n+1)}{n},\ \mu=2-2n.$$

For the case $\kappa=\mu=0$, M must be locally isometric to the product $\mathbf{E}^{n+1}\times S^n(4)$ (see [1] p.121). Since $\kappa<1$, the case $\kappa=\mu=3+n$ is not possible. But the case $\kappa=\dfrac{(n-1)(n+1)}{n}$, $\mu=2-2n$ is possible only for $n=1$. Thus M is 3-dimensional in this case and by Theorem 3 in [2], M is locally isometric to $E(2)$ (the rigid motions of the Euclidean 2-space).

(**Case II**). Suppose $\varphi hY=0$. Then we have $\nabla_Y\xi=-\varphi Y$ which implies that M is $K-$contact. Therefore $h=0$. Since $h^2=(\kappa-1)\varphi^2$, we obtain $k=1$ which is contradicting the fact that $\kappa<1$ so this case does not occur.

ii) If $\kappa=1$, then M is an Einstein-Sasakian manifold.

First, using the relation $(\nabla_X S)(Y,Z)=\nabla_X S(Y,Z)-S(\nabla_X Y,Z)-S(Y,\nabla_X Z)$ and the symmetric property of Q one can write $g(Y,(\nabla_X Q)Z)=g((\nabla_X Q)Y,Z)$ and similarly $g(X, (\nabla_Y Q)Z) = g((\nabla_Y Q)X, Z)$. Since M is R-harmonic, by Corollary 3.2 we obtain

$$g((\nabla_X Q)Y, Z)) = g((\nabla_Y Q)X, Z). \tag{11}$$

Setting $Z = \xi$ into (11) and using the relations $\nabla_X \xi = -\varphi X$ and $Q\xi = 2n\xi$ (see [1]), we have

$$-2ng(Y, \varphi X) + g(Y, Q\varphi X) = -2ng(X, \varphi Y) + g(X, Q\varphi Y). \tag{12}$$

Since M is Sasakian, we have $Q\varphi = \varphi Q$. So the equation (12) becomes

$$2ng(X, \varphi Y) - g(X, Q\varphi Y) = 0. \tag{13}$$

Interchanging Y with φY in (13) one finds $S(X, Y) = 2ng(X, Y)$, i.e., M is an Einstein manifold. This completes the proof of the theorem.

Corollary 3.4. *Let M be a contact metric manifold with ξ belonging to (κ, μ)-nullity distribution . If M is R-harmonic on the distribution $D = \{X \mid \eta(X) = 0,\ X \in \chi(M)\}$, then M is either Einstein or Einstein-Sasakian manifold.*

Proof. Suppose M is a contact metric manifold with ξ belonging to (κ, μ)-nullity distribution.

First we suppose that $\kappa < 1$. If M is a R-harmonic on the distribution D, then the equations (5) and (7) respectively become

$$QX = [2(n-1) - n\mu]\, X + [2(n-1) + \mu]\, hX, \tag{14}$$

$$(\nabla_X Q)Y - (\nabla_Y Q)X = [2(n-1) + \mu]\, [(\nabla_X h)Y - (\nabla_Y h)X] = 0.$$

So we have the following cases.

Case I. $2(n-1) + \mu = 0$,

Case II. $(\nabla_X h)Y - (\nabla_Y h)X = 0$.

Let us consider these in turn.

(**Case I**). Suppose $2(n-1) + \mu = 0$. Then the equation (14) becomes $QX = [2(n-1) - n\mu]\, X$, which implies that M is an Einstein manifold.

(**Case II**). Suppose $(\nabla_X h)Y - (\nabla_Y h)X = 0$. Then by Lemma 2.1 we have $(\nabla_X h)Y - (\nabla_Y h)X = 2(1-\kappa)g(X, \varphi Y) = 0$, which implies $\kappa = 1$. This contradicts the fact that $\kappa < 1$. So this case does not occur.

If $\kappa = 1$, then by the same discussion given in Theorem 3.3 ii) it is easy to show that M is an Einstein manifold. This completes the proof of the corollary.

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