Determination of Invariant Inner Product on Seven Dimensional Nilpotent Lie Algebras

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Abstract

The aim of the present paper is to study the invariant bilinear forms on the Nilpotent Lie algebras of dimension seven.These Nilpotent Lie algebras are over **C**.

Mathematics Subject Classification: 17B40
Key words: Nilpotent Lie algebra, bilinear symmetric form, invariant bilinear form and nullspace.

1 Introduction

Let g be a Nilpotent Lie algebra over a field K of characteristic zero. We assume that the dimension of g is n. The classification of g is known up to dimension eight [5], [8].

The study of the invariant symmetric bilinear forms on g is an open problem.

This problem has been completely solved until to dimension six and partially for dimension eight [8]. The purpose of this article is to determine the invariant symmetric bilinear forms on the Nilpotent Lie algebras of dimension seven and over the field **C**. It is also determined the null spaces of these bilinear forms.

The paper contains two paragraphs. The first paragraph is the introduction. The determination of invariant symmetric bilinear forms on the Nilpotent Lie algebras of dimension seven are contained in the second paragraph. It also includes the null spaces of these bilinear forms.

2. Let g be a Lie algebra over a field of characteristic zero, whose dimension is n; that is $dimg = n$. Let f be an invariant symmetric bilinear form on the Lie algebra g, that means a bilinear form f which satisfies the relation:

$$f([x,y],z) = f(x,[y,z]), \quad \forall x,y,z \in g.$$

Each bilinear form f on g can be represented by a matrix $M = (a_{ij})$, $a_{ij} \in \mathbf{C}$ (with respect of the base $\{e_1, e_2, ..., e_n\}$ of the Lie algebra g, where $a_{ij} = f(e_i, e_j)$ and $a_{ij} = a_{ji}$. The nullspace N_f of f is defined as follows

$$N_f := \{x \in g / f(g,x) = 0 = f(x,g)\}.$$

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In this paper we consider invariant symmetric bilinear forms f on a Nilpotent Lie algebra g of dimension seven, and we shall determine all the invariant symmetric bilinear forms f on g and their nullspaces.

We use the classification which is given in [5]. We have estimated all the corresponding matrices of the bilinear forms and the nullspaces of the seven dimensional Nilpotent Lie algebras. These matrices and nullspaces are given below:

$$M_1 = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & a_{15} & 0 & 0 \\ a_{12} & a_{22} & 0 & -a_{15} & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 & 0 & 0 & 0 \\ 0 & -a_{15} & 0 & 0 & 0 & 0 & 0 \\ a_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & a_{16} & 0 \\ a_{12} & a_{22} & 0 & -a_{16} & 0 & 0 & 0 \\ 0 & 0 & a_{16} & 0 & 0 & 0 & 0 \\ 0 & -a_{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_5 = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & a_{16} & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 & a_{26} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{16} & a_{26} & 0 & 0 & 0 & a_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & a_{15} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & a_{25} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & a_{35} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{15} & a_{25} & a_{35} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_7 = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & a_{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{14} & a_{24} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_8 = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & a_{15} & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 & a_{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{15} & a_{25} & 0 & 0 & a_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_9 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & a_{34} & 0 & 0 & 0 \\ a_{14} & a_{24} & a_{34} & a_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{10} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & a_{14} & 0 \\ a_{13} & a_{23} & a_{33} & 0 & -a_{14} & 0 & 0 \\ a_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_{14} & 0 & 0 & 0 & 0 \\ 0 & a_{14} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & a_{17} \\ a_{12} & a_{22} & 0 & 0 & 0 & a_{17} & 0 \\ 0 & 0 & 0 & 0 & -a_{17} & 0 & 0 \\ 0 & 0 & 0 & a_{17} & 0 & 0 & 0 \\ 0 & 0 & -a_{17} & 0 & 0 & 0 & 0 \\ 0 & a_{17} & 0 & 0 & 0 & 0 & 0 \\ a_{17} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & a_{16} & 0 \\ a_{12} & a_{22} & a_{23} & 0 & -a_{16} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{16} & 0 & 0 & 0 \\ 0 & -a_{16} & 0 & 0 & 0 & 0 & 0 \\ a_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{13} = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & a_{15} & 0 & 0 \\ a_{12} & a_{22} & 0 & a_{24} & a_{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{14} & a_{24} & 0 & a_{44} & a_{45} & 0 & 0 \\ a_{15} & a_{25} & 0 & a_{45} & a_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{14} = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & a_{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{14} & a_{24} & 0 & a_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{15} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & a_{16} & 0 \\ a_{12} & a_{22} & a_{23} & 0 & -a_{16} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & a_{16} & 0 & 0 & 0 \\ 0 & 0 & a_{16} & 0 & 0 & 0 & 0 \\ 0 & -a_{16} & 0 & 0 & 0 & 0 & 0 \\ a_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{16} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & a_{16} & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & a_{26} & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & a_{36} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{16} & a_{26} & a_{36} & 0 & 0 & a_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{17} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & 0 & 0 \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & 0 & 0 \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{18} = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & 0 & a_{16} & 0 \\ a_{12} & a_{22} & 0 & a_{24} & -a_{16} & 0 & 0 \\ 0 & 0 & a_{16} & 0 & 0 & 0 & 0 \\ a_{14} & a_{24} & 0 & a_{44} & 0 & 0 & 0 \\ 0 & -a_{16} & 0 & 0 & 0 & 0 & 0 \\ a_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{19} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & 0 \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} & 0 \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} & 0 \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} & 0 \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Table 1

Matrix	**Nullspace**	**Algebras**
M_1	$\{e_3, e_4, e_5, e_6, e_7\}$	0.1,0.2,0.3,1.1(i_λ),1.1(ii),1.1(iii),1.4,1.5,1.6,2.3
M_2	$\{e_6, e_7\}$	0.4_λ,0.5,0.6,0.7,1.02,1.1(iv),1.10,1.13,1.14,1.17, 2.5,2.6,2.7,2.8,2.9
M_3	$\{e_4, e_5, e_6, e_7\}$	0.8,1.01(i),1.01(ii),1.1(v),1.2(i_λ)(*),1.2(iii),1.2(iv), 1.3(i_λ),1.3(ii),1.3(iii),1.3(iv),1.7,1.8,1.9,1.11,1.12, 1.15,1.18,1.19,1.21,2.1(i_λ),2.1(ii),2.1(v),2.10,2.11, 2.12,2.13,2.14,2.15,2.16,2.17,2.19,2.20,2.21,2.22, 2.24,2.26,2.31,2.32,2.33,2.34,2.35,2.37,2.39,2.42, 2.43,3.2,3.3,3.4,3.5,3.20,3.21,3.22
M_4	$\{e_5, e_7\}$	1.03
M_5	$\{e_3, e_4, e_5, e_7\}$	1.1(vi)
M_6	$\{e_4, e_6, e_7\}$	1.3(v),2.1(iii)
M_7	$\{e_3, e_5, e_6, e_7\}$	1.16,2.40
M_8	$\{e_3, e_4, e_6, e_7\}$	1.20
M_9	$\{e_5, e_6, e_7\}$	2.1(iv),2.23,2.25,2.27,2.28,2.29,2.30,2.36,2.45, 3.7,3.8,3.9,3.10,3.11.,3.12,3.13,3.14,3.15,3.16, 3.17,3.24,4.1,4.2,4.5
M_{10}	$\{e_7\}$	2.2
M_{11}	$\{0\}$	2.4
M_{12}	$\{e_7\}$	2.18
M_{13}	$\{e_3, e_6, e_7\}$	2.38
M_{14}	$\{e_3, e_5, e_6, e_7\}$	2.41
M_{15}	$\{e_7\}$	2.44,$3.1(i_\lambda), 3.1(ii), 3.6$
M_{16}	$\{e_4, e_5, e_6\}$	3.1(iii)
M_{17}	$\{e_6, e_7\}$	3.18,3.19,4.3
M_{18}	$\{e_7\}$	3.23
M_{19}	$\{e_7\}$	4.4

Consequently, we have proved the following

Theorem 2.1. *Let g be a Nilpotent Lie algebra of dimension seven. The invariant symmetric bilinear form on each of these algebras is described by one of the matrices* $M_1 - M_{19}$.

From the above theorem we conclude

Corollary 2.2. *Let g be a Nilpotent Lie algebra of dimension seven. The nullspaces of the invariant symmetric bilinear forms of these Lie algebras are given by the table 1. There are 13 different nullspaces.*

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