Discrete Euler-Poincaré and Euler-Poincaré-Poisson Equations for Semidirect Products and Principal Bundles

D. Opriş and I.D. Albu

Abstract

A variational approach for a discrete Lagrangian on a semidirect product and on a princpal bundle is developed.

Mathematics Subject Classification: 17B30
Key Words: Semidirect product, principal bundle, discrete Euler-Poincaré equations, reduction-reconstruction.

1 Introduction

Dynamical systems with symmetry play an important role in mathematical modelling for several processes in physics, mechanics, economy etc. In the last time the variational description of systems with symmetries seems to be more interesting ([4], [3], [2] etc). In [2], [3], the theory is developed for Lagrangian systems on Lie groups. A symmetry of the Lagrangian with respect to a group action leads to a reduced Lagrangian system which is described by the so called Euler-Poincaré equations.

Independently by all appearances, [5] and [1] propose a discrete analogues of Euler-Poincaré reduction theory for systems on Lie groups. Our study is maked knowing only the first paper, [5].

We propose a variational approach for a discrete Lagrangian system whose configuration space is a semidirect product or a principal bundle. By following the construction in [5] we consider a discrete Lagrangian $L : S \times S \longrightarrow R$, where $S = G \bigcirc V$ is a semidirect product and we write the discrete Euler-Lagrange equations. If L is left G-invariant, then we define a reduced discrete Lagrangian $l : S \times V \longrightarrow R$ and we deduced the discrete Euler-Poincré equations. In Section 3 a reduction-reconstruction theorem about the algorithms satisfying these discrete equations is formulated. A discrete Euler-Poincaré-Poisson algorithm is discussed in Section 4. In Section 5 a discrete variant of the Kelvin-Noether Theorem is established. The last Section is an attempt in to approach the discrete variational description for a G-invariant Lagrangian on a principal bundle and to obtain dicrete Euler-Poincaré equations and discrete Euler-Poincaré algorithms.

Balkan Journal of Geometry and Its Applications, Vol.4, No.2, 1999, pp. 93-102

2 Generalities on semidirect products

Let V be a finite dimensional vector space and assume that a finite dimensional Lie group G acts on the left by linear maps on V and also on its dual space V^*. The action of $g \in G$ on $v \in V$ is denoted by simple concatenation gv. The semidirect product $S = G \bigcirc V$ is the Cartezian product $G \times V$ endowed with the group multiplication given by

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2). \tag{1}$$

If $e \in G$ is the identity and $o \in V$ is the null vector of V, then $(e, o) \in G \times V$ is the identity element of S. The inverse of the element (g, v) is

$$(g, v)^{-1} = (g^{-1}, -g^{-1}v). \tag{2}$$

The Lie algebra of S is the semidirect product Lie algebra $\mathcal{S} = \mathcal{G} \bigcirc V$, whose bracket is given by

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1), \tag{3}$$

where the induced action of $\mathcal{G}$ on V is also denoted by concatenation. As well the action of S on $\mathcal{S}$ is given by

$$(g, v)(\xi, u) = (g\xi, gu - (g\xi)v), \tag{4}$$

where $(g, v) \in S, (\xi, u) \in \mathcal{S}$ and the action of S on $\mathcal{S}^* = \mathcal{G}^* \bigcirc V^*$ is given by

$$(g, v)(\mu, a) = (g\mu + \rho_v^*(ga), ga), \tag{5}$$

where $(\mu, a) \in \mathcal{S}^*$, $\rho_v^* : V^* \longrightarrow \mathcal{G}^*$ is the dual of the linear map $\rho_v : \mathcal{G} \longrightarrow V$, $\rho_v(\xi) = \xi v$. In (4) and (5) we denote $g\xi = \mathrm{Ad}_g \xi$, $g\mu = \mathrm{Ad}^*_{g^{-1}} \mu$ (the adjoint and coadjoint actions).

For $a \in V^*$ and $v \in V$ we shall denote $\rho_v^* = v \diamond a \in \mathcal{G}^*$ and the formula (5) becomes

$$(g, v)(\mu, a) = (g\mu + v\diamond(ga), ga). \tag{6}$$

Using the concatenation notations for Lie algebra actions on $\mathcal{G}^*$ and V^*, $\xi\mu, \xi a$, where $\xi \in \mathcal{G}$, $a \in V^*$, we obtain an alternative definition of $v \diamond a \in \mathcal{G}^*$: for $v \in V$, $a \in V^*$, $\xi \in \mathcal{G}^*$, $< \xi a, v >= - < v\diamond, a\xi >$.

Sometimes it is convenient to use the right representations of G on V which changes unessentially the formulas (we note the adjoint and coadjoint actions are left actions).

3 The discrete Euler-Poincaré algorithm

Let $L : S \times S \longrightarrow R$ be a discrete Lagrangian on $S = G \bigcirc V$ and let $\mathcal{A} : S^{N+1} \longrightarrow R$ be the associated action,

$$\mathcal{A} = \sum_{k=0}^{N-1} L(g_k, v_k, g_{k+1}, v_{k+1}), \tag{7}$$

where $(g_k, v_k) \in S$, $k = 0, 1, \ldots, N$ with $(g_0, v_0), (g_N, v_N)$ fixed.

The discrete Euler-Lagrange equations (DEL) are given by

$$\begin{aligned} &D_3L(g_{k-1},v_{k-1},g_k,v_k)+D_1L(g_k,v_k,g_{k+1},v_{k+1})=0,\\ &D_4L(g_{k-1},v_{k-1},g_k,v_k)+D_2L(g_k,v_k,g_{k+1},v_{k+1})=0, \end{aligned} \tag{8}$$

where D_iL, $i=1,2,3,4$, is the matrix containing the partial derivatives with respect to the $i-$th variable.

Consider the quotient map $\pi : S\times S \longrightarrow^{S\times S}/_G \cong S\times V$ given by

$$\pi((g_k,v_k),(g_{k+1},v_{k+1})=(g_{k+1}^{-1}g_k,g_{k+1}^{-1}v_k,g_{k+1}^{-1}v_{k+1}). \tag{9}$$

Using the notations $f_{k+1k}=g_{k+1}^{-1}g_k$, $u_k=g_k^{-1}v_k$ it results $g_{k+1}^{-1}v_k=f_{k+1k}u_k$ and π is given by

$$\pi((g_k,v_k),(g_{k+1},v_{k+1})=(f_{k+1k},f_{k+1k},u_k,u_{k+1}). \tag{10}$$

Assume that the discrete Lagrangian $L: S\times S\longrightarrow R$ is left $G-$invariant. The projection (10) defines the reduced discrete Lagrangian $l: S\times V\longrightarrow R$ by $l\circ\pi=L$. Consequently

$$\begin{aligned} &L(g_k,v_k,g_{k+1},v_{k+1})=L(g_{k+1}^{-1}g_k,g_{k+1}^{-1}v_k,e,g_{k+1}^{-1}v_{k+1})=\\ &=L(f_{k+1k},f_{k+1k}u_k,e,u_{k+1})=l(f_{k+1k},f_{k+1k}u_k,u_{k+1}). \end{aligned} \tag{11}$$

The action associated to the reduced discrete Lagrangian l is $a:(S\times V)^{N+1}\longrightarrow R$,

$$a=\sum_{k=0}^{N-1} l(f_{k+1k},f_{k+1k}u_k,u_{k+1}). \tag{12}$$

The DEL equations for reduced discrete Lagrangian l are called the discrete Euler-Poincaré equations (DEP).

Theorem 3.1. *Let L be a left $G-$invariant discrete Lagrangian on $S\times S$ and let l be the reduced discrete Lagrangian associated to L. For a finite sequence $\{(g_k,v_k),(g_{k+1},v_{k+1})\}$, $k=0,1,\dots,N-1$, in $S\times S$, with $(g_0,v_0),(g_N,v_N)$ fixed we define a corresponding sequence $f_{k+1k}=g_{k+1k}^{-1}g_k$ in G and a sequence $u_k=g_k^{-1}v_k$ in V. The following statements are equivalent:*

1. *The sequence $\{(g_k,v_k),(g_{k+1},v_{k+1})\}, k=0,\dots,N-1$, is an extremum for the action $\mathcal{A}$ for arbitrary variations $\delta g_k,\delta v_k$, $k=1,2,\dots,N-1$.*
2. *The sequence $\{(g_k,v_k),(g_{k+1},v_{k+1})\}, k=0,\dots,N-1$, satisfies the DEL(8).*
3. *The sequence $\{f_{k+1k},f_{k+1k}u_k,u_{k+1}\}$, $k=0,\dots,N-1$, is an extremum for the reduced action a with respect to variations $\delta f_{k+1k},\delta u_k$ induced by the variations $\delta g_k,\delta v_k$.*
4. *The sequence $\{f_{k+1k},f_{k+1k}u_k,u_{k+1}\}$, $k=0,\dots,N-1$, satisfies the DEP:*

$$\begin{aligned} &D_1l(f_{k+1k},f_{k+1k}u_k,u_{k+1})\mathrm{Ad}_{f_{k+1k}}T_eR_{f_{k+1k}}-D_1l(f_{kk-1},f_{kk-1}u_{k-1},u_k)\\ &T_eR_{f_{kk-1}}=\mathrm{Ad}^*_{f_{k+1k}}(f_{k+1k}u_k\diamond D_2l(f_{k+1k},f_{k+1k}u_k,u_{k+1}))-f_{kk-1}u_{k-1}\\ &\diamond D_2l(f_{kk-1},f_{kk-1}u_{k-1},u_k),f_{k+1k}D_2l(f_{k+1k},f_{k+1k}u_k,u_{k+1})+\\ &D_3l(f_{kk-1},f_{kk-1}u_{k-1},u_k)=0. \end{aligned} \tag{13}$$

Proof. Let $(g_k, v_k) : \varepsilon \in (-a, a) \mapsto (g_k(\varepsilon), v_k(\varepsilon)) \in S$, $k = 0, \ldots, N$, a family of curves on S such that $g_k(0) = g_k, v_k(0) = v_k$. For this family the action of L is

$$\mathcal{A}(\varepsilon) = \sum_{k=0}^{N-1} L(g_k(\varepsilon), v_k(\varepsilon), g_{k+1}(\varepsilon), v_{k+1}(\varepsilon)).$$

By considering the variations of g_k, v_k,

$$\delta g_k = \left.\frac{\mathrm{d}g_k(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}, \delta v_k = \left.\frac{\mathrm{d}v_k(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}$$

and using the discrete analogue of integration by parts it results

$$\delta\mathcal{A} = \left.\frac{\mathrm{d}\mathcal{A}(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \sum_{k=1}^{N-1} \left[D_1 L(g_k, v_k, g_{k+1}, v_{k+1}) + D_3 L(g_{k-1}, v_{k-1}, g_k, v_k)\right] \delta g_k$$

$$+ \sum_{k=1}^{N-1} \left[D_2 L(g_k, v_k, g_{k+1}, v_{k+1}) + D_4 L(g_{k-1}, v_{k-1}, g_k, v_k)\right] \delta v_k.$$

Since the variations $\delta g_k, \delta v_k$ are arbitrary it follows $\delta\mathcal{A} = 0$ if and only if the DEL(8) holds.

Now consider the family of curves $f_{k+1k}(\varepsilon) = g_{k+1}^{-1}(\varepsilon) g_k(\varepsilon), u_k(\varepsilon) = g_k^{-1}(\varepsilon) v_k(\varepsilon)$, with $f_{k+1k}(0) = f_{k+1k}, u_k(0) = u_k$. Using the concatenation notation it results

$$\delta f_{k+1k} = \left.\frac{\mathrm{d}f_{k+1k}(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \delta g_{k+1}^{-1} g_k + g_{k+1}^{-1} \delta g_k = -g_{k+1}^{-1} \delta g_{k+1} g_{k+1}^{-1} g_k +$$

$$g_{k+1}^{-1} \delta g_k = -(g_{k+1}^{-1} \delta g_{k+1}) f_{k+1k} + f_{k+1k} (g_{k+1}^{-1} \delta g_k) f_{k+1k}^{-1} f_{k+1k} =$$

$$-T_e R_{f_{k+1k}} (g_{k+1}^{-1} \delta g_{k+1}) + \mathrm{Ad}_{f_{k+1k}} T_e R_{f_{k+1k}} (g_k^{-1} \delta g_k);$$

$$\delta u_k = \left.\frac{\mathrm{d}u_k(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = -g_k^{-1} \delta g_k g_k^{-1} v_k + g_k^{-1} \delta v_k = -(g_k^{-1} \delta g_k) u_k + g_k^{-1} \delta g_k; \tag{14}$$

$$\delta(f_{k+1k} u_k) = \left.\frac{\mathrm{d}f_{k+1k}(\varepsilon) u_k(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = -(g_{k+1}^{-1} \delta g_{k+1})(f_{k+1k} u_k) +$$

$$\left[f_{k+1k} (g_k^{-1} \delta g_k) f_{k+1k}^{-1}\right] (f_{k+1k} u_k) + f_{k+1k} \delta u_k = -(g_{k+1}^{-1} \delta g_{k+1})(f_{k+1k} u_k) +$$

$$\mathrm{Ad}_{f_{k+1k}} (g_k^{-1} \delta g_k)(f_{k+1k} u_k) + f_{k+1k} \delta u_k.$$

From $L = l \circ \pi$ it results $a \circ \pi = \mathcal{A}$ and $\left.\frac{\mathrm{d}a(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \left.\frac{\mathrm{d}\mathcal{A}(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}$, where

$$a(\varepsilon) = \sum_{k=0}^{N-1} l(f_{k+1k}(\varepsilon), f_{k+1k}(\varepsilon) u_k(\varepsilon), u_{k+1}(\varepsilon)).$$

It follows

$$\delta a = \left.\frac{\mathrm{d}a(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \sum_{k=0}^{N-1} [D_1 l(f_{k+1k}, f_{k+1k} u_k, u_{k+1}) \delta f_{k+1k} +$$

$$+D_2l(f_{k+1k}, f_{k+1k}u_k, u_{k+1})\delta(f_{k+1k}u_k) + D_3l(f_{k+1k}, f_{k+1k}u_k, u_{k+1})\delta u_{k+1}].$$

Using the discrete analogue of integration by parts we obtain

$$\delta a = \sum_{k=0}^{N-1}[D_1l(f_{k+1k}, f_{k+1k}u_k, u_{k+1})\mathrm{Ad}_{f_{k+1k}}T_eR_{f_{k+1k}}(g_k^{-1}\delta g_k)-$$

$$D_1l(f_{k+1k}, f_{k+1k}u_k, u_{k+1})T_eR_{f_{k+1k}}(g_{k+1}^{-1}\delta g_{k+1}) + f_{k+1k}u_k\diamond$$

$$D_2l(f_{k+1k}, f_{k+1k}u_k, u_{k+1})(g_{k+1}^{-1}\delta g_{k+1}^{-1}) - f_{k+1k}u_k\diamond D_2l(f_{k+1k}, f_{+1k}u_k, u_{k+1})$$

$$(\mathrm{Ad}_{f_{k+1k}}g_k^{-1}\delta g_k) + f_{k+1k}D_2l(f_{k+1k}, f_{k+1k}u_k, u_{k+1})\delta u_k+ \tag{15}$$

$$D_3l(f_{k+1k}, f_{k+1k}u_k, u_{k+1})\delta u_{k+1}] = \sum_{k=1}^{N-1}[D_1l(f_{k+1k}, f_{k+1k}u_k, u_{k+1})\mathrm{Ad}_{f_{k+1k}}$$

$$T_eR_{f_{k+1k}} - D_1l(f_{kk-1}, f_{kk-1}u_{k-1}, u_k)T_eR_{f_{kk-1}} + f_{kk-1}u_{k-1}\diamond$$

$$D_2l(f_{kk-1}, f_{kk-1}u_{k-1}, u_k) - \mathrm{Ad}_{f_{k+1k}}(f_{k+1k}u_k\diamond D_2l(f_{k+1k}, f_{k+1k}u_k, u_{k+1}))]$$

$$(g_k^{-1}\delta g_k) + \sum_{k=1}^{N-1}[f_{k+1k}D_2l(f_{k+1k}, f_{k+1k}u_k, u_{k+1}) + D_3l(f_{kk-1}, f_{kk-1}u_{k-1}, u_k)]\delta u_k.$$

Because the variations $\delta f_{k+1k}, \delta u_k$ are abitrary it follows $\delta a = 0$ if and only if the DEP(13) holds.

A map

$$\Phi : ((g_{k-1}, v_{k-1}), (g_k, v_k)) \in S^2 \mapsto ((g_k, v_k), (g_{k+1}, v_{k+1})) \in S^2 \tag{16}$$

which satisfies (8) is called the discrete Euler-Lagrange algorithm.

A map $\varphi : (f_{kk-1}, f_{kk-1}u_{k-1}, u_k) \in S \times V \mapsto (f_{k+1k}, f_{k+1k}u_k, u_{k+1}) \in S \times V$ which satisfies (13) is called the discrete Euler-Poincaré algorithm.

Theorem 3.2. (reduction, reconstruction). *The canonical projection π given by (10) applies the DEL algorithm in the DEP algorithm. From the DEP algorithm it can be constructed the DEL algorithm.*

Proof. The first part of the theorem derives from Theorem 3.1. The DEL algorithm can be reconstructed from the DEP algorithm by

$$((g_{k-1}, v_{k-1}), (g_k, v_k)) \longrightarrow ((g_k, v_k), (g_{k+1}, v_{k+1})) =$$

$$((g_{k-1}f_{kk-1}^{-1}, g_{k-1}f_{kk-1}^{-1}u_k), (g_kf_{k+1k}^{-1}, g_kf_{k+1k}^{-1}u_{k+1})), \tag{17}$$

where (f_{k+1k}, u_k) is a solution of the DEP equations given by 13. The algorithm 17 is the DEL algorithm because

$$g_{k-1}f_{kk-1}^{-1} = g_{k-1}g_{k-1}^{-1}g_k = g_kg_{k-1}f_{kk-1}^{-1}u_k = g_{k-1}g_{k-1}^{-1}g_ku_k = v_k.$$

4 The discrete Euler-Poincaré-Poisson algorithm

The discrete Euler-Poicaré-Poisson are obtained for Lagrange functions which contain adequate parameters by using the Lagrange-d'Alembert principle.

Let V be a finite dimensional vector space, G a finite dimensional Lie group acting on the left on V. The action is denoted by $gv \in V$, for $g \in G, v \in V$. The left action on the cartesian product $G \times V$ is given by $h(g,v) = (hg, hv), \forall (g,v) \in G \times V, \forall h \in G$. The action of G on V^* is given by

$$(ha)(v) = a(hv), \forall a \in V^*, \forall h \in G, \forall v \in V.$$

Let $L : G \times G \times V^* \longrightarrow R$ be a left $G-$invariant function, that is

$$L(g_1, g_2, a) = L(hg_1, hg_2, ha), \forall h \in G, (g_1, g_2) \in G \times G, a \in V^*.$$

For $a_0 \in V^*$ fixed, we define $L_{a_0} : G \times G \longrightarrow R$ by $L_{a_0}(g_1, g_2) = L(g_1, g_2, a_0)$. L_{a_0} is left $G-$invariant with respect to the left action of G_{a_0} on G, where $G_{a_0} = \{h \in G | ha_0 = a_0\}$ is the isotropy group of a_0. The invariance of L permits us to define the reduced Lagrangian $l : G \times V^* \longrightarrow R$ by

$$l(g_2^{-1}g_1, g_2^{-1}a_0) = L(g_1, g_2, a_0) = L(g_2^{-1}g_1, e, g_2^{-1}a_0). \tag{18}$$

If we consider a sequence $(g_k, g_{k+1}) \in G \times G, k = 0, \dots, N-1$, then we can form the sequences $f_{k+1k} = g_{k+1}^{-1}g_k \in G$ and $a_k = g_k^{-1}a_0 \in V^*, k = 0, \dots, N-1$.

Theorem 4.1. *With the preceding notation, the following are equivalent:*

1. *The sequence* $(g_k, g_{k+1}), k = 0, \dots, N-1$, *is an extremum for the action* $\mathcal{A}_{a_0} : G^{N+1} \longrightarrow R$ *associated to* $L_{a_0} : G^2 \longrightarrow R$ *with* a_0 *fixed.*

2. *The sequence* $(g_k, g_{k+1}), k = 0, \dots, N-1$ *satifies the DEL for* L_{a_0} *and for arbitrary variation* $\delta g_k, k = 1, \dots, N-1$, $\delta g_0 = \delta g_N = 0$.

3. *The sequence* $(f_{k+1k}, a_k), k = 0, \dots, N-1$ *satisfies the Lagrange-d'Alembert on* $G \times V^*$, *associated to* $l : G \times V^* \longrightarrow R$ *for the variations* $\delta f_{k+1k}, \delta a_k$.

4. *The sequance* $(f_{k+1k}, a_k), k = 0, \dots, N-1$, *satisfies the discrete Euler-Poincaré-Poisson equations (DEPP):*

$$D_1 l(f_{k+1k}, a_k) \mathrm{Ad}_{f_{k+1k}} T_e R_{f_{k+1k}} - D_1 l(f_{kk-1}, a_{k-1}) T_e R_{f_{kk-1}} = \\ = D_2 l(f_{k+1k}, a_k) \diamond a_k. \tag{19}$$

Proof. Let $g_k : \varepsilon \in (-a, a) \to g_k(\varepsilon) \in G$ be with $g_k(0) = g_k$. The action of L_{a_0} for $g_k(\varepsilon)$ is

$$\mathcal{A}_{a_0}(\varepsilon) = \sum_{k=0}^{N-1} L_{a_0}(g_k(\varepsilon), g_{k+1}(\varepsilon)).$$

Denoting $\delta g_k = \dfrac{\mathrm{d}g_k(\varepsilon)}{\mathrm{d}\varepsilon}|_{\varepsilon=0}$ and using the discrete analogue of the integration by parts it results that $\delta \mathcal{A}_{a_0} = 0$, for arbitrary variations $\delta g_k, \delta g_0 = \delta g_N = 0$ if, and only if the DEL hold:

$$D_2 L_{a_0}(g_{k-1}, g_k) + D_1 L_{a_0}(g_{k-1}, g_{k+1}) = 0. \tag{20}$$

If $f_{k+1k}(\varepsilon) = g_{k+1}^{-1}(\varepsilon) g_k(\varepsilon)$, then we denote $\delta f_{k+1k} = \dfrac{\mathrm{d}f_{k+1k}(\varepsilon)}{\mathrm{d}\varepsilon}|_{\varepsilon=0}$ and using the concatenation operations it results

$$\delta f_{k+1k} = -T_e R_{f_{k+1k}}(g_{k+1}^{-1}\delta g_{k+1}) + \mathrm{Ad}_{f_{k+1k}} T_e R_{f_{k+1k}}(g_k^{-1}\delta g_k). \tag{21}$$

From $a_k = g_k^{-1}a_0$ for $a_k(\varepsilon) = g_k^{-1}(\varepsilon)a_0$, we have

$$\delta a_k = \delta g_k^{-1}a_0 = -(g_k^{-1}\delta g_k g_k^{-1})a_0 = -(g_k^{-1}\delta g_k)a_k. \tag{22}$$

We shall prove 3) is equivalent to 4). The Lagrange-d'Alembert principle, for $l : G \times V^* \longrightarrow R$ and the variations $\delta f_{k+1k}, \delta a_k$, is

$$\sum_{k=0}^{N-1} D_1 l(f_{k+1k}, a_k)\delta f_{k+1k} + D_2 l(f_{k+1k})\delta a_k = 0.$$

Substituting (21) and (22) we obtain:

$$\sum_{k=1}^{N-1}[D_1 l(f_{k+1k}, a_k)\mathrm{Ad}_{f_{k+1k}} T_e R_{f_{k+1k}} - D_1 l(f_{kk-1}, a_{k-1})T_e R_{f_{kk-1}} -$$

$$D_2 l(f_{k+1k}, a_k)\diamond a_k](g_k^{-1}\delta g_k) = 0. \tag{23}$$

For each variation of the form $g_k^{-1}\delta g_k$, from (23) follows (19). The equivalence 1) $\Longleftrightarrow$ 3) results from the $G-$invariance of L and from the fact that the variation of $g_k a_k = a_0$ cancels, which proves that L_{a_0} depends only of g_k, g_{k+1}.

5 The Kelvin-Noether theorem

Let $L_{a_0} : G \times G \longrightarrow R$ depending on a parameter $a_0 \in V^*$ as above. Consider a manifold $\mathcal{C}$ on which G acts to the left and an equivariant map $\mathcal{K} : \mathcal{C} \times V^* \longrightarrow \mathcal{G}$. A discrete Kelvin-Noether quantity associated to the discrete reduced Lagrangian l is the map $I : \mathcal{C} \times G \times V^* \longrightarrow R$ given by:

$$I(c_k, f_{kk-1}, a_k) = D_1 l(f_{kk-1}, a_{k-1})T_e R_{f_{kk-1}}\mathcal{K}(c_k, a_k). \tag{24}$$

Theorem 5.1. (discrete Kelvin-Noether). *Let $c_0 \in \mathcal{C}$ fixed, f_{k+1k}, a_k and g_k the solution of the equation $g_{k+1} = g_k f_{k+1k}^{-1}$ with $g_0 = e \in G$. $c_k = g_k^{-1}c_0$ and $I_k = I(c_k, f_{kk-1}, a_k)$ the following relatio holds:*

$$I_{k+1} - I_k = (D_2 l(f_{k+1k}, a_k)\diamond a_k)(\mathcal{K}(c_k, a_k)). \tag{25}$$

Proof. Applying to $\mathcal{K}(c_k, a_k)$ the operator (25), it results

$$D_1 l(f_{k+1k}, a_k)\mathrm{Ad}_{f_{k+1k}} T_e R_{f_{k+1k}}\mathcal{K}(c_k, a_k) - D_1 l(f_{kk-1}, a_{k-1})T_e$$

$$R_{f_{kk-1}}\mathcal{K}(c_k, a_k) = [D_2 l(f_{k+1k}, a_k)\diamond a_k]\,(\mathcal{K}(c_k, a_k)). \tag{26}$$

$\mathcal{K}$ is equivariant, therefore (26) implies

$$D_1 l(f_{k+1k}, a_k)T_e R_{f_{k+1k}}\mathcal{K}(f_{k+1k}c_k, f_{k+1k}a_k) - D_1 l(f_{kk-1}, a_{k-1})T_e$$

$$R_{f_{kk-1}}\mathcal{K}(c_k, a_k) = [D_2 l(f_{k+1k}, a_k)\diamond a_k]\,(\mathcal{K}(c_k, a_k)). \tag{27}$$

Since $f_{k+1k}c_k = g_{k+1}^{-1}c_0 = c_{k+1}$; $f_{k+1k}a_k = a_{k+1}$, from (27) we deduce (25).

6 The discrete Euler-Poincaré equations for principal bundles

Let (Q, M, π, G) be a principal $G-$bundle, where $M =^Q /_G$ is the quotient manifold of Q by the action of the Lie group G on Q nd $\pi : Q \longrightarrow M$ is the canonical projection. A $M-$morphism $(F, \varphi) : (Q \times Q, M \times M, \pi \times \pi, G \times G) \longrightarrow (Q, M, \pi, G)$ is gives by the maps $F : Q \times Q \longrightarrow Q$, $\varphi : G \times G \longrightarrow G$, such that

$$F(q_k, q_{k+1}) = B(q_k, q_{k+1})q_k,$$

(28) $$\varphi(g, h) = h^{-1}g,$$

where $B : Q \times Q \longrightarrow G$ satisfies the condition

(29) $$B(gq_k, hq_{k+1}) = h^{-1}gB(q_k, q_{k+1})g{-}1.$$

Indeed, from the $M-$morphism property

(30) $$F((g, h)(q_k, q_{k+1})) = \varphi(g, h)F(q_k, q_{k+1})$$

it follows $B(gq_k, hq_{k+1})gq_k = h^{-1}gB(q_k, q_{k+1})q_k$, therefore (29).

By considering a local trivalization on Q the element $q_k \in Q$ has the form $q_k = (r_k, g_k)$, where $r_k = \pi(q_k), q_k = g_k(r_k, e)$. From (29) it results

$$B(q_k, q_{k+1}) = B((r_k, g_k), (r_{k+1}, g_{k+1})) = B(g_k(r_k, e), g_{k+1}(r_{k+1}, e)) =$$

(31) $$g_{k+1}^{-1}g_kB((r_k, e), (r_{k+1}, g_{k+1}))g_k^{-1} = f_{k+1k}b(r_k, r_{k+1})g_k^{-1} = \omega_{k+1k}g_k^{-1},$$

where

(32) $$f_{k+1k} = g_{k+1}^{-1}g_k, g(r_k, r_{k+1}) = B((r_k, e), (r_{k+1}, e)).$$

Let $\tilde{\pi}: Q \times Q \longrightarrow Q \times M$ be the map given by

(33) $$\tilde{\pi}\,(q_k, q_{k+1}) = (B(q_k, q_{k+1})q_k, \pi(q_{k+1})).$$

It follows

(34) $$\tilde{\pi}\,(q_k, q_{k+1}) = (\omega_{k+1k}g_k^{-1}(r_k, g_k), (r_{k+1}, e)) = ((r_k, \omega_{k+1k}), r_{k+1}).$$

Now let $L : Q \times Q \longrightarrow R$ be a $G-$invariant discrete Lagrangian and let l be the reduced Lagrangian defined by $l \circ \tilde{\pi} = L$. Thus

$$l(r_k, \omega_{k+1k}, r_{k+1}) = L((r_k, g_{k+1}^{-1}g_k), (r_{k+1}, e)) = L((r_k, \omega_{k+1k}b(r_k, r_{k+1})^{-1})(r_{k+1}, e)).$$

Theorem 6.1. *The following statements are equivalent:*

1. *The sequence* $\{q_k = (r_k, g_k)\}, k = 0, \ldots, N,$ *is an extremum for the action* $\mathcal{A} : Q^{N+1} \longrightarrow R$ *associated to* L*, where*

$$\mathcal{A}(q_0, \ldots, q_N) = \sum_{k=0}^{N-1} L((r_k, g_k), (r_{k+1}, g_{k+1})).$$

2. *The sequence* $\{q_k = (r_k, g_k)\}, k = 0, \dots, N,$, *satisfies the DEL for* L *and for any variation* $\delta q_k = (\delta r_k, \delta g_k)$, *with* $\delta r_0 = \delta r_N = 0$, $\delta g_0 = \delta g_N = 0$.

3. *The sequence* $(r_k, \omega_{k+1k}, r_{k+1}), k = 0, \dots, N-1$, *is an extremum for the action* $a : \left({}^{Q\times Q}/_G\right)^{N+1} \longrightarrow R$ *associated to* l;

$$a(r_0, \omega 10, r_1, \dots, r_{N-1}, \omega_{NN-1}, r_N) = \sum_{k=0}^{N-1} l(r_k, \omega_{k+1k}, r_{k+1}).$$

4. *The sequence* $(r_k, \omega_{k+1k}, r_{k+1}), k = 0, \dots, N-1$, *satisfies the discrete Euler-Poncaré equations (DEP):*

$$D_1 l(r_k, \omega_{k+1k}, r_{k+1}) + D_3 l(r_{k-1}, \omega_{kk-1}, r_k) + D_2 l(r_k, \omega_{k+1k}, r_{k+1})$$
$$\omega_1(k+1, k) + D_2 l(r_{k-1}, \omega_{kk-1}, r_k)\omega_2(k, k-1) = 0$$
$$D_2 l(r_k, \omega_{k+1k}, r_{k+1}) T_e R_{\omega_{k+1k}} \mathrm{Ad}_{\omega_{k+1k}} b^{-1}(r_k, r_{k+1}) -$$
$$D_2 l(r_{k-1}, \omega_{kk-1}, r_k) T_e R_{\omega_{kk-1}} = 0, \tag{35}$$

where $D_i l(r_k, \omega_{k+1k}, r_{k+1}), i = 1, 2, 3$, *represents the matrix of the partial derivatives with respect to the argument* i *and*

$$\omega_1(k+1, k) = T_e L_{\omega_{k+1k}} T_{(r_k,)} L_{b^{-1}(r_k, r_{k+1})} D_1 b(r_k, r_{k+1}),$$
$$\omega_2(k+1, k) = T_e L_{\omega_{k+1k}} T_{(r_k,)} L_{b^{-1}(r_k, r_{k+1})} D_2 b(r_k, r_{k+1}), \tag{36}$$

represent the $1-$*forms associated to the* $M-$*morphism* (F, φ).

Proof. From the properties of this type for differentiable manifolds it results that the statements 1) and 2) are equivalent. Consider the family $\omega_{k+1k}(\varepsilon) = f_{k+1k}(\varepsilon) \cdot b(r_k(\varepsilon), r_{k+1}(\varepsilon))$ with $f_{k+1k}(\varepsilon) = g_{k+1}^{-1}(\varepsilon) g_k(\varepsilon), \varepsilon \in (-a, a)$ and $\omega_{k+1k}(0) = \omega_{k+1k}$. By denoting $\delta\omega_{k+1k} = \dfrac{\mathrm{d}\omega_{k+1k}(\varepsilon)}{\mathrm{d}\varepsilon}|_{\varepsilon=0}$, we have

$$\delta\omega_{k+1k} = -T_e R_{\omega_{k+1k}}(g_{k+1}^{-1}\delta g_{k+1}) + T_e R_{\omega_{k+1k}} \mathrm{Ad}_{\omega_{k+1k} b^{-1}(r_k, r_{k+1})}(g_k^{-1}\delta g_k) +$$
$$+ T_e L_{\omega_{k+1k}} T_{(r_k, e)} L_{b^{-1}(r_k, r_{k+1})} D_1 b(r_k, r_{k+1})(\delta r_k) + T_e L_{\omega_{k+1k}} T_{r_{k+1}} L_{b^{-1}(r_k, r_{k+1})}$$
$$D_2 b(r_k, r_{k+1})(\delta r_{k+1}).$$

We deduce

$$\delta a = \frac{\mathrm{d}a(\varepsilon)}{\mathrm{d}\varepsilon}|_{\varepsilon=0} = \sum_{k=1}^{N-1} [D_1 l(r_k, \omega_{k+1k}, r_{k+1}) + D_3 l(r_{k-1}, \omega_{kk-1}, r_k) +$$
$$+ D_2 l(r_k, \omega_{k+1k}, r_{k+1})\omega_1(k+1, k) + D_2 l(r_{k-1}, \omega_{kk-1}, r_k)\omega_2(k-1, k)](\delta r_k) +$$
$$+ \sum_{k=1}^{N-1} [D_2 l(r_k, \omega_{k+1k}, r_{k+1}) T_e R_{\omega_{k+1k}}, r_{k+1}) T_e R_{\omega_{k+1k}}$$
$$\mathrm{Ad}_{\omega_{k+1k} b^{-1}(r_k, r_{k+1})} - D_2 l(r_{k-1}, \omega_{k-1k}, r_k) T_e R_{\omega_{kk-1}}](g_k^{-1}\delta g_k).$$

Thus $\delta a = 0$ iff the relations (35) holds. Therefore 3) and 4) are equivalent. The equivalence of 1) and 3) follows from the $G-$invariance of L.

A map $\Phi : (q_{k-1}, q_k) \in Q^2 \mapsto (q_k, q_{k+1}) \in q^2$ satisfying the relation

$$D_1 L(q_k, q_{k+1}) + D_2 L(q_{k-1}, q_k) = 0 \tag{37}$$

is called a discrete Euler-Lagrange algorithm (DELA).

A map $\varphi : (r_{k-1}, \omega_{kk-1}, r_k) \in^{Q\times Q} /_G \mapsto (r_k, \omega_{k+1k}, r_{k+1}) \in^{Q\times Q} /_G$ satisfaying the relations 35 is called a discrete Euler-Poincaré algorithm (DEPA).

Theorem 6.2. (reduction, reconstruction). *The canonical projection $\widetilde{\pi}$ given by (33) applies the algorithm DELA on the algorithm DEPA. The algorithm DELA can be reconstructed from the algorithm DEPA.*

Proof. The first statement of the theorem results from Theorem 6.1.. The algorithm DELA is constructed from the algorithm DEPA as follows: let (r_k, ω_{k+1k}) be a solution of the algorithm DEPA and $f_{k+1k} = \omega_{k+1k}\ b^{-1}(r_k, r_{k+1})$; since $f_{k+1k} = g_{k+1}^{-1} g_k$ for g_k fixed, it results $g_{k+1} = g_k f_{k+1k}^{-1}$. In this way the algorithm DELA is given by

$$((r_{k-1}, g_{k-1}), (r_k, g_k)) \longrightarrow ((r_k, g_k), (r_{k+1}, g_{k+1})) = ((r_k, g_{k-1} f_{kk+1}^{-1}), (r_{k+1}, g_k f_{k+1k}^{-1}).$$

Remark. If the map B has the property $B(q_k, q_{k+1}) = e \in G \forall (q_k, q_{k+1}) \in Q^2$, then the EDP are

$$D_1 l(r_k, f_{k+1k}, r_{k+1}) + D_3 l(r_{k-1}, f_{kk-1}, r_k) = 0,$$

$$D_2 l(r_k, f_{k+1k}, r_{+1}) T_e R_{f_{k+1k}} \cdot \mathrm{Ad}_{f_{k+1k}} - D_2 l(r_{k-1}, f_{kk-1}, r_k) T_e R_{f_{kk-1}} = 0,$$

where $f_{k+1k} = g_{k+1}^{-1} g_k$.

Acknowledgements. A version of this paper was presented at the Workshop on Electromagnetic Flows and Dynamics, Oct 15-19, 1999, University Politehnica of Bucharest.

Supported by MEN Grant No 21815/28.09.1999, CNCSU-31.

References

[1] A. Bobenko, Y. B. Suris, *Discrete Lagrangian reduction, discrete Euler-Poincaré equations and semidirect products*, Lett. Math. Phys. 49(1999), 79-93.

[2] H. Cendra, D. D. Holm, J. E. Marsden, T. Ratiu, *Lagrangian reduction, the Euler-Poincaré equations and semidirect products*, AMS Arnold Volume II (1999).

[3] D. D. Holm, J. E. Marsden, T. Ratiu, *The Euler-Poincaré equations and semidirect products with applications to continuum theories*, Adv. Math. (1998).

[4] J. E. Marsden, G. Patrick, S. Shkoller, *Multisymplectic geometry, variational integrators and nonlinear PDE´s*, Comm. Math. Phys. (1998).

[5] J. E. Marsden, S. Pekarsky, S. Shkoller, *Discrete Euler-Poincaré and Lie-Poisson equations*, (1998) (15 pag.)

Universitatea de Vest din Timişoara, Catedra de Algebră
Bulevardul V. Pârvan 4, 1900 Timişoara
E-mail addresss: opris@tim1.uvt.ro
E-mail addresss: albud@tim1.uvt.ro