Riemannian Manifolds with Almost Constant Scalar Curvature

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Abstract

We find sufficient conditions for a Riemannian manifold with almost constant scalar curvature s (i.e. satisfying (R(X,Y)Z)s = 0) to have constant scalar curvature. In particular, this is true for compact or conformally flat manifolds, or for hypersurfaces in spaceforms. Examples are given which show a quite exotic behaviour of the scalar curvature.

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1 Introduction

Defined by E.Cartan [4] as a generalization of locally symmetric manifolds, semi-symmetric manifolds are characterized by the condition (involving the curvature tensor) $RR = 0$, where $R(X,Y)$ acts on the tensorial algebra like a derivation, for every vector fields X and Y. Semi-symmetric manifolds were classified by Z.Szabo (see [14] for further details and references).

When contracting the previous formula, we obtain $RQ = 0$, where Q is the Ricci tensor. The manifolds which satisfy this condition were studied by several authors (K. Nomizu [9], S. Tanno [15], Sekigawa and Takagi [13], G. Pripoae and the author [8]).

This process may continue (even if not by contraction !), for the class of Riemannian manifolds satisfying $Rs = 0$, where s is the scalar curvature; that is, $R(X,Y)Zs = 0$, for every vector fields X, Y and Z. We say that the scalar curvature is *almost constant.*

Despite its (maybe too) formal appearence, this property has a strong geometric flavour: generically, it means that the scalar curvature is or constant, or its level hypersurfaces are invariant submanifolds. Moreover, the almost constant scalar curvature seems to be useful in at least two difficult topics. First, we have the posibility to formulate an extended Yamabe problem, "intermediating" between the constant and the arbitrary prescribed scalar curvature:

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(*) *Given a Riemannian manifold* (M, g), *is there a metric* g_1, *conformal to* g, *with almost constant scalar curvature* s_1 ?

When M is compact and s_1 is asked to be constant, we recover the classical problem of Yamabe, with positive answer given by Schoen ([12]). As we prove in Theorem 5, on compact manifolds there do not exist *any* metric with non-constant almost constant scalar curvature, so there Problem (*) is irrelevant.

On noncompact manifolds, existence and non-existence results were proven for constant ([1]) or prescribed ([6]) scalar curvatures in some conformal class. A common feature of these results are some strong non-positivity hypothesis put on the sectional and the Ricci curvatures of the apriori metric. In §5 we consider an example of Riemannian manifold with non-constant almost constant sectional curvature, non-conformally flat and without the previously quoted negativity conditions. So, there is no doubt that Problem (*) is non-trivial on non-compact manifolds but, as yet, it is not clear if the system of PDE's governing it is workable.

Secondly, the almost constant scalar curvature condition is equivalent to the fact that the Hessian of the scalar curvature is a Codazzi tensor. It is known ([5]) that the eigenspace distributions of a Codazzi tensor are integrable, and the integral manifolds are totally umbilical. In §4 *we prove that these leaves are also invariant submanifolds, so all their second fundamental forms are also Codazzi tensors. For them we can repeat the construction, considering their integral manifolds; this process of (generic) reduction stops only when every umbilical leaf is totally geodesic.*

We also provide examples of (warped product) Riemannian manifolds having (*non-constant*) almost constant scalar curvature (§2). In order to distinguish between the constant and the (non-constant) almost constant cases, we prove (§3):

Each of the following conditions are sufficient for an almost constant scalar curvature to be constant : (1) M is compact; (ii) M is conformally flat; (iii) the Ricci tensor is non-degenerated; (iv) the sectional curvature has constant sign; (v) M is a hypersurface in a space form.

2 Examples

Let (M, g) a connected, n-dimensional Riemannian manifold. We denote by ∇ the Levi-Civita connection of g, and by R the curvature tensor, defined for every vector fields X, Y by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Consider Q the Ricci tensor of (0,2)-type and r the associated (1,1)-tensor field, defined by $g(rX, Y) = Q(X, Y)$. The trace of Q is the scalar curvature s. If s is almost constant, we say that (M, g) is an *ACSC-manifold.*

Remarks. (i) Every two-dimensional Riemanian manifold with almost constant scalar curvature has constant curvature.

(ii) In the case $n = 3$, the curvature tensor may be expressed as

$$R(X, Y) = \frac{s}{2} X \wedge Y - rX \wedge Y - X \wedge rY.$$

If $Rs = 0$, by contracting twice we obtain $rX(s) = 0$, and $Y(s)X - X(s)Y = 0$ for every vector fields X, Y on M. Thus s is constant.

So, in what follows, we may suppose the dimension of manifolds be greater than three.

Examples. (i) A Riemannian manifold with parallel ds has almost constant scalar curvature s. It is known that any differentiable manifold admits a metric with constant scalar curvature (see [12] for the compact and [3] for the non-compact cases).

(ii) Let A be an open half plane in R^2 and $f : A \to R$ a non-vanishing linear function. Consider the warped product metric g on $M = A \times_f R^2$, given by

$$g = (dx^1)^2 + (dx^2)^2 + f^2((dx^3)^2 + (dx^4)^2).$$

Then the scalar curvature $s = -2|df|^2/f^2$ is a non-constant function defined on A.

On another hand, the curvature tensor vanishes for all vector fields on M, but for those tangent to the second plane. This proves that $R(X,Y)Zs = 0$, for every vector fields X, Y, Z on M.

So, (M, g) is ACSC-manifold with non-constant scalar curvature (and ds non-parallel).

(iii) More generally, let (B, g_B) and (F, g_F) be two Riemannian manifolds (the "base" and the "fiber" respectively) and f a differentiable function on B. Denote the dimension of F by d and the projections on B and F by π and σ respectively. Consider the warped product manifold $B\times_f$F, with the metric

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F).$$

Then

$$s = s_B - \frac{2d}{f}\Delta_B f + \frac{s_F}{f^2} - d(d-1)\frac{|df|^2}{f^2}$$

where s_B and s_F are the scalar curvatures of B and F and Δ_B is the Laplacian of g_B. Denote by X, Y, Z and by U, V, W arbitrary vector fields tangent to the base and to the fiber, respectively. Denote H_s the Hessian of f on B. Then, the condition $Rs = 0$ writes

$$[{}^B R(X,Y)Z] \;\; s = 0 \quad , \quad H_f(X,Y)V(s_F) = 0 \quad , \quad (\nabla_X gradf)s = 0$$

$$\left[{}^F R(V,W)U + \frac{|df|^2}{f^2} g(V,U)W - g(W,U)V\right](s_F) = 0$$

A sufficient condition for $B \times_f F$ to be an ACSC-manifold is: "s_F is a constant function and f satisfies the (previous) first and third differential equations on B".

Finally, we remark that a Riemannian product of ACSC-manifolds is an ACSC-manifold too.

3 Basic properties of ACSC-manifolds

Consider a Riemannian manifold M, with scalar curvature s. Denote ξ the vector field associated to the one-form ds. Then we have some general properties.

Lemma 1. *The following assertions are equivalent :*

(i) $Rs = 0$;

(ii) $R(X,Y)\xi = 0$;

(iii) $R(\xi, X)Y = 0$;

(iv) $\delta Q(R(X,Y)Z) = 0$;

(v) H_s *is a Codazzi tensor.*

Proof. The condition $Rs = 0$ writes $g(\xi, R(X,Y)Z) = 0$, hence the first three assertions are equivalent.

For the next equivalence we recall the classical identity $ds = 2\delta Q$, where δ is the divergence operator. We have

$$g(R(X,Y)Z,\xi) = X[(\nabla_Y ds)Z] - (\nabla_Y ds)(\nabla_X Z) - Y[(\nabla_X ds)Z] +$$

$$+(\nabla_X ds)(\nabla_Y Z) - (\nabla_{[X,Y]}Z)(s) = (\nabla_X H_s)(Y,Z) - (\nabla_Y H_s)(X,Z)$$

which proves the last equivalence. □

Lemma 2. *If* $Rs = 0$, *then*

(i) $(rX)s = 0$;

(ii) $r(\nabla_\xi \xi) = r(\xi) = 0$;

(iii) $(\delta Q)(rX) = 0$;

(iv) $(\nabla_X \nabla_Y r)\xi = (\nabla_Y \nabla_X r)\xi$;

(v) $(\nabla_\xi R)(X,Y,Z) = R(\nabla_Y \xi, X)Z - R(\nabla_X \xi, Y)Z$.

Proof. By contracting in Lemma 1,(i) we get (i) and (iii). Formula (ii) results from $g(r\xi, X) = g(\xi, rX) = 0$.

Consider the second identity of Bianchi

$$\sum_{XYZ} (\nabla_X R)(Y,Z,V) = 0 \tag{1}$$

for arbitrary vector fields X, Y, Z, V. We replace $X = V = \xi$ and we deduce $R(Y,Z)\nabla_\xi \xi = 0$, for every Y and Z. By contraction, we obtain (ii).

If in (1) we replace $X = \xi$, from the previous relations we get (v).

Finally, we use the identity

$$(\nabla_X \nabla_Y Q)(Z,W) - (\nabla_Y \nabla_X Q)(Z,W) = Q(R(Z,W)X,Y) - Q(R(Z,W)Y.X)$$

and we obtain (iv) . □

Remarks. The definition of ACSC-manifolds might look a bit formal. The following geometric interpretation shows that this new class has a non-trivial behaviour:

(i) Consider an ACSC-manifold whose scalar curvature has no critical points. Then, the level sets of the scalar curvature foliate M. From Prop.1, (iii) we see that *each such hypersurface is invariant* (that is, the curvature of M, restricted to each leaf, remains tangent to the respective leaf).

Generically, this property characterizes the ACSC-manifolds. Indeed,let (M,g) be a Riemannian manifold and suppose the scalar curvature has no critical points (so, we exclude the constant scalar curvature case and we consider only the generic regular points of s). *If all the level sets of s are invariant hypersurfaces, then (M,g) is an ACSC-manifold.*

This characterization is similar to a result by Ogiue [10]: a Riemannian manifold has constant (sectional) curvature if and only if all its submanifolds are invariant.

(ii) For the larger class of manifolds satisfying only the condition $rs = 0$, we have a similar characteristic property: the leaves of the respective foliation are Ricci-invariant hypersurfaces (i.e. the restriction of the Ricci tensor of$M$, to each leaf, remains tangent to the leaf).

The next two results are easy consequences of the lemmas.

Proposition 3. *An ACSC-manifold with non-degenerate Ricci tensor has constant scalar curvature.*

Proposition 4. *An ACSC-manifold with non-vanishing sectional curvature has constant scalar curvature.*

The next result shows there are no interesting *compact* ACSC-manifolds.

Theorem 5. *A compact ACSC-manifold has constant scalar curvature.*

Proof. Apply the Bochner-Lichnerowicz formula to the scalar curvature function s

$$\frac{1}{2}\Delta(\mid ds \mid^2) = \mid H_s \mid^2 + d\Delta s(\xi) - Q(\xi,\xi)$$

So, on the compact manifold M we have the integral formula

$$\int_M \mid H_s \mid^2 d\sigma = 0.$$

We deduce $H_s = 0$, so the differential of s is a parallel 1-form. If s is not constant, then (generically) M decomposes as a product with a line factor, which leads to a contradiction.□

When a new geometric condition appears, it is interesting to investigate it in the framework of the submanifolds theory.The next result shows that the ACSC condition is irrelevant for a large class of hypersurfaces.

Theorem 6. *If a hypersurface in a spaceform is an ACSC-manifold, then it has constant scalar curvature.*

Proof. Let (M, g) be an immersed hypersurface in a spaceform of constant curvature c. Denote by A the Weingarten tensor of M. The Gauss equation gives the curvature tensor of M

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\} + g(AY,Z)AX - g(AX,Z)AY.$$

By hypothesis, $Rs = 0$, so

$$c\{X(s)Y - Y(s)X\} + AX(s)AY - AY(s)AX = 0.$$

Fix a point of M, and e_1, ... , e_n a basis of eigenvectors of A, corresponding to the eigenvalues λ_1, ... ,λ_n respectively. The last formula becomes

$$(c - \lambda_i\lambda_j)e_j(s) = 0 \qquad (1)$$

for every indices i and j.

Case I: $c < 0$. For $i = j$ in (1), we obtain $e_j = 0$, for every j.

Case II: $c > 0$. If all the eigenvalues are equal to $\sqrt{c}$, then M is a totally umbilical hypersurface; a standard result implies M has constant (sectional) curvature.

Consider i fixed, and another indice j such that the corresponding eigenvalues are distinct. Then at least one of the two quantities $(c-(\lambda_i)^2)$ and $(c-\lambda_i\lambda_j)$ is non-null. From formula (1) we derive $e_i(s)=0$.

Case III: $c=0$. Consider a dense open subset U of M, on which the number of distinct eigenvalues of A is locally constant. On U, the eigenvalue functions $x \to \lambda_i(x)$ are smooth, and the corresponding eigenspace distributions V_{λ_i} are smooth and with constant rank.

Suppose V_0 (the eigenspace distribution corresponding to the null eigenvalue) has rank k ($k \geq 1$); if $\lambda_1 = ... = \lambda_k = 0$, then formula (1) implies $e_i(s)=0$ for $i>k$.

One knows V_0 is a totally geodesic distribution. If (N,h) is an integral manifold of V_0, it is flat. So, the scalar curvature of M, restricted to N, vanishes. We deduce s is constant along the distribution V_0, so is constant on the whole U. By continuity, s is constant on M. □

The next step is to investigate the relevance of the ACSC property in the conformal geometry.

Theorem 7. *A conformally flat ACSC-manifold has constant scalar curvature.*

Proof. If (M,g) is conformally flat, we have the well-known formula, given by the vanishing of the conformal curvature tensor

$$R(X,Y)Z - \frac{s}{(n-1)(n-2)}\{g(X,Z)Y - g(Y,Z)X\} - \tag{2}$$

$$-\frac{1}{n-2}\{g(Y,Z)rX - Q(X,Z)Y - g(X,Z)rY + Q(Y,Z)X\} = 0.$$

The condition $Rs=0$ becomes

$$\frac{s}{(n-1)}\{X(s)Y - Y(s)X\} - X(s)rY + Y(s)rX = 0 \tag{3}$$

since $rX(s)=0$ by Prop.2,(ii).

Suppose that the gradient ξ of the scalar curvature is non-null. In (3), we replace $X=\xi$ and obtain

$$rY = \frac{s}{n-1}\{Y - \frac{Y(s)}{\xi(s)}\xi\}. \tag{4}$$

Suppose λ be a non-null eigenvalue of r and Y a corresponding eigenvector. Then, from (4) we deduce $Y(s)=0$.

If Y is an eigenvector of the null eigenvalue of r, then (4) implies Y is a multiple of ξ. Hence, the multiplicity of the null eigenvalue is one.

Let λ be a non-null eigenvalue or r and Y a corresponding eigenvector. Since Y cannot be colinear with ξ, from (4) it follows $\lambda = \dfrac{s}{n-1}$. Hence, all the $n-1$ non-null eigenvalues of r are equal to $\dfrac{s}{n-1}$.

For X,Y,Z orthogonal to ξ, there exists a function k such that (2) writes

$$R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}.$$

If $k=0$, then the curvature tensor vanishes, hence a contradiction.

On another hand, for each point p where $k \neq 0$, the set

$$\{R(x,y)z \mid x,y,z \in T_pM\}$$

spannes the whole tangent space T_pM. From $R(s) = 0$ we deduce s is constant.□

4 A Codazzi digression

Consider an ACSC-manifold. Suppose ds is non-parallel, so the Hessian H_s is a non-null Codazzi tensor. Let U be a dense open set in M such that on U the number of the eigenvalues for H_s is locally constant. Denote by λ such an eigenvalue function and by V_λ the corresponding eigenspace distribution. From the general theory of Codazzi tensors (for exemple, ch.16,C in [2]), it is known that each distribution V_λ is integrable, and has umbilical leaves. Moreover, we have

Proposition 8. *For an arbitrary Codazzi tensor on M, the leaves of eigenspace distributions are invariant submanifolds, so all the second fundamental forms are Codazzi tensors.*

Proof. Let b a (0,2)-tensor of Codazzi type. Let $X \in V_\alpha$, $Y \in V_\beta$, $Z \in V_\gamma$. One knows ([5]) that for $\alpha \neq \gamma$, and for $\beta \neq \gamma$ we have $R(X,Y)Z = 0$. From the symmetries of the curvature tensor we derive easily that $R(X,Y)Z \in V_\alpha$, for $X,Y,Z \in V_\alpha$ and for $X,Z \in V_\beta$, $Y \in V_\alpha$ with $\alpha \neq \beta$.

So, for every integral manifold N of V_α, the curvature tensor of (M,g) leaves invariant the tangent spaces of N. By the Codazzi-Mainardi equation, for every vector field $W \in V_\beta$, orthonormal to N, the second fundamental form h^W is a Codazzi tensor on N. □

Remarks. (i) Considering an arbitrary non-null Codazzi tensor field b, we can perform an "umbilical reduction" of the manifold M, in a neighboorhood of each generic point $p \in M$; so, we can obtain a totally umbilical leaf N containing p, for each eigenspace distribution of b. If every N is totally geodesic, the algorithm stops. If there exists at least a leaf N, with non-null second fundamental form h, this one is again a Codazzi tensor, but on N. Now we can repeat the reduction, starting with h on N, instead with b on M.

(ii) If the Codazzi tensor is just H_s, then Prop.8 implies that the Ricci tensor Q of (M,g) vanishes on each product $V_\alpha \times V_\beta$, with different α and β.This remark, together with the Weitzenböck formula ([2], p.436)

$$\delta\nabla H_s - H_{\Delta s} = R^0(H_s) - H_s \circ Q$$

prove the

Corollary 9. *For an ACSC-manifold, $H_{\Delta s} = \delta\nabla H_s$, on every product of different eigenspace distributions.*

5 A Yamabe digression

In [1], Aviles and McOwen solved the Yamabe problem for noncompact complete Riemannian manifolds, with upper bounded negative scalar curvature, and found conformally metrics with constant negative scalar curvature. Jin ([6]) started with noncompact, complete Riemannian manifolds with upper bounded negative sectional curvature and Ricci curvature bounded from below by a negative constant. For a prescribed function K, satisfying various bound conditions, he proved existence or non-existence theorems for conformally related metrics having scalar curvature K.

Consider Exemple 2 from §2. In view of Theorem 7, it is clear that the ACSC-manifold (M, g) is not conformally flat. Moreover, its sectional, scalar and Ricci curvatures can reach arbitrary negative values. So, we have a non-trivial solution for Problem (*), which cannot be recovered from the previously quoted results.

Remark. Let (M, g) be a Riemannian manifold, f a real differentiable function on M and the metric $g_1 = e^{2f}g$. We denote by

$$A_f = H_f - df \circ df + \left(\frac{|df|^2}{2}\right) g$$

and by a_f its associated (1,1)-tensor field. Let h_f the associated (1,1)-tensor for the Hessian H_f. It is known that the (1,3)-curvature tensor R^1 and the scalar curvature s^1 on (M, g_1) are related to those on (M, g) by

$$R^1 = R - g \otimes a_f - A_f \otimes Id + a_f \otimes g + Id \otimes A_f$$

and

$$s^1 = e^{-2f}(s - 2(n-1)tracea_f))$$

where

$$2tracea_f = -\Delta f + (n-2)|df|^2.$$

Imposing the condition $R^1 s^1 = 0$ leads to a third order system of nonlinear PDE's, whose dominant term is (in local coordinates)

$$(R^1)^i_{jkl} \frac{\partial}{\partial x^i} \Delta f.$$

Integrating it is much more chalenging than the classical Yamabe problem, whose solutions (constant s^1) are (maybe) the simplest.

6 Comments

Remarks. (i) The notion of ACSC-manifold can be defined also in semi-Riemannian geometry (i.e. for non-degenerate metrics), without major difficulties. Most of our results admit extensions to this more general case, with some precautions due to the signature and to the possible degeneracy of induced metric tensors.

(ii) The class of ACSC-manifolds is strictly contained in the class of Riemannian manifolds satisfying $rs = 0$ (for example, choose an appropriate function f in the warped products of §2).

(iii) Consider an ACSC-manifold M and, for every point $p \in M$, define the subspace D_p spanned bt the set $\{R(x, y)z \mid x, y, z \in T_pM\}$. If, for each point p, D_p "filles" the tangent space T_pM, then the scalar curvature is constant.

But, in general, the distribution $p \to D_p$ may have non-constant rank. Denote $D^\perp$ its complementary distribution. (In general, D and $D^\perp$ may be singular and even non-differentiable). If s is constant "along" $D^\perp$, then again s is constant on M.

This fact and the examples of §2 suggest that, on ACSC-manifolds, the obstruction to the constancy of s is contained in the distribution $D^{\perp}$.

(iv) In the proof of Theorem 7 we need not (M, g) to be conformally flat; it suffices that the conformal curvature tensor C satisfy the condition $Cs = 0$.

A similar result is true for the projective curvature tensor P : if an ACSC-manifold satisfies also $Ps = 0$, then its scalar curvature is constant.

(v) Proposition 4 can be improved: if p is a regular point of s, then the sectional curvature of every 2-plane π containing ξ_p vanishes.

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