# Semi-Symmetric Conformal Metrical $N$-Linear Connections in the Bundle of Accelerations

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**Abstract**

In the present paper we determine all semi-symmetric conformal metrical $N$-linear connections, which preserve the nonlinear connection N, in the bundle of accelerations. We study the group of transformations of these connections and its invariants.

**AMS subject classification**: 53C05
**Key words**: osculator bundle, curvature, torsion, semi-symmetric conformal metrical N-linear connection.

## 1 Introduction

The differential geometry of higher order Lagrange spaces was introduced and studied by R.Miron and Gh.Atanasiu in [8] – [13].

The applications of the Lagrange geometry of order $k$ in Physics and Mechanics are quite numerous and important, [8].

The study of higher order Lagrange spaces is grounded on the $k$-osculator bundle notion. The bundle of accelerations corresponds in this study to $k = 2, [1], [10]$.

In the present paper we define the notion of semi-symmetric conformal metrical $N$-linear connection on $E = Osc^2M$ and we determine the set of all semi-symmetric conformal metrical $N$-linear connections, which preserve the nonlinear connection N, on E. (§2) The group of their transformations preserving a nonlinear connection $N$, gives us important invariants (§3).

As to the terminology and notations we use those from [14], which are essentially based on M.Matsumoto's book [6].

## 2 Notion of semi-symmetric conformal metrical $N$-linear connection in the bundle of accelerations

Let $M$ be a real $n$-dimensional $C^\infty$-differentiable manifold and $(Osc^2M, \pi, M)$ its 2-osculator bundle, or the bundle of accelerations.

Balkan Journal of Geometry and Its Applications, Vol.4, No.2, 1999, pp. 103-108

The local coordinates on $E = Osc^2M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$.

If $N$ is a nonlinear connection on $E$, with the coefficients $N_{(1)}{}^i{}_j$, $N_{(2)}{}^i{}_j$, then let $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ be an $N$- linear connection on $E$.

We consider a metrical $d$-structure on E, defined by a $d$-tensor field of the type $(0,2)$, marked as $g_{ij}(x^i, y^{(1)i}, y^{(2)i})$. This $d$-tensor field is symmetric and nondegenerate.

To the metrical $d$-structure $g_{ij}$ on E, we associate Obata's operators

$$\Omega^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - g_{sj}g^{ir}),\ \Omega^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + g_{sj}g^{ir}), \tag{2.1}$$

where $(g^{ij})$ is the inverse matrix of $(g_{ij})$. They have the same properties as the ones associated with a Finsler space [14].

Let $\mathcal{S}_2(E)$ be the set of all symmetric $d$-tensor fields of the type $(0,2)$ on $E$. It is easy to show that, the relation

$$a_{ij} \sim b_{ij} \Leftrightarrow \exists\rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E) \mid a_{ij} = e^{2\rho}b_{ij};\quad a_{ij}, b_{ij} \in \mathcal{S}_2(E) \tag{2.2}$$

is an equivalence relation on $\mathcal{S}_2(E)$.

**Definition 2.1.** The equivalence class $\hat{g}$ of $\mathcal{S}_2(E)/_\sim$, to which the metrical $d$-structure $g_{ij}$ belongs, is called *conformal metrical* $d$-structure on E.

**Definition 2.2.** An $N$-linear connection $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ on E is said to be *compatible* with the conformal metrical $d$-structure $\hat{g}$, or a conformal metrical $N$-linear connection on E, if

$$g_{ij|k} = 2\omega_k g_{ij},\ g_{ij}\overset{(\alpha)}{|}_k = 2\lambda_{(\alpha)k}g_{ij},\ (\alpha = 1,2), \tag{2.3}$$

where $\omega_k = \omega_k(x, y^{(1)}, y^{(2)})$ and $\lambda_{(\alpha)k} = \lambda_{(\alpha)k}(x, y^{(1)}, y^{(2)})$, $(\alpha = 1,2)$ are covariant $d$-vector fields and $|$, $\overset{(\alpha)}{|}$, $(\alpha = 1,2)$ denote the $h$-and $v_\alpha$-covariant derivatives, $(\alpha = 1,2)$ with respect to $D\Gamma(N)$.

**Definition 2.3.** An $N$-linear connection $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ on E, is called *semi-symmetric* if the torsion $d$-tensor fields $T_{(0)}{}^i{}_{jk}$, $S_{(\alpha)}{}^i{}_{jk}$, $(\alpha = 1,2)$ have the form

(2.4)

$$T_{(0)}{}^i{}_{jk} = \frac{1}{n-1}(T_{(0)j}\delta^i_k - T_{(0)k}\delta^i_j), S_{(\alpha)}{}^i{}_{jk} = \frac{1}{n-1}(S_{(\alpha)j}\delta^i_k - S_{(\alpha)k}\delta^i_j),\ \alpha = 1,2),$$

where $T_{(0)j} = T_{(0)}{}^i{}_{ji}$, $S_{(\alpha)j} = S_{(\alpha)}{}^i{}_{ji}$, $(\alpha = 1,2)$.

**Definition 2.4.** An $N$-linear connection $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ on E is called a *semi-symmetric conformal* metrical $N$-linear connection if the relations (2.3) and (2.4) are verified.

If $\sigma_j = \frac{1}{n-1}T_{(0)j}$, $\tau_{(\alpha)j} = \frac{1}{n-1}S_{(\alpha)j}$, $(\alpha = 1,2)$ and if we apply the Theorem 5.4.3., [8], we obtain:

**Theorem 2.1** *The set of all semi-symmetric conformal metrical $N$-linear connections $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$, which preserve the nonlinear connection $N$, on $E$ is given by*

(2.5) $$L^{i}_{jk} = \overset{0}{L}{}^{i}_{jk} + 2\Omega^{ir}_{kj}\sigma_r, C_{(\alpha)}{}^{i}_{jk} = \overset{0}{C}_{(\alpha)}{}^{i}_{jk} + 2\Omega^{ir}_{kj}\tau_{(\alpha)r}, \ (\alpha = 1,2),$$

*where* $D\overset{0}{\Gamma}(N) = (\overset{0}{L}{}^{i}_{jk}, \overset{0}{C}_{(1)}{}^{i}_{jk}, \overset{0}{C}_{(2)}{}^{i}_{jk})$ *is an arbitrary conformal metrical $N$-linear connection on E, whose d-tensor fields $T_{(0)}$, $S_{(\alpha)}$, $(\alpha = 1,2)$ are vanish.*

# 3 Group of transformations of semi- symmetric conformal metrical $N$-linear connections on $E = Osc^2M$, which preserve the nonlinear connection N.

Let us consider the transformations $D\Gamma(N) \longrightarrow D\bar{\Gamma}(N)$ of semi-symmetric conformal metrical $N$-linear connections on E, which preserve the nonlinear connection $N$.

**Theorem 3.1** *Two semi-symmetric conformal metrical $N$-linear connections on E,* $D\Gamma(N) = (L^{i}_{jk}, C_{(1)}{}^{i}_{jk}, C_{(2)}{}^{i}_{jk})$, $D\bar{\Gamma}(N) = (\bar{L}^{i}_{jk}, \bar{C}_{(1)}{}^{i}_{jk}, \bar{C}_{(2)}{}^{i}_{jk})$, *are related as follows:*

(3.1) $$\bar{L}^{i}_{jk} = L^{i}_{jk} - \delta^i_j\omega_k + 2\Omega^{ir}_{kj}\theta_r, \bar{C}_{(\alpha)}{}^{i}_{jk} = C_{(\alpha)}{}^{i}_{jk} - \delta^i_j\lambda_{(\alpha)k} + 2\Omega^{ir}_{kj}\gamma_{(\alpha)r},$$
$$(\alpha = 1,2),$$

*where we have* $\theta_r = \sigma_r - \omega_r$, $\gamma_{(\alpha)r} = \tau_{(\alpha)r} - \lambda_{(\alpha)r}$, $(\alpha = 1,2)$.

**Theorem 3.2** *The set $\mathcal{C}^s{}_N$ of all the transformations given by* (3.1) *is a transformation group of the set of all conformal metrical N-linear connections on E, which preserve the nonlinear connection N, together with the mapping product.*

The transformation $t : D\Gamma \longrightarrow D\bar{\Gamma}$ given by (3.1) is expresed by the product of the following two transformations:

(3.2) $$\bar{L}^{i}_{jk} = L^{i}_{jk} - \delta^i_j\omega_k, \ \bar{C}_{(\alpha)}{}^{i}_{jk} = C_{(\alpha)}{}^{i}_{jk} - \delta^i_j\lambda_{(\alpha)k}, \ (\alpha = 1,2),$$

(3.3) $$\bar{L}^{i}_{jk} = L^{i}_{jk} + 2\Omega^{ir}_{kj}\theta_r, \ \bar{C}_{(\alpha)}{}^{i}_{jk} = C_{(\alpha)}{}^{i}_{jk} + 2\Omega^{ir}_{kj}\gamma_{(\alpha)r}, \ (\alpha = 1,2).$$

**Theorem 3.3** *The group $\mathcal{C}^s{}_N$ is the direct product of the group $\mathcal{C}^p{}_N$ (of all transformations* (3.2)*) and the group $\mathcal{C}^m{}_N$ (of all transformations* (3.3)*).*

One can notice that the invariants of the group $\mathcal{C}^s{}_N$ will be invariants of each of these subgroups, and reciprocally.

In our previous paper [16], starting from the tensor fields $K_h{}^{i}{}_{jk}$, $\mathcal{P}_{(\alpha)h}{}^{i}{}_{jk}$, $(\alpha = 1,2)$, $S_{(22)h}{}^{i}{}_{jk}$, where

$$(3.4)\quad \begin{cases} K_h{}^i{}_{jk} = R_h{}^i{}_{jk} - C_{(1)}{}^i_{hs} R_{(01)}{}^s{}_{jk} - C_{(2)}{}^i_{hs} R_{(02)}{}^s{}_{jk}, \\ \mathcal{P}_{(1)h}{}^i{}_{jk} = \mathcal{A}_{jk}\{P_{(1)h}{}^i{}_{jk} - C_{(1)}{}^i_{hs}\frac{\delta N_{(1)}{}^s{}_j}{\delta y^{(1)k}} - C_{(2)}{}^i_{hs}(N_{(1)}{}^s{}_m \frac{\delta N_{(1)}{}^m{}_j}{\delta y^{(1)k}} + \\ + \frac{\delta N_{(2)}{}^s{}_j}{\delta y^{(1)k}} - \frac{\delta N_{(2)}{}^s{}_k}{\delta y^{(1)j}})\}, \\ \mathcal{P}_{(2)h}{}^i{}_{jk} = \mathcal{A}_{jk}\{P_{(2)h}{}^i{}_{jk} - C_{(1)}{}^i_{hs}\frac{\partial N_{(1)}{}^s{}_j}{\delta y^{(2)k}} - C_{(2)}{}^i_{hs}(N_{(1)}{}^s{}_m \frac{\partial N_{(1)}{}^m{}_j}{\delta y^{(2)k}} - \\ - \frac{\partial N_{(2)}{}^s{}_j}{\partial y^{(2)k}})\}, \end{cases}$$

we obtained the following important invariants of the group of transformations of semi-symmetric metrical $N$-linear connections, which preserve the nonlinear connection $N$, on E:

$$(3.5)\quad \begin{cases} H_h{}^i{}_{jk} = K_h{}^i{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(K_{rk} - \frac{K g_{rk}}{2(n-1)})\}, \\ N_{(\alpha)h}{}^i{}_{jk} = \mathcal{P}_{(\alpha)h}{}^i{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(\mathcal{P}_{(\alpha)rk} - \frac{\mathcal{P}_{(\alpha)} g_{rk}}{2(n-1)})\},\ (\alpha = 1,2), \\ M_{(22)h}{}^i{}_{jk} = S_{(22)h}{}^i{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(S_{(22)rk} - \frac{S_{(22)} g_{rk}}{2(n-1)})\}, \end{cases}$$

where

$$K_{hj} = K_h{}^i{}_{ji},\ \mathcal{P}_{(\alpha)hj} = \mathcal{P}_{(\alpha)h}{}^i{}_{ji},\ S_{(22)hj} = S_{(22)h}{}^i{}_{ji},$$

$$K = g^{hj}K_{hj},\ \mathcal{P}_{(\alpha)} = g^{hj}\mathcal{P}_{(\alpha)hj},\ S_{(22)} = g^{hj}S_{(22)hj},\ (\alpha = 1,2).$$

If we replace $K_h{}^i{}_{jk}$, $\mathcal{P}_{(\alpha)h}{}^i{}_{jk}$, $(\alpha = 1,2)$, $S_{(22)h}{}^i{}_{jk}$ by the tensor fields $K^*{}_h{}^i{}_{jk}$, $\mathcal{P}^*{}_{(\alpha)h}{}^i{}_{jk}$, $(\alpha = 1,2), S^*{}_{(22)h}{}^i{}_{jk}$ defined by:

$$(3.6)\quad \begin{cases} K^*{}_h{}^i{}_{jk} = K_h{}^i{}_{jk} - \frac{1}{n}\delta^i_h K_s{}^s{}_{jk}, \\ \mathcal{P}^*_{(\alpha)h}{}^i{}_{jk} = \mathcal{P}_{(\alpha)h}{}^i{}_{jk} - \frac{1}{n}\delta^i_h \mathcal{P}_{(\alpha)s}{}^s{}_{jk},\ (\alpha = 1,2), \\ S^*{}_{(22)h}{}^i{}_{jk} = S_{(22)h}{}^i{}_{jk} - \frac{1}{n}\delta^i_h S_{(22)s}{}^s{}_{jk}, \end{cases}$$

we can obtain the invariants of the group of transformations of semi-symmetric conformal metrical $N$-linear connections on E, which preserve the nonlinear connection $N$

**Theorem 3.4** *Let*
$K^*{}_{hj} = K^*{}_h{}^i{}_{ji}$, $\mathcal{P}^*_{(\alpha)hj} = \mathcal{P}^*_{(\alpha)h}{}^i{}_{ji}$, $(\alpha = 1,2)$, $S^*_{(22)hj} = S_{(22)h}{}^i{}_{ji}$, $K^* = g^{hj}K^*{}_{hj}$
$\mathcal{P}^*_{(\alpha)} = g^{hj}\mathcal{P}^*_{(\alpha)hj}$, $(\alpha = 1,2)$, $S^*_{(22)} = g^{hj}S^*_{(22)hj}$.
*For $n > 2$ the following tensor fields*

$$(3.7)\quad \begin{cases} H^{*}{}_{h}{}^{i}{}_{jk} = K^{*}{}_{h}{}^{i}{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(K^{*}{}_{rk} - \frac{K^{*}g_{rk}}{2(n-1)})\}, \\ N^{*}_{(\alpha)h}{}^{i}{}_{jk} = \mathcal{P}^{*}_{(\alpha)h}{}^{i}{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(\mathcal{P}^{*}_{(\alpha)rk} - \frac{\mathcal{P}^{*}_{(\alpha)}g_{rk}}{2(n-1)})\},\ (\alpha = 1,2), \\ M^{*}_{(22)h}{}^{i}{}_{jk} = S^{*}_{(22)h}{}^{i}{}_{jk} + \frac{2}{n-2}\mathcal{A}_{jk}\{\Omega^{ir}_{jh}(S^{*}_{(22)rk} - \frac{S^{*}_{(22)}g_{rk}}{2(n-1)})\}, \end{cases}$$

*determined by of semi-symmetric conformal metrical N-linear connections, which preserve the nonlinear connection N, on E, are invariants of the group* $\mathcal{C}^{s}{}_{N}$*:*

**Proof**. We can easily show that $H^{*}{}_{h}{}^{i}{}_{jk}$, $N^{*}_{(\alpha)h}{}^{i}{}_{jk}$, $(\alpha = 1,2)$, $M^{*}_{(22)h}{}^{i}{}_{jk}$ are invariants of $\mathcal{C}^{p}{}_{N}$. Owing to Theorem 3.3, it suffices to prove the theorem for $\mathcal{C}^{m}{}_{N}$.

From Theorem 2.3 of [16], the tensor fields $K_{h}{}^{i}{}_{jk}$, $\mathcal{P}_{(\alpha)h}{}^{i}{}_{jk}$, $(\alpha = 1,2)$, $S_{(22)h}{}^{i}{}_{jk}$ of a semi-symmetric metrical $N$-linear connection $D\Gamma(N)$, are transformed on the basis of the relations (3.3) of $D\Gamma(N)$ to $D\bar{\Gamma}(N)$ as follows:

$$(3.8)\quad \begin{cases} \bar{K}_{h}{}^{i}{}_{jk} = K_{h}{}^{i}{}_{jk} + 2\mathcal{A}_{jk}\{\Omega^{ir}_{jh}\sigma_{rk}\}, \\ \bar{\mathcal{P}}_{(\alpha)h}{}^{i}{}_{jk} = \mathcal{P}_{(\alpha)h}{}^{i}{}_{jk} + 2\mathcal{A}_{jk}\{\Omega^{ir}_{jh}\rho_{(\alpha)rk}\},\ (\alpha = 1,2), \\ \bar{S}_{(22)h}{}^{i}{}_{jk} = S_{(22)h}{}^{i}{}_{jk} + 2\mathcal{A}_{jk}\{\Omega^{ir}_{jh}\tau_{(2)rk}\}, \end{cases}$$

where $\sigma_{rk}$, $\rho_{(\alpha)rk}$, $\tau_{(2)rk}$, $(\alpha = 1,2)$ are some $d$-tensor fields determined from $D\Gamma(N)$. Since $\Omega^{sr}_{ks} = 0$, the tensor fields $K^{*}{}_{h}{}^{i}{}_{jk}$, $\mathcal{P}^{*}_{(\alpha)h}{}^{i}{}_{jk}$, $(\alpha = 1,2)$, $S^{*}_{(22)h}{}^{i}{}_{jk}$ obey the same transformation laws as (3.8), Hence, (3.7) follows from the well-known elimination method used in Theorem 2.4 of [16].

**Acknowledgements**. A version of this paper was presented at the Workshop on Electromagnetic Flows and Dynamics, Oct 15-19, 1999, University Politehnica of Bucharest.

Supported by MEN Grant No 21815/28.09.1999, CNCSU-31.

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